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Research Article

Extensions of Ostrowski Type Inequalities via *h*-Integrals and *s*-Convexity

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In this paper, Hölder, Minkowski, and power mean inequalities are used to establish Ostrowski type inequalities for s-convex functions via h-calculus. The new inequalities are generalized versions of Ostrowski type inequalities available in literature.

1. Introduction

In mathematics, the quantum calculus is equivalent to usual infinitesimal calculus without depending upon the concept of limit. It has two major branches, q-calculus and the h-calculus. It is really the calculus of finite differences, but a more systematic analogy with classical calculus makes it additionally transparent. The definite h-integral is a Riemann sum so that the fundamental theorem of h-calculus allows one to evaluate finite sums and h-integration by parts which is simply the Abel transform. The theory of h-discrete calculus is the rapidly developing area of great interest both from theoretical and applied point of view. This calculus is the study of the definitions, properties, and applications of the related concepts, the fractional calculus and discrete fractional calculus.

1.1. h-Derivative [1]. The h-derivative is defined as follows: let $\phi\colon T_\nu \longrightarrow \mathbb{R}, T_\nu = \{\nu, \nu + h, \nu + 2h, \ldots\}$, $h > 0$.

$$D_h\phi(\theta) = \frac{d_h\phi(\theta)}{d_h\theta} = \frac{\phi(\theta + h) - \phi(\theta)}{h}, \tag{1}$$

where, classically, $\lim_{h\longrightarrow 0} D_h\phi(\theta) = \mathrm{d}\phi(\theta)/\mathrm{d}\theta$. Also, h-differential is $d_h\phi(\theta) = \phi(\theta + h) - \phi(\theta)$, in particular $d_h\theta = h$.

1.2. h-Integral [1]. The h-integral is defined as follows: let $\phi\colon T_\nu \longrightarrow \mathbb{R}, T_\nu = \{\nu, \nu + h, \nu + 2h, \ldots\}$, $h > 0$.

$$\int_\nu^\mu \phi(\theta) d_h\theta = \sum_{j=0}^{\mu-\nu/h-1} \phi(\nu + jh)h, \tag{2}$$

where $\nu, \mu \in \mathbb{R}, \nu < \mu$.

1.3. Definition of h-Integral [1]. If $\mu - \nu \in h\mathbb{Z}$, we define h-integral to be

$$\int_{\nu}^{\mu}\phi(\theta)d_h\theta = \begin{cases} h(\phi(\nu)+\phi(\nu+h)+\phi(\nu+2h)+\cdots+\phi(\mu-h)), & \text{if } \nu<\mu, \\ 0 \quad \text{if } \nu=\mu, \\ -h(\phi(\mu)+\phi(\mu+h)+\phi(\mu+2h)+\cdots+\phi(\nu-h)) & \text{if } \nu>\mu. \end{cases} \tag{3}$$

With this definition, the definite h-integral is Riemann sum of $\phi(\theta)$ on the interval $[\nu,\mu]$, which is proportioned to subintervals of equal width.

1.4. Formula of h-Integration by Parts [1]. Let $\phi, g\colon [\nu,\mu] \longrightarrow \mathbb{R}$ be the continuous functions and $\theta \in [\nu,\mu]$, then the formula of h-integration by parts is stated as

$$\int_{\nu}^{\mu}\phi(\theta)d_h g(\theta) = \phi(\mu)g(\mu)-\phi(\nu)g(\nu)-\int_{\nu}^{\mu}g(\theta+h)d_h\phi(\theta). \tag{4}$$

1.5. Properties of h-Calculus [1]. The h-analogue of a binomial expansion $(\theta-\nu)^n$ is defined as

$$(\theta-\nu)_h^n = (\theta-\nu)(\theta-\nu-h)\ldots(\theta-\nu-(n-1)h). \tag{5}$$

For $n\geqslant 1$ and $(\theta-\nu)_h^0 = 1$,

$$\begin{aligned} D_h(\theta-\nu)_h^n &= n(\theta-\nu)_h^{n-1}, \\ D_h(\nu-\theta)_h^n &= -n(\nu-h-\theta)_h^{n-1}, \\ D_h\frac{1}{(\theta-\nu)_h^n} &= -\frac{n}{(\nu+h-\theta)_h^{n+1}}, \\ D_h\frac{1}{(\nu-\theta)_h^n} &= \frac{n}{(\nu-\theta)_h^{n+1}}. \end{aligned} \tag{6}$$

Note that h-analogues of an integer n is still n, and $(\theta-0)_h^n \neq \theta^n$.

1.6. h-Fractional Function [2]. Let $t,\alpha \in \mathbb{R}$, then the h-fractional function $t_h^{(\alpha)}$ is defined by

$$t_h^{(\alpha)} = h^{\alpha}\frac{\Gamma(t/h+1)}{\Gamma(t/h+1-\alpha)}, \tag{7}$$

where Γ is the Euler gamma function $t/h \notin \{-1,-2,-3,-4\ldots\}$. Note that $\lim_{h\longrightarrow 0}t_h^{(\alpha)} = t^{\alpha}$, hence we define

$$t_h^{(\alpha-1)} = \frac{1}{\alpha}D_h\left[t_h^{(\alpha)}\right]. \tag{8}$$

Basic inequalities have a massive role both in pure and applied sciences in the light of their wide applications in mathematics and physical sciences, while convexity theory has stayed as a significant instrument in the foundation of the hypothesis of integral inequalities.

The following Ostrowski inequality [3] is notable to read.

Theorem 1. *Let $\phi\colon I \longrightarrow \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping differentiable in the interior I° of I, and let $\nu,\mu \in I^{\circ}$ with $\nu<\mu$. If $|\phi'(\theta)| \leq M$ for all $\theta \in [\nu,\mu]$, then the following inequality holds:*

$$\left|\phi(\theta)-\frac{1}{\mu-\nu}\int_{\nu}^{\mu}\phi(t)\mathrm{d}t\right| \leq M(\mu-\nu)\left[\frac{1}{4}+\frac{(\theta-\nu+\mu/2)^2}{(\mu-\nu)^2}\right], \tag{9}$$

for all $\theta \in [\nu,\mu]$.

From the invention of (9), it is being studied extensively by many researchers (see [4–9]). Generalizations, extensions, and variants of this inequality exist in literature (see [10–17]) for different classes of convex functions. More on s-convex functions and on the conformable functions can be seen in [4, 18–24].

In [25], the class of functions which are called s-convex in the second sense has been introduced by Hudzik and Maligranda as follows:

1.7. s-Convex Function. A function $\phi\colon \mathbb{R}^+ \longrightarrow \mathbb{R}$ is said to be s-convex in second sense if

$$\phi(\Omega\theta+(1-\Omega)\phi) \leq \Omega^s\phi(\theta)+(1-\Omega)^s\phi(\phi), \tag{10}$$

for each $\theta,\phi \in \mathbb{R}^+$, $\Omega \in [0,1]$ and for unique $s \in (0,1]$.

The integral equality is established by Alomari et al. in [26].

Lemma 1. *Suppose that $\phi\colon J \subset \mathbb{R} \longrightarrow \mathbb{R}$ is a mapping such that $\phi' \in L[\nu,\mu]$, then*

$$\phi(\theta)-\frac{1}{\mu-\nu}\int_{\nu}^{\mu}\phi(u)\mathrm{d}u = \frac{(\theta-\nu)^2}{\mu-\nu}\int_0^1 u\phi'(u\theta+(1-u)\nu)\mathrm{d}u - \frac{(\mu-\theta)^2}{\mu-\nu}\int_0^1 u\phi'(u\theta+(1-u)\mu)\mathrm{d}u, \tag{11}$$

for each $\theta \in [\nu,\mu]$.

Using Lemma 1, Alomari et al. in [26] presented the following integral inequalities.

Theorem 2. *Consider the function $\phi\colon J \subset \mathbb{R}^+ \longrightarrow \mathbb{R}$ such that $\phi' \in L[\nu,\mu]$ for $\nu,\mu \in J$. If $|\phi'|$ in term of second sense is s-convex on $[\nu,\mu]$, for unique $s \in (0,1]$ and $|\phi'(\theta)| \leq M$, $\theta \in [\nu,\mu]$, the following result holds:*

$$\left|\phi(\theta)-\frac{1}{\mu-\nu}\int_{\nu}^{\mu}\phi(u)\mathrm{d}u\right|\leq\frac{M}{\mu-\nu}\left[\frac{(\theta-\nu)^2+(\mu-\theta)^2}{s+1}\right], \tag{12}$$

for each $\theta\in[\nu,\mu]$.

Theorem 3. *Consider the function* $\phi: J\subset\mathbb{R}^+\longrightarrow\mathbb{R}$ *such that* $\phi'\in L[\nu,\mu]$ *for* $\nu,\mu\in J$. *If* $|\phi'|^m$ *is s-convex in second sense on* $[\nu,\mu]$, *for unique* $s\in(0,1]$, $m>1$, $n=m/m-1$ *and* $|\phi'(\theta)|\leq M$, $\theta\in[\nu,\mu]$, *the following integral inequality holds:*

$$\left|\phi(\theta)-\frac{1}{\mu-\nu}\int_{\nu}^{\mu}\phi(u)\mathrm{d}u\right|\leq\frac{M}{(1+n)^{1/n}}\left(\frac{2}{s+1}\right)^{1/m}\left[\frac{(\theta-\nu)^2+(\mu-\theta)^2}{\mu-\nu}\right], \tag{13}$$

for each $\theta\in[\nu,\mu]$.

Theorem 4. *Consider the function* $\phi: J\subset\mathbb{R}^+\longrightarrow\mathbb{R}$ *such that* $\phi'\in L[\nu,\mu]$ *for* $\nu,\mu\in J$. *If* $|\phi'|^m$ *is a s-convex in second sense on* $[\nu,\mu]$, *for some static* $s\in(0,1]$, $m\geq1$ *and* $|\phi'(\theta)|\leq M$, $\theta\in[\nu,\mu]$, *then the following result holds:*

$$\left|\phi(\theta)-\frac{1}{\mu-\nu}\int_{\nu}^{\mu}\phi(u)\mathrm{d}u\right|\leq M\left(\frac{2}{s+1}\right)^{1/m}\left[\frac{(\theta-\nu)^2+(\mu-\theta)^2}{2(\mu-\nu)}\right], \tag{14}$$

for each $\theta\in[\nu,\mu]$.

2. Main Results

Initially, we establish the following identity.

Lemma 2. *Suppose* $\phi: J\longrightarrow\mathbb{R}$ *be h-differentiable mapping on interior of interval* J *in which* $\nu,\mu\in J$ *and* $\nu<\mu$. *If* $D_h\phi\in L[\nu,\mu]$, *then the following h-integral equality is valid:*

$$\begin{aligned}&\phi(\theta)+\frac{h(\theta-\nu)\phi(\nu)+(\mu-\theta)\phi(\mu)}{\mu-\nu}-\frac{1}{\mu-\nu}\int_{\nu}^{\mu}\phi(u)_{\nu}\mathrm{d}_h u\\&=\frac{(\theta-\nu)^2}{\mu-\nu}\int_0^1 uD_h\phi(u\theta+(1-u)\nu)_0 d_h u-\frac{(\mu-\theta)^2}{\mu-\nu}\int_0^1 uD_h\phi(u\theta+(1-u)\mu)_0\mathrm{d}_h u,\end{aligned} \tag{15}$$

for each $\theta\in[\nu,\mu]$.

Proof. By formula (4) of h-integration by parts, the first term of right hand side of (15) becomes

$$\begin{aligned}&\frac{(\theta-\nu)^2}{\mu-\nu}\int_0^1 uD_h\phi(u\theta+(1-u)\nu)\mathrm{d}_h u\\&\frac{(\theta-\nu)^2}{\mu-\nu}\left(u\frac{(\phi(u\theta+(1-u)\nu)}{\theta-\nu}\Big|_0^1-\int_0^1\frac{\phi(\theta(u+h)+(1-(u+h))\nu)}{\theta-\nu}\mathrm{d}_h u\right)\\&=\frac{\theta-\nu}{\mu-\nu}\phi(\theta)-\frac{\theta-\nu}{\mu-\nu}\left(h\sum_{j=0}^{1/h-1}\phi(\theta(jh+h)+(1-(jh+h))\nu)\right)\\&=\frac{\theta-\nu}{\mu-\nu}\phi(\theta)-\frac{\theta-\nu}{\mu-\nu}\left(h\sum_{j=0}^{1/h-1}\phi(\theta(jh+h)+(1-(jh+h))\nu)\right)\\&=\frac{\theta-\nu}{\mu-\nu}\phi(\theta)-\frac{\theta-\nu}{\mu-\nu}\left(h\sum_{j=1}^{1/h-1}\phi(jh\theta+(1-jh)\nu)\right)\\&=\frac{\theta-\nu}{\mu-\nu}\phi(\theta)-\frac{1}{\mu-\nu}\left(h(\theta-\nu)\sum_{j=1}^{1/h-1}\phi(\nu+(\theta-\nu)jh)\right)-\frac{h(\theta-\nu)\phi(\nu)}{(\mu-\nu)}+\frac{h(\theta-\nu)\phi(\nu)}{(\mu-\nu)}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\theta-\nu}{\mu-\nu}\phi(\theta) - \frac{1}{\mu-\nu}\left(h(\theta-\nu)\sum_{j=1}^{1/h-1}\phi(\nu+(\theta-\nu)jh) + \frac{(\theta-\nu)\phi(\nu)}{(\mu-\nu)}\right) + \frac{(\theta-\nu)\phi(\nu)}{(\mu-\nu)}\\
&= \frac{\theta-\nu}{\mu-\nu}\phi(\theta) - \frac{1}{\mu-\nu}\left(h(\theta-\nu)\sum_{j=0}^{1/h-1}\phi(\nu+(\theta-\nu)jh)\right) + \frac{h(\theta-\nu)\phi(\nu)}{(\mu-\nu)}\\
&= \frac{\theta-\nu}{\mu-\nu}\phi(\theta) - \frac{1}{\mu-\nu}\int_{\nu}^{\theta}\phi(u)\mathrm{d}_h u + \frac{h(\theta-\nu)\phi(\nu)}{(\mu-\nu)},
\end{aligned}
\tag{16}
$$

and the second term of right hand side of (15) becomes

$$
\begin{aligned}
&\frac{(\mu-\theta)^2}{\mu-\nu}\int_0^1 uD_h\phi(u\theta+(1-u)\mu)\mathrm{d}_h u\\
&\quad= \frac{(\mu-\theta)^2}{\mu-\nu}\left(u\frac{(\phi(u\theta+(1-u)\mu)}{\theta-\mu}\Big|_0^1 + \int_0^1 \frac{\phi(\theta(u+h)+(1-(u+h))\mu)}{\mu-\theta}\mathrm{d}_h u\right)\\
&\quad= -\frac{\mu-\theta}{\mu-\nu}\phi(\theta) + \frac{\mu-\theta}{\mu-\nu}\left(h\sum_{j=0}^{1/h-1}\phi(\theta(jh+h)+(1-(jh+h))\mu)\right)\\
&\quad= -\frac{\mu-\theta}{\mu-\nu}\phi(\theta) + \frac{\mu-\theta}{\mu-\nu}\left(h\sum_{j=0}^{1/h-1}\phi(\theta(jh+h)+(1-(jh+h))\mu)\right)\\
&\quad= \frac{\theta-\nu}{\mu-\nu}\phi(\theta) - \frac{\theta-\nu}{\mu-\nu}\left(h\sum_{j=1}^{1/h-1}\phi(jh\theta+(1-jh)\mu)\right)\\
&\quad= \frac{\theta-\nu}{\mu-\nu}\phi(\theta) - \frac{1}{\mu-\nu}\left(h(\theta-\mu)\sum_{j=1}^{1/h-1}\phi(\mu+(\theta-\mu)jh)\right) - \frac{h(\theta-\mu)\phi(\mu)}{(\mu-\nu)} + \frac{h(\theta-\mu)\phi(\mu)}{(\mu-\nu)}\\
&\quad= -\frac{\mu-\theta}{\mu-\nu}\phi(\theta) + \frac{1}{\mu-\nu}\left(h(\theta-\mu)\sum_{j=1}^{1/h-1}\phi(\mu+(\theta-\mu)jh) + \frac{(\theta-\mu)\phi(\nu)}{(\mu-\nu)}\right) - \frac{h(\theta-\mu)\phi(\mu)}{(\mu-\nu)}\\
&\quad= -\frac{\mu-\theta}{\mu-\nu}\phi(\theta) + \frac{1}{\mu-\nu}\left(h(\theta-\mu)\sum_{j=0}^{1/h-1}\phi(\nu+(\theta-\mu)jh)\right) - \frac{h(\theta-\mu)\phi(\mu)}{(\mu-\nu)}\\
&\quad= -\frac{\mu-\theta}{\mu-\nu}\phi(\theta) - \frac{1}{\mu-\nu}\int_{\mu}^{\theta}\phi(u)d_h u + \frac{h(\theta-\mu)\phi(\mu)}{(\mu-\nu)}.
\end{aligned}
\tag{17}
$$

From (16) and (17),

$$
\begin{aligned}
&\frac{\theta-\nu}{\mu-\nu}\phi(\theta) - \frac{1}{\mu-\nu}\int_{\nu}^{\theta}\phi(u)\mathrm{d}_h u + \frac{h(\theta-\nu)\phi(\nu)}{(\mu-\nu)} + \frac{\mu-\theta}{\mu-\nu}\phi(\theta) - \frac{1}{\mu-\nu}\int_{\theta}^{\mu}\phi(u)\mathrm{d}_h u + \frac{h(\mu-\theta)\phi(\mu)}{(\mu-\nu)}\\
&\frac{\theta-\nu+\mu-\theta}{\mu-\nu}\phi(\theta) + \frac{h(\theta-\nu)\phi(\nu)+h(\mu-\theta)\phi(\mu)}{\mu-\nu} - \frac{1}{\mu-\nu}\int_{\nu}^{\mu}\phi(u)\mathrm{d}_h u\\
&\quad= \phi(\theta) + \frac{h(\theta-\nu)\phi(\nu)+h(\mu-\theta)\phi(\mu)}{\mu-\nu} - \frac{1}{\mu-\nu}\int_{\nu}^{\mu}\phi(u)\mathrm{d}_h u,
\end{aligned}
\tag{18}
$$

which is the required result. □

Using Lemma 2, we prove the following results.

Theorem 5. *Suppose $\phi: J \longrightarrow \mathbb{R}$ be a h-differentiable mapping on interior of a positive interval J in such a way that $D_h\phi \in L[\nu,\mu]$, for $\nu,\mu \in J$. If $|D_h\phi|$ is the s-convex in second sense on $[\nu,\mu]$ for some fixed $s \in (0,1]$ and $|D_h\phi(\theta)| \le M$, $\theta \in [\nu,\mu]$, we have the following h-integral inequality:*

$$\begin{aligned}
&\left|\phi(\theta) + h\frac{(\theta-\nu)\phi(\nu)+(\mu-\theta)\phi(\mu)}{\mu-\nu} - \frac{1}{\mu-\nu}\int_\nu^\mu \phi(u)_\nu \mathrm{d}_h u\right| \\
&\leqslant M\frac{(\theta-\nu)^2+(\mu-\theta)^2}{\mu-\nu}\left[-\frac{h^{s+1}}{s+1}+\frac{1}{s+1}\left(\frac{-h^{s+2}+(1+h)^{s+2}}{s+2}\right)+\frac{1}{s+2}\right],
\end{aligned} \tag{19}$$

for each $\theta \in [\nu,\mu]$.

Proof. Using Lemma 2 for $s-$convex mapping $|D_h\phi|$ defined on $[\nu,\mu]$, we see

$$\begin{aligned}
&\left|\phi(\theta) + h\frac{(\theta-\nu)\phi(\nu)+(\mu-\theta)\phi(\mu)}{\mu-\nu} - \frac{1}{\mu-\nu}\int_\nu^\mu \phi(u)_\nu \mathrm{d}_h u\right| \\
&\leqslant \frac{(\theta-\nu)^2}{\mu-\nu}\int_0^1 u\left|D_h\phi(u\theta+(1-u)\nu)\right|\mathrm{d}_h u + \frac{(\mu-\theta)^2}{\mu-\nu}\int_0^1 u\left|D_h\phi(u\theta+(1-u)\mu)\right|\mathrm{d}_h u \\
&\leqslant \frac{(\theta-\nu)^2}{\mu-\nu}\left[\int_0^1 u_h^{(s+1)}\left|D_h\phi(\theta)\right|\mathrm{d}_h u + \int_0^1 u(1-u)^s\left|D_h\phi(\nu)\right|\mathrm{d}_h u\right] + \frac{(\mu-\theta)^2}{\mu-\nu}\left[\int_0^1 u_h^{(s+1)}\left|D_h\phi(\theta)\right|\mathrm{d}_q u + \int_0^1 u(1-u)^s\left|D_h\phi(\mu)\right|\mathrm{d}_h u\right] \\
&\leqslant \frac{(\theta-\nu)^2}{\mu-\nu}\left[\int_0^1 u_h^{(s+1)}\left|D_h\phi(\theta)\right|\mathrm{d}_h u \int_0^1 u(1-u)^s\left|D_h\phi(\nu)\right|\mathrm{d}_h u\right] + \frac{(\mu-\theta)^2}{\mu-\nu}\left[\int_0^1 u_h^{(s+1)}\left|D_h\phi(\theta)\right|\mathrm{d}_h u + \int_0^1 u(1-u)^s\left|D_h\phi(\mu)\right|\mathrm{d}_h u\right] \\
&= \frac{M(\theta-\nu)^2}{\mu-\nu}\left[\int_0^1 u_h^{(s+1)}\mathrm{d}_h u + \int_0^1 u(1-u)^s \mathrm{d}_h u\right] + \frac{(\mu-\theta)^2}{\mu-\nu}\left[\int_0^1 u_h^{(s+1)}\mathrm{d}_h u + \int_0^1 u(1-u)^s \mathrm{d}_h u\right] \\
&= M\frac{(\theta-\nu)^2+(\mu-\theta)^2}{\mu-\nu}\left[\int_0^1 u_h^{(s+1)}\mathrm{d}_h u + \int_0^1 u(1-u)^s \mathrm{d}_h u\right] \\
&\int_0^1 u_h^{(s+1)}\mathrm{d}_h u = \frac{1}{s+2} \\
&\int_0^1 u(1-u)^s \mathrm{d}_h u = \left|u.\frac{-(1-u+h)^{s+1}}{s+1}\right|_0^1 + \frac{1}{s+1}\int_0^1 (1-(u+h)+h)^{s+1}\mathrm{d}_h u \\
&= -\frac{h^{s+1}}{s+1} + \frac{1}{s+1}\int_0^1 (1-u)^{s+1}\mathrm{d}_h u \\
&= -\frac{h^{s+1}}{s+1} + \frac{1}{s+1}\left|-\frac{1}{s+2}(1+h-u)^{s+1}\right|_0^1 \\
&= -\frac{h^{s+1}}{s+1} + \frac{1}{s+1}\left(-\frac{h^{s+2}}{s+2} + \frac{(1+h)^{s+2}}{s+2}\right) \\
&\leq M\frac{(\theta-\nu)^2+(\mu-\theta)^2}{\mu-\nu}\left[-\frac{h^{s+1}}{s+1}+\frac{1}{s+1}\left(\frac{-h^{s+2}+(1+h)^{s+2}}{s+2}\right)+\frac{1}{s+2}\right].
\end{aligned} \tag{20}$$

□

Theorem 6. *Suppose ϕ: $J \subset \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a h-differentiable mapping on J° in such a way that $D_h\phi \in L[\nu, \mu]$, in which $\nu, \mu \in J$ for $\nu < \mu$. If $|D_h\phi|^m$ is s-convex $[\nu, \mu]$ for some static $s \in (0, 1]$, $m > 1$, $n = m/m - 1$, and $|D_h\phi(\theta)| \leq M$, $\theta \in [\nu, \mu]$, then we have the h-integral inequality in discrete calculus:*

$$\left|\phi(\theta) + h\frac{(\theta-\nu)\phi(\nu) + (\mu-\theta)\phi(\mu)}{\mu-\nu} - \frac{1}{\mu-\nu}\int_\nu^\mu \phi(u)_\nu \mathrm{d}_h u\right| \leqslant M\left[\frac{(\theta-\nu)^2 + (\mu-\theta)^2}{\mu-\nu}\right]\left(\frac{1}{n+1}\right)^{1/n}\left(\frac{1-h^{s+1}+(1+h)^{s+1}}{s+1}\right)^{1/m}, \tag{21}$$

for each $\theta \in [\nu, \mu]$.

Proof. From Lemma 2 and keeping the familiar Hölder inequality, we have

$$\begin{aligned}
&\left|\phi(\theta) + h\frac{(\theta-\nu)\phi(\nu) + (\mu-\theta)\phi(\mu)}{\mu-\nu} - \frac{1}{\mu-\nu}\int_\nu^\mu \phi(u)_\nu \mathrm{d}_h u\right| \\
&\leqslant \frac{(\theta-\nu)^2}{\mu-\nu}\int_0^1 u\left|D_h\phi(u\theta+(1-u)\nu)\right|\mathrm{d}_h u + \frac{(\mu-\theta)^2}{\mu-\nu}\int_0^1 u\left|D_h\phi(u\theta+(1-u)\mu)\right|\mathrm{d}_h u \\
&\leqslant \frac{(\theta-\nu)^2}{\mu-\nu}\left(\int_0^1 u_h^{(n)}\mathrm{d}_h u\right)^{1/n}\left(\int_0^1 \left|D_h\phi(u\theta+(1-u)\nu)\right|^m \mathrm{d}_h u\right)^{1/m} + \frac{(\mu-\theta)^2}{\mu-\nu}\int_0^1\left(\int_0^1 u_h^{(n)}\mathrm{d}_h u\right)^{\frac{1}{n}}\left(\int_0^1 \left|D_h\phi(u\theta+(1-u)\mu)\right|^m \mathrm{d}_h u\right)^{1/m} \\
&\cdot \int_0^1 \left|D_h\phi(u\theta+(1-u)\nu)\right|^m \mathrm{d}_h u \leqslant \int_0^1 u_h^{(s)}\left|D_h\phi(\theta)\right|^m \mathrm{d}_h u + \int_0^1 (1-u)^s\left|D_h\phi(\nu)\right|^m \mathrm{d}_h u \\
&\leqslant M^m\left(\int_0^1 u_h^{(s)}\mathrm{d}_h u + \int_0^1 (1-u)^s \mathrm{d}_h u\right) \leqslant M^m\left(\left|\frac{u_h^{(s+1)}}{s+1}\right|_0^1 + \left|-\frac{(1+h-u)^{s+1}}{s+1}\right|_0^1\right) = M^m\left(\frac{1}{s+1} - \frac{h^{s+1}}{s+1} + \frac{(1+h)^{s+1}}{s+1}\right) \\
&= M^m\left(\frac{1-h^{s+1}+(1+h)^{s+1}}{s+1}\right),
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\int_0^1 \left|D_h\phi(u\theta+(1-u)\mu)\right|^m \mathrm{d}_h u \leqslant \int_0^1 u_h^{(s)}\left|D_h\phi(\theta)\right|^m \mathrm{d}_h u + \int_0^1 (1-u)^s\left|D_h\phi(\mu)\right|^m \mathrm{d}_h u \\
&\leqslant M^m\left(\left|\frac{u_h^{(s+1)}}{s+1}\right|_0^1 + \left|-\frac{(1+h-u)^{s+1}}{s+1}\right|_0^1\right) = M^m\left(\frac{1}{s+1} - \frac{h^{s+1}}{s+1} + \frac{(1+h)^{s+1}}{s+1}\right) \\
&= M^m\left(\frac{1-h^{s+1}+(1+h)^{s+1}}{s+1}\right) \\
&\leqslant \left[\frac{(\theta-\nu)^2 + (\mu-\theta)^2}{\mu-\nu}\right]\left(\int_0^1 u_h^{(n)}\mathrm{d}_h u\right)^{1/n} M^m\left(\int_0^1 u_h^{(s)}\mathrm{d}_q u + \int_0^1 (1-u)^s \mathrm{d}_h u\right)^{1/m} \\
&\leqslant \left[\frac{(\theta-\nu)^2}{\mu-\nu} + \frac{(\mu-\theta)^2}{\mu-\nu}\right]\left(\frac{1}{n+1}\right)^{1/n}\left(M^m\left(\frac{1-h^{s+1}+(1+h)^{s+1}}{s+1}\right)\right)^{1/m} \\
&\leqslant M\left[\frac{(\theta-\nu)^2}{\mu-\nu} + \frac{(\mu-\theta)^2}{\mu-\nu}\right]\left(\frac{1}{n+1}\right)^{1/n}\left(\frac{1-h^{s+1}+(1+h)^{s+1}}{s+1}\right)^{1/m},
\end{aligned} \tag{23}$$

which completes the proof. □

Theorem 7. *Suppose $\phi: J \longrightarrow \mathbb{R}$ be a h-differentiable mapping on interior of a positive interval J in such a way that $D_h\phi \in L[\nu,\mu]$, for $\nu,\mu \in J$. If $|D_h\phi|^m$ is a s-convex in second sense on $[\nu,\mu]$ for some static $s \in (0,1]$, $m \geq 1$ and $|D_h\phi(\theta)| \leq M$, $\theta \in [\nu,\mu]$, then we have the following h-integral inequality:*

$$\begin{aligned}&\left|\phi(\theta) + h\frac{(\theta-\nu)\phi(\nu) + (\mu-\theta)\phi(\mu)}{\mu-\nu} - \frac{1}{\mu-\nu}\int_\nu^\mu \phi(u)_\nu \mathrm{d}_h u\right| \\ &\leqslant M\left[\frac{(\theta-\nu)^2 + (\mu-\theta)^2}{\mu-\nu}\right]\left[\frac{1}{2}\right]^{1-1/m}\left[-\frac{h^{s+1}}{s+1} + \frac{1}{s+1}\left(\frac{-h^{s+2} + (1+h)^{s+2}}{s+2}\right) + \frac{1}{s+2}\right]^{1/m},\end{aligned} \tag{24}$$

for each $\theta \in [\nu,\mu]$.

Proof. From Lemma 2 and keeping in view the familiar power mean inequality, we get

$$\begin{aligned}&\left|\phi(\theta) + \frac{h(\theta-\nu)\phi(\nu) + (\mu-\theta)\phi(\mu)}{\mu-\nu} - \frac{1}{\mu-\nu}\int_\nu^\mu \phi(u)_\nu \mathrm{d}_h u\right| \\
&\leqslant \frac{(\theta-\nu)^2}{\mu-\nu}\int_0^1 u\left|D_h\phi(u\theta + (1-u)\nu)\right|\mathrm{d}_h u + \frac{(\mu-\theta)^2}{\mu-\nu}\int_0^1 u\left|D_h\phi(u\theta + (1-u)\mu)\right|\mathrm{d}_h u \\
&\leqslant \frac{(\theta-\nu)^2}{\mu-\nu}\left(\int_0^1 u \mathrm{d}_h u\right)^{1-1/m}\left(\int_0^1 \left|D_h\phi(u\theta + (1-u)\nu)\right|^m \mathrm{d}_h u\right)^{1/m} \\
&\quad + \frac{(\mu-\theta)^2}{\mu-\nu}\left(\int_0^1 u \mathrm{d}_h u\right)^{1-1/m}\left(\int_0^1 \left|D_h\phi(u\theta + (1-u)\mu)\right|^m \mathrm{d}_h u\right)^{1/m} \\
&\leqslant \frac{M(\theta-\nu)^2}{\mu-\nu}\left(\int_0^1 u \mathrm{d}_h u\right)^{1-1/m}\left[\int_0^1 u_h^{(s+1)} \mathrm{d}_h u + \int_0^1 u(1-u)^s \mathrm{d}_h u\right]^{1/m} \\
&\quad + \frac{(\mu-\theta)^2}{\mu-\nu}\left(\int_0^1 u \mathrm{d}_h u\right)^{1-1/m}\left[\int_0^1 u_h^{(s+1)} \mathrm{d}_h u + \int_0^1 u(1-u)^s \mathrm{d}_h u\right]^{1/m} \\
&\cdot \int_0^1 \left|D_h\phi(u\theta + (1-u)\nu)\right|^m \mathrm{d}_h u \leqslant \int_0^1 u_h^{(s+1)}\left|D_h\phi(\theta)\right|^m \mathrm{d}_h u + \int_0^1 u(1-u)^s\left|D_h\phi(\nu)\right|^m \mathrm{d}_h u \\
&\leqslant M^m\left(\int_0^1 u_h^{(s+1)} \mathrm{d}_h u + \int_0^1 u(1-u)^s \mathrm{d}_h u\right) \\
&\cdot \int_0^1 u(1-u)^s \mathrm{d}_h u = \left|u \cdot \frac{-(1-u+h)^{s+1}}{s+1}\right|_0^1 + \frac{1}{s+1}\int_0^1 (1-(u+h)+h)^{s+1} \mathrm{d}_h u \\
&= -\frac{h^{s+1}}{s+1} + \frac{1}{s+1}\int_0^1 (1-u)^{s+1} \mathrm{d}_h u \\
&= -\frac{h^{s+1}}{s+1} + \frac{1}{s+1}\left|-\frac{1}{s+2}(1+h-u)^{s+2}\right|_0^1 \\
&= -\frac{h^{s+1}}{s+1} + \frac{1}{s+1}\left(-\frac{h^{s+2}}{s+2} + \frac{(1+h)^{s+2}}{s+2}\right), \\
&\leqslant M^m\left(-\frac{h^{s+1}}{s+1} + \frac{1}{s+1}\left(-\frac{h^{s+2}}{s+2} + \frac{(1+h)^{s+2}}{s+2}\right) + \frac{1}{s+2}\right),\end{aligned} \tag{25}$$

$$\begin{aligned}&\int_0^1 \left|D_h\phi(u\theta+(1-u)\mu)\right|^m \mathrm{d}_h u \leqslant \int_0^1 u_h^{(s+1)}\left|D_h\phi(\theta)\right|^m \mathrm{d}_h u + \int_0^1 u(1-u)^s\left|D_h\phi(\mu)\right|^m \mathrm{d}_h u\\&\leqslant M^m\left(\int_0^1 u_h^{(s+1)} d_h u + \int_0^1 u(1-u)^s \mathrm{d}_h u\right)\\&\leqslant M^m\left(-\frac{h^{s+1}}{s+1}+\frac{1}{s+1}\left(-\frac{h^{s+2}}{s+2}+\frac{(1+h)^{s+2}}{s+2}\right)+\frac{1}{s+2}\right)\\&\leqslant M\left[\frac{(\theta-\nu)^2+(\mu-\theta)^2}{\mu-\nu}\right]\left[\frac{1}{2}\right]^{1-1/m}\left[-\frac{h^{s+1}}{s+1}+\frac{1}{s+1}\left(\frac{-h^{s+2}+(1+h)^{s+2}}{s+2}\right)+\frac{1}{s+2}\right]^{1/m}.\end{aligned} \tag{26}$$

□

Remark 1.

(a) In Theorem 5, if we take $h=0$, then (19) diminishes the inequality (12) of Theorem 2

(b) In Theorem 6, if we take $h=0$, then (21) diminishes the inequality (13) of Theorem 3

(c) In Theorem 7, if we take $h=0$, then (24) diminishes the inequality (14) of Theorem 4

In [27], if $\top=\mathbb{Z}$,

$$\int_\nu^\mu \varphi(u)\Delta u = \begin{cases}\sum_{u=\nu}^{\mu-1}\varphi(u), & \text{if } \nu<\mu,\\ 0, & \text{if } \nu=\mu,\\ -\sum_{u=\mu}^{\nu-1}\varphi(u), & \text{if } \nu>\mu.\end{cases} \tag{27}$$

Example 1. In case of time scale $\top=\mathbb{Z}$ in Lemma 2, we have

$$\begin{aligned}&\varphi(\theta)+\frac{(\theta-\nu)\varphi(\nu)+(\mu-\theta)\varphi(\mu)}{\mu-\nu}-\frac{1}{\mu-\nu}\int_\nu^\mu \varphi(u)_\nu \mathrm{d}_1 u\\&=\frac{(\theta-\nu)^2}{\mu-\nu}\int_0^1 uD_1\varphi(u\theta+(1-u)\nu)_0\mathrm{d}_1 u-\frac{(\mu-\theta)^2}{\mu-\nu}\int_0^1 uD_1\varphi(u\theta+(1-u)\mu)_0\mathrm{d}_1 u.\end{aligned} \tag{28}$$

By taking $h=1$, in Theorem 5, we have

$$\begin{aligned}&\left|\varphi(\theta)+\frac{(\theta-\nu)\varphi(\nu)+(\mu-\theta)\varphi(\mu)}{\mu-\nu}-\frac{1}{\mu-\nu}\int_\nu^\mu \varphi(u)_\nu \mathrm{d}_1 u\right|\\&\leq M\frac{(\theta-\nu)^2+(\mu-\theta)^2}{\mu-\nu}\left[-\frac{1}{s+1}+\left(\frac{-1+(2)^{s+2}}{(s+1)(s+2)}\right)+\frac{1}{s+2}\right].\end{aligned} \tag{29}$$

By taking $h=1$, in Theorem 6, we have

$$\begin{aligned}&\left|\varphi(\theta)+\frac{(\theta-\nu)\varphi(\nu)+(\mu-\theta)\varphi(\mu)}{\mu-\nu}-\frac{1}{\mu-\nu}\int_\nu^\mu \varphi(u)_\nu \mathrm{d}_1 u\right|\\&\leq M\left[\frac{(\theta-\nu)^2+(\mu-\theta)^2}{\mu-\nu}\right]\left(\frac{1}{n+1}\right)^{1/n}\left(\frac{(2)^{s+1}}{s+1}\right)^{1/m}.\end{aligned} \tag{30}$$

By taking $h=1$, in Theorem 7, we have

$$\begin{aligned}&\left|\varphi(\theta)+\frac{(\theta-\nu)\varphi(\nu)+(\mu-\theta)\varphi(\mu)}{\mu-\nu}-\frac{1}{\mu-\nu}\int_\nu^\mu \varphi(u)_\nu \mathrm{d}_1 u\right|\\&\leq M\left[\frac{(\theta-\nu)^2+(\mu-\theta)^2}{\mu-\nu}\right]\left[\frac{1}{2}\right]^{1-1/m}\left[-\frac{1}{s+1}+\left(\frac{-1+(2)^{s+2}}{(s+1)(s+2)}\right)+\frac{1}{s+2}\right]^{1/m}.\end{aligned} \tag{31}$$

By taking $h = 1$ and $s = 1$ in Theorem 5, we have

$$\left|\varphi(\theta) + \frac{(\theta-\nu)\varphi(\nu) + (\mu-\theta)\varphi(\mu)}{\mu-\nu} - \frac{1}{\mu-\nu}\int_{\nu}^{\mu}\varphi(u)_{\nu}\mathrm{d}_1 u\right| \leq M\left[\frac{(\theta-\nu)^2 + (\mu-\theta)^2}{\mu-\nu}\right]. \quad (32)$$

By taking $h = 1$ and $s = 1$ in Theorem 6, we have

$$\left|\varphi(\theta) + \frac{(\theta-\nu)\varphi(\nu) + (\mu-\theta)\varphi(\mu)}{\mu-\nu} - \frac{1}{\mu-\nu}\int_{\nu}^{\mu}\varphi(u)_{\nu}\mathrm{d}_1 u\right| \leq M(2)^{1/m}\left[\frac{(\theta-\nu)^2 + (\mu-\theta)^2}{\mu-\nu}\right]\left(\frac{1}{n+1}\right)^{1/n}. \quad (33)$$

By taking $h = 1$ and $s = 1$ in Theorem 7, we have

$$\left|\varphi(\theta) + \frac{(\theta-\nu)\varphi(\nu) + (\mu-\theta)\varphi(\mu)}{\mu-\nu} - \frac{1}{\mu-\nu}\int_{\nu}^{\mu}\varphi(u)_{\nu}\mathrm{d}_1 u\right| \leq M\left[\frac{(\theta-\nu)^2 + (\mu-\theta)^2}{\mu-\nu}\right]\left[\frac{1}{2}\right]^{1-1/m}. \quad (34)$$

3. Conclusion

Our results extend and generalize the results of Alomari et al. In this work, some important Ostrowski type inequalities are established in the context of h-calculus. The derived results constitute contributions to the theory of h-integral and can be specialized to yield numerous interesting integral inequalities including some known results. An interesting feature of our results is that they provide new estimates and best approximation on Ostrowski type of inequalities for h-integral. If we take limit $h \longrightarrow 0$, Ostrowski type of h-integral inequalities reduces to simple inequalities present in [26]. The presented results motivate scientists to stimulate more work in such directions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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