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Research Article

Periodic Points of Asymmetric Bernoulli Shifts

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It is well-known that Sharkovskii's theorem gives a complete structure of periodic order for a continuous self-map on a closed bounded interval. As a further study, a natural problem is how to determine the location and number of periodic points for a specific map. This paper considers the periodic points of asymmetric Bernoulli shift, which is a piecewise linear chaotic map.

1. Introduction

In 1964, Sharkovskii [1] firstly introduced a special ordering on the set of positive integers. This ordering implies that if $p \lhd q$ and a continuous self-map of a closed bounded interval has a point of period p; then it has a point of period q. The least number with respect to this ordering is 3. Thus, if a map has a point of period 3, then it has points of any periods. In 1975, the latter result was rediscovered by Li and Yorke [2]. Then numerous papers are devoted to the study of interval maps (see e.g., [3–5] and references therein).

Bifurcation points of some interval maps were studied in [6], and the limit behavior of orbits and probabilistic some problems were considered in [7, 8]. Recently, Ivanov in [9] considered an exact lower bound for the number of orbits of a given period for a self-map of a closed bounded interval.

Consider the asymmetric Bernoulli shift F: $[0,1] \longrightarrow [0,1]$ with a parameter $0 < a < 1$, defined by

$$F(x) = \begin{cases} \dfrac{x}{a}, & 0 \le x \le a, \\ \\ \dfrac{x-a}{1-a}, & a < x \le 1. \end{cases} \tag{1}$$

Specially, when $a = 1/2$, it is the Bernoulli Shift or the binary transformation, also known as doubling map or the binary transformation. Conjugacies between asymmetric Bernoulli shifts are constructed in [10].

Given a positive integer n, one interesting question is how to find all n-periodic points of F. The other is how many n-periodic points of F.

In this paper, we study periodic orbits of F. In the next section, we present dynamics of jumps of F^n. Section 3 recalls the real number representation, i.e, F-expansion. In Section 4, we use the F-expansion to give explicit formulas of $F^n(x)$ for $n \in \mathbb{N}$, explicit formulas of jumps of $F^n(x)$, explicit formulae of fixed points of $F^n(x)$, and explicit formulas of all n-periodic points of $F(x)$. The last section gives the number $h(n)$ of periodic orbits of a given period n for F and the limit behavior of $h(n)$.

2. Dynamics of Jumps of F^n

For $n \in \mathbb{N}$, let $F^n(x)$ denote the n-th iterate of F, which is recursively defined by $F^0(x) = x$ and $F^n = F(F^{n-1}(x))$ for $x \in [0,1]$.

A point $c \in (0,1)$ is called a jump of F if the one-sided limits, $F(c-)$ and $F(c+)$, exist and are finite, but are not equal. The set of jumps of F is denoted by $\mathcal{J}(F)$. One can see that

$$\mathcal{J}(F) \subseteq \mathcal{J}\left(F^2\right) \subseteq \cdots \subseteq \mathcal{J}(F^n) \subseteq \mathcal{J}\left(F^{n+1}\right) \subseteq \cdots. \tag{2}$$

Each element of $\mathcal{J}(F^{n+1}) \backslash \mathcal{J}(F^n)$ must be a preimage under F of a point from $\mathcal{J}(F^n)$. More precisely,

$$\mathcal{J}\left(F^{n+1}\right)\backslash\mathcal{J}\left(F^{n}\right) = F^{-1}\left(\mathcal{J}\left(F^{n}\right)\right)\backslash\mathcal{J}\left(F^{n}\right). \quad (3)$$

The map F has the unique jump a. Put $x_{1,0}: = 0, x_{1,1}: = a$, and $x_{1,2}: = 1$. Let I denote the unit interval $[0,1]$, $I_{1,1}: = (x_{1,0}, x_{1,1})$, and $I_{1,2}: = (x_{1,1}, x_{1,2})$. One can see that F^n has $2^n - 1$ jumps for $n \geq 2$ by induction. For $i, j \in \mathbb{N}^+$, let $x_{i,0}: = 0, x_{i,2^i}: = 1$, and $x_{i,j}$ denote the j th jumps of F^i in the following order:

$$0 = x_{i,0} < x_{i,1} < x_{i,2} < \cdots < x_{i,j} < \cdots < x_{i,2^i-1} < x_{i,2^i} = 1. \quad (4)$$

Put $I_{i,j}: = (x_{i,j-1}, x_{i,j})$ for every $j \in \{1, 2, 3, \ldots, 2^i\}$. It is clear that $I_{i,j}$ is the j-th monotonic interval of F^i.

Lemma 1. *For $n \geq 1$, the jumps of F^n and F^{n-1} have the following relationship:*

(i) $F(x_{n,k}) = F(x_{n,2^{n-1}+k}) = x_{n-1,k}$ for $1 \leq k \leq 2^{n-1} - 1$
(ii) $x_{n,2^{n-1}} = x_{n-1,2^{n-2}} = a$
(iii) $F^i(I_{n,k}) = F^i(I_{n,2^{n-1}+k}) = I_{n-i,k}$ for $1 \leq k \leq 2^{n-i}$ and $1 \leq i \leq n-1$

Proof. We first claim that a is a jump of F^n for every $n \geq 1$. In fact, since a is a jump of $F(x)$, a is also a jump of $F^n(x)$ for $n \geq 2$. Moreover, it is easy to check that $F^n(a) = 1$ for $n \geq 2$.

Next, we prove (i) and (ii) by induction. It is clear that these results holds for $n = 2$.

Assume that these results hold for $n = m \geq 2$, i.e.,

(i) $F(x_{m,k}) = F(x_{m,2^{m-1}+k}) = x_{m-1,k}$ for $1 \leq k \leq 2^{m-1} - 1$
(ii) $x_{m,2^{m-1}} = x_{m-1,2^{m-2}} = a$

Now we shall prove these results hold for $n = m + 1$. Denote $2^m - 1$ jumps of F^m by

$$0 < x_{m,1} < x_{m,2} < \cdots < x_{m,k} < \cdots < x_{m,2^m-1} < 1. \quad (5)$$

Since F is strictly increasing on the subinterval $I_{1,1}$ and $F(I_{1,1}) = (0,1)$, for each $k \in \{1, 2, \ldots, 2^m - 1\}$, there exists the unique point, denoted by $x_{m+1,k}$, in $I_{1,1}$ such that $F(x_{m+1,k}) = x_{m,k}$. Since F is strictly increasing on $I_{1,1}$, one can see that

$$0 < x_{m+1,1} < x_{m+1,2} < \cdots < x_{m+1,k} < \cdots < x_{m+1,2^m-1} < a. \quad (6)$$

Further, by the definition of jump, $x_{m+1,k}$ is a jump of $F^{m+1} = F^m \circ F$ for each $k \in \{1, 2, \ldots, 2^m - 1\}$.

Similarly, since F is strictly increasing on the subinterval $I_{1,2}$ and $F(I_{1,2}) = (0,1)$, for each $k \in \{1, 2, \ldots, 2^m - 1\}$, there exists the unique point, denoted by $x_{m+1,2^m+k}$, in $I_{1,2}$ such that $F(x_{m+1,2^m+k}) = x_{m,k}$. Since F is strictly increasing on $I_{1,2}$, one can see that

$$\begin{aligned} a &< x_{m+1,2^m+1} < x_{m+1,2^m+2} < \cdots < x_{m+1,2^m+k} \\ &< \cdots < x_{m+1,2^{m+1}-1} < 1. \end{aligned} \quad (7)$$

Further, by the definition of jump, $x_{m+1,2^m+k}$ is a jump of $F^{m+1} = F^m \circ F$ for each $k \in \{1, 2, \ldots, 2^m - 1\}$. Let $x_{m+1,2^m}$ denote a. Therefore,

(i) $F(x_{m+1,k}) = F(x_{m+1,2^m+k}) = x_{m,k}$ for $1 \leq k \leq 2^m - 1$
(ii) $x_{m+1,2^m} = x_{m,2^{m-1}} = a$

It follows from (i) that for $1 \leq k \leq 2^{n-1}$,

$$F\left(I_{n,k}\right) = F\left(I_{n,2^{n-1}+k}\right) = I_{n-1,k}. \quad (8)$$

Then for $1 \leq k \leq 2^{n-i}$ and $1 \leq i \leq n-1$,

$$F^i\left(I_{n,k}\right) = F^i\left(I_{n,2^{n-1}+k}\right) = I_{n-i,k}. \quad (9)$$

This completes the proof. □

3. F-Expansion

In this section, we will introduce a new real number representation.

Definition 1. A sequence $\{\varepsilon_k\}_{k\in\mathbb{N}^+}$ of 0 and 1 is called the itinerary of $x \in [0,1]$ with respect to the asymmetric Bernoulli shift F: $[0,1] \longrightarrow [0,1]$ and $a \in (0,1)$, if, for $k \geq 1$,

$$\varepsilon_k = \begin{cases} 0, & F^{k-1}(x) \leq a, \\ 1, & F^{k-1}(x) > a. \end{cases} \quad (10)$$

In fact, the itinerary of $x \in [0,1]$ with respect to F and $a \in (0,1)$ is just the F-expansion of a real $x \in [0,1]$. According to [10], or these two classic papers [11, 12], we have an expansion for x in powers of the numbers a and $1 - a$:

$$x = \sum_{k=1}^{\infty} \varepsilon_k a^{k-s_{k-1}} (1-a)^{s_{k-1}} = \sum_{k=1}^{\infty} \varepsilon_k a^k \left(\frac{1-a}{a}\right)^{s_{k-1}}, \quad (11)$$

where $s_0 = 0$ and $s_k: = \sum_{j=1}^{k} \varepsilon_j$ for $k \geq 1$.

Thus, every $x \in [0,1]$ can be represented through its digit sequence $\{\varepsilon_k\}_{k\in\mathbb{N}^+}$. In this situation, write $x = [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k, \ldots]$ for short. One can see that every infinite F-expansion is unique, whereas each $x \in (0,1)$ with a finite F-expansion can be expanded in exactly two ways, namely, one immediately verifies that

$$x = [\varepsilon_1, \ldots, \varepsilon_{k-1}, 1] = [\varepsilon_1, \ldots, \varepsilon_{k-1}, 0, 1, 1, 1, \ldots]. \quad (12)$$

In the following, we employ a convention in which finite fractions such as

$$\begin{aligned} &[1, 1, 0, 0, 0, 0, \ldots] = [1, 0, 1, 1, 1, 1, \ldots], \\ &[1, 0, 1, 0, 0, 0, \ldots] = [1, 0, 0, 1, 1, 1, \ldots], \end{aligned} \quad (13)$$

are represented as finite fractions with infinite zeros, as $[1, 1, 0, 0, 0, \ldots]$ or $[1, 0, 1, 0, 0, 0, \ldots]$, unless otherwise stated.

Lemma 2. *If* $x = [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k, \ldots]$, *then* $F(x) = [\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_k, \ldots]$.

Proof. It follows from a property of the asymmetric Bernoulli shift $F(x)$ that $x \le a$ provided that $\varepsilon_1 = 0$ in $x = [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k, \ldots]$. One then finds

$$\begin{aligned} F(x) &= \frac{x}{a} = \sum_{k=1}^{\infty} \varepsilon_k a^{k-1} \left(\frac{1-a}{a}\right)^{s_{k-1}} \\ &= \sum_{k=2}^{\infty} \varepsilon_k a^{k-1} \left(\frac{1-a}{a}\right)^{s_{k-1}} \\ &= \sum_{k=1}^{\infty} \varepsilon_{k+1} a^{k} \left(\frac{1-a}{a}\right)^{s_{k}} \\ &= [\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_k, \ldots]. \end{aligned} \tag{14}$$

On the other hand, one has $a < x \le 1$ if $\varepsilon_1 = 1$ and hence

$$\begin{aligned} F(x) &= \frac{x-a}{1-a} = \sum_{k=2}^{\infty} \varepsilon_k a^{k-s_{k-1}} (1-a)^{s_{k-1}-1} \\ &= \sum_{k=1}^{\infty} \varepsilon_{k+1} a^{k+1-s_k} (1-a)^{s_k-1} \\ &= [\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_k, \ldots]. \end{aligned} \tag{15}$$

□

This shows that, from the perspective of symbolic dynamics, F corresponds to the shift map on the space $\{0,1\}^{\mathbb{N}^+}$, at least for those points with an infinite F-expansion.

One easily finds that the periodicity of the orbits is related to recurring F-expansions. For example,

$$[1, 0, 1, 1, 1, 1, 0, 1, 1, 1, \ldots], \tag{16}$$

is a recurring F-expansion with the recurring unit of the length 5, and hence, it is a 5-periodic point of F.

4. The Explicit Formula of F^n

Since F^n is a piecewise linear map, and F^n is strictly increasing on each subinterval $I_{n,k}$. One can obtain the explicit formula of F^n.

Theorem 1. *If* $x = [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k, \ldots]$, *then*

$$F^n(x) = \frac{x}{a^{n-s_n}(1-a)^{s_n}} - \sum_{j=1}^{n} \frac{\varepsilon_j}{a^{n-j-s_n+s_j-1}(1-a)^{s_n-s_j+1}}, \quad \text{for } n \in \mathbb{N}^+. \tag{17}$$

Proof. We prove this result by mathematical induction.

We firstly consider the trivial case $n = 1$. If $x \in I_{1,1}$, then $\varepsilon_1 = 0$ and $s_1 = 0$. Thus,

$$F(x) = \frac{x}{a^{1-s_1}(1-a)^{s_1}} - \sum_{j=1}^{1} \frac{\varepsilon_j}{a^{1-j-s_1+s_j-1}(1-a)^{s_1-s_j+1}} = \frac{x}{a}. \tag{18}$$

If $x \in I_{1,2}$, then $\varepsilon_1 = 1$ and $s_1 = 1$. Thus,

$$\begin{aligned} F(x) &= \frac{x}{a^{1-s_1}(1-a)^{s_1}} - \sum_{j=1}^{1} \frac{\varepsilon_j}{a^{1-j-s_1+s_j-1}(1-a)^{s_1-s_j+1}} \\ &= \frac{x}{1-a} - \frac{a}{1-a}. \end{aligned} \tag{19}$$

Therefore, the result holds for $n = 1$.

Assume that the result holds for $n = m \ge 1$, i.e.,

$$F^m(x) = \frac{x}{a^{m-s_m}(1-a)^{s_m}} - \sum_{j=1}^{m} \frac{\varepsilon_j}{a^{m-j-s_m+s_j-1}(1-a)^{s_m-s_j+1}}. \tag{20}$$

Now we shall prove that the result holds for $n = m+1$. If $F^m(x) \le a$; then $\varepsilon_{m+1} = 0$ and $s_{m+1} = s_m$. Thus,

$$\begin{aligned} F^{m+1}(x) &= \frac{1}{a} \cdot \left(\frac{x}{a^{m-s_m}(1-a)^{s_m}} - \sum_{j=1}^{m} \frac{\varepsilon_j}{a^{m-j-s_m+s_j-1}(1-a)^{s_m-s_j+1}} \right) \\ &= \frac{x}{a^{m+1-s_{m+1}}(1-a)^{s_{m+1}}} - \sum_{j=1}^{m+1} \frac{\varepsilon_j}{a^{m+1-j-s_{m+1}+s_j-1}(1-a)^{s_{m+1}-s_j+1}}. \end{aligned} \tag{21}$$

If $F^m(x) > a$; then $\varepsilon_{m+1} = 1$ and $s_{m+1} = s_m + 1$. Thus,

$$
\begin{aligned}
F^{m+1}(x) &= \frac{1}{1-a}\cdot\left(\frac{x}{a^{m-s_m}(1-a)^{s_m}} - \sum_{j=1}^{m}\frac{\varepsilon_j}{a^{m-j-s_m+s_j-1}(1-a)^{s_m-s_j+1}}\right) - \frac{a}{1-a} \\
&= \frac{x}{a^{m-s_m}(1-a)^{s_m+1}} - \sum_{j=1}^{m}\frac{\varepsilon_j}{a^{m-j-s_m+s_j-1}(1-a)^{s_m-s_j+1+1}} - \frac{a}{1-a} \\
&= \frac{x}{a^{m+1-s_{m+1}}(1-a)^{s_{m+1}}} - \sum_{j=1}^{m}\frac{\varepsilon_j}{a^{m+1-j-s_{m+1}+s_j-1}(1-a)^{s_{m+1}-s_j+1}} - \frac{a}{1-a} \\
&= \frac{x}{a^{m+1-s_{m+1}}(1-a)^{s_{m+1}}} - \sum_{j=1}^{m+1}\frac{\varepsilon_j}{a^{m+1-j-s_{m+1}+s_j-1}(1-a)^{s_{m+1}-s_j+1}}.
\end{aligned}
\tag{22}
$$

Therefore, the result holds for $n = m + 1$. The proof is completed. □

As a corollary, we present the exact formulas of these jumps of F^n.

Corollary 1. *All jumps of F^n are given by*

$$\sum_{j=1}^{n}\varepsilon_j a^{j+1-s_j}(1-a)^{s_j-1}, \tag{23}$$

where $\varepsilon_j = 0$ or 1, and not all are ε_j equal to 0.

Proof. If all ε_j are zero, then $x_{1,0} = 0$, and it is not a jump.

From Theorem 2, solving $F^n(x) = 0$, we can obtain all these jumps of F^n. □

Definition 2. A point x in X is called a periodic point of a self-mapping $f\colon X \longrightarrow X$ if there exists an positive integer n such that

$$f^n(x) = x. \tag{24}$$

The smallest positive integer n satisfying the above is called the prime period or least period of the point x, the point x is called an n-periodic point of f, and the sequence $\{x, f(x), \ldots, f^{n-1}(x)\}$ is called an n-periodic orbit.

In particularly, an 1-periodic point is called a fixed point.

The following corollary presents the exact formulas of all fixed points of F^n.

Corollary 2. *All fixed points of F^n are given by*

$$\frac{1}{1-a^{n-s_n}(1-a)^{s_n}}\sum_{j=1}^{n}\varepsilon_j a^{j+1-s_j}(1-a)^{s_j-1}, \tag{25}$$

where $\varepsilon_j = 0$ or 1.

Proof. Since the curve of $y = F^n(x)$ intersects the line of $y = x$ at 2^n points, $F^n(x)$ has 2^n fixed points. Solving $F^n(x) = x$, we have

$$x = \frac{1}{1-a^{n-s_n}(1-a)^{s_n}}\sum_{j=1}^{n}\varepsilon_j a^{j+1-s_j}(1-a)^{s_j-1}. \tag{26}$$

□

5. The Number of n-Periodic Points

The fixed points of F are the intersections of $y = F(x)$ and $y = x$, namely, two points $x = 0, 1$. The intersections of $y = F^2(x)$ and $y = x$ have four points where there are two 2-periodic points, namely, two points

$$
\begin{gathered}
\frac{a^2}{1-a+a^2}, \\
\frac{a}{1-a+a^2}.
\end{gathered}
\tag{27}
$$

The other two intersections $x = 0$ and 1 are the fixed points. The intersections of $y = F^3(x)$ and $y = x$ have eight points where there are six 3-periodic points and two fixed points.

In general, the intersections of $y = F^n(x)$ and $y = x$ have 2^n periodic points. If x is a p-periodic point of F, then $p \mid n$. Let $h(p)$ denote the number of the p-periodic points. Then,

$$\sum_{p\mid n} h(p) = 2^n, \quad \text{for every integer } n \geq 1, \tag{28}$$

where the sum extends over all positive divisors p of n.

In order to obtain the exact number $h(n)$ of n-periodic points of F, we need to introduce the Möbius function and Möbius inversion formula (see, for example, [13, 14]).

Define the Möblius function $\mu\colon \mathbb{N} \longrightarrow \{1, 0, 1\}$ by

$$
\mu(n) = \begin{cases} 1, & n = 1, \\ (-1)^r, & n = q_1, q_2, \ldots, q_r, q_1 < q_2 < \cdots < q_r, \\ 0, & \text{others}. \end{cases}
\tag{29}
$$

Thus, if $(n_1, n_2) = 1$, then $\mu(n_1 n_2) = \mu(n_1)\mu(n_2)$, and for any $n \in \mathbb{N}$, there holds

$$\sum_{k\mid n}\mu(k) = \left\lfloor \frac{1}{n} \right\rfloor. \tag{30}$$

Lemma 3 (Möbius inversion formula). *If h and g are arithmetic functions, i.e., from $\mathbb{N}$ to $\mathbb{C}$, satisfying*

$$g(n) = \sum_{d|n} h(d), \quad \text{for every integer } n \geq 1. \tag{31}$$

Then,

$$h(n) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right), \quad \text{for every integer } n \geq 1, \tag{32}$$

where μ is the Möbius function and the sums extend over all positive divisors d of n.

In effect, the original $h(n)$ can be determined given $g(n)$ by using the inversion formula.

Corollary 3. *The number of n-periodic points of F is given by,*

$$h(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) 2^d, \quad \text{for every integer } n \geq 1. \tag{33}$$

Let $N(n)$ denote the number of n-periodic orbits of F. Then,

$$N(n) = \frac{h(n)}{n} = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) 2^d, \quad \text{for every integer } n \geq 1. \tag{34}$$

The first several $N(n)$ are

$$\begin{aligned} N(1) &= 2, \\ N(2) &= 1, \\ N(3) &= 2, \\ N(4) &= 3, \\ N(5) &= 6, \\ N(6) &= 9, \\ N(7) &= 18, \\ N(8) &= 30, \end{aligned} \tag{35}$$

while $N(n)$ for larger n are

$$\begin{aligned} N(16) &= 4080, \\ N(20) &= 52377, \\ N(32) &= 134215680, \\ N(64) &= 288230376084602880. \end{aligned} \tag{36}$$

If the values just above are compared to

$$\begin{aligned} \frac{2^{16}}{16} &= 4096, \\ \frac{2^{20}}{20} &= 52428.8, \\ \frac{2^{32}}{32} &= 134217728, \\ \frac{2^{64}}{64} &= 288230376151711744, \end{aligned} \tag{37}$$

one finds that the ratio of $N(n)$ and $(2^n/n)$ approaches 1 as $n \longrightarrow +\infty$.

Now we shall prove that the ratio of $h(n)$ and 2^n approaches 1 as $n \longrightarrow +\infty$.

Theorem 2. *Let $h(p)$ be the number of the p-periodic points of F. Then,*

$$\lim_{n \longrightarrow +\infty} \frac{h(n)}{2^n} = \lim_{n \longrightarrow +\infty} \frac{\sum_{d|n} \mu(n/d) 2^d}{2^n} = 1, \tag{38}$$

where μ is the Möbius function.

Proof. On one hand,

$$\begin{aligned} \sum_{d|n} \mu\left(\frac{n}{d}\right) 2^d &\leq \sum_{d|n} 2^d \\ &= 2^n + \sum_{d|n, d \neq n} 2^d \\ &\leq 2^n + \sum_{1 \leq d \leq \lfloor n/2 \rfloor} 2^d \\ &\leq 2^n + 2^{(n/2)+1} - 2. \end{aligned} \tag{39}$$

On the other hand,

$$\begin{aligned} \sum_{d|n} \mu\left(\frac{n}{d}\right) 2^d &= 2^n + \sum_{d|n, d \neq n} \mu\left(\frac{n}{d}\right) 2^d \\ &\geq 2^n - \sum_{d|n, d \neq n} 2^d \\ &\geq 2^n - \sum_{1 \leq d \leq \lfloor n/2 \rfloor} 2^d \\ &\geq 2^n - 2^{(n/2)+1} + 2. \end{aligned} \tag{40}$$

Consequently,

$$1=\lim_{n\longrightarrow+\infty}\frac{2^n-2^{(n/2)+1}+2}{2^n}\leq\lim_{n\longrightarrow+\infty}\frac{\sum_{d|n}\mu(n/d)2^d}{2^n}\leq\lim_{n\longrightarrow+\infty}\frac{2^n+2^{(n/2)+1}-2}{2^n}=1. \tag{41}$$

By the squeeze theorem,

$$\lim_{n\longrightarrow+\infty}\frac{h(n)}{2^n}=\lim_{n\longrightarrow+\infty}\frac{\sum_{d|n}\mu(n/d)2^d}{2^n}=1. \tag{42}$$

□

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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