

Research Article

Existence and Uniqueness Results for Two-Term Nonlinear Fractional Differential Equations via a Fixed Point Technique

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The work addressed in this paper is to ensure the existence and uniqueness of positive solutions for initial value problems for nonlinear fractional differential equations with two terms of fractional orders. By virtue of recent fixed point theorems on mixed monotone operators, we get some new straightforward results with a wide range of applications.

1. Introduction

In recent years, we see a great interest to consider fractional differential equations in various kinds of studies. Namely, these equations have striking applications in diverse science. Moreover, during the past decades, new areas for applications of fractional differential and integral equations with initial boundary conditions have been appeared. In the modeling of significant numbers of phenomena occurring in engineering and scientific subjects, we encountered a problem of such kinds. In multiple cases, modeling by fractional operators can also be done better than modeling by ordinary derivative operators. One can see fractional operators in applications of rheology, control, viscoelasticity, porous structures, chemical physics, electrochemistry, and significant number of other branches of science (see [1–3] for more details). For these reasons, there exist many articles about the existence and uniqueness of solutions of problems containing operators of fractional orders

and using fixed point techniques [4–27]. But, the number of research studies on the study of existence and uniqueness of solutions for FDEs with two or more terms of fractional operators is limited. Fujita [28] studied the following specific Cauchy problem:

$$\frac{\partial^{\alpha_1}}{\partial t} w(t, x) = \frac{\partial^{\alpha_2}}{\partial t} w(t, x), \quad 1 \leq \alpha_1, \alpha_2 \leq 2, \quad (1)$$

and established some existence and uniqueness results. In [29], the author considered an initial value problem of the following form:

$$D^\alpha y(t) = g(t, D^\beta y(t)), \quad 0 < t \leq 1, \quad (2)$$

$$y^{(k)}(0) = \mu_k, \quad 0 \leq k \leq n-1, k \in \mathbb{Z}, \quad (3)$$

where $n-1 < \beta < \alpha < n$, and studied existence and uniqueness of related solutions. Based on the results of the above paper, the following exacting conditions should be satisfied on g for the existence of solutions:

- (C1) g is a continuous and differentiable function on $[0, 1] \times \mathbf{R} \rightarrow \mathbf{R}$.
- (C2) For $t \in I, g(t, 0) \neq 0$ and $g(0, 0) = 0$, where I is any compact subinterval of $[0, 1]$.
- (C3) There exists a continuous function q which satisfies

$$C \sup_{t \in [0,1]} \int_0^t (t-r)^{\alpha-\beta-1} q(r) dr < 1, \quad (n-1 < \beta < \alpha < n), \tag{4}$$

where $C = 1/\Gamma(\alpha - \beta)$.

To detect a unique solution other than (C1)–(C3), the following restriction should hold.

- (C4) Let w_1 and w_2 be any real numbers and $q \in C([0, 1], [0, \infty))$ be the continuous function appeared in (4), then

$$|g(t, w_1) - g(t, w_2)| \leq q(t)|w_1 - w_2|, \quad 0 \leq t \leq 1. \tag{5}$$

The authors in [16, 30] have studied the same problem. In [16], we observe that the necessary conditions to exist at least one solution for (2) and (3) are (C1), (C2), and the following condition.

- (C5) For a continuous function $y \in C[0, \infty)$, there exist positive numbers c and d such that

$$|g(t, y(t))| \leq c|y(t)| + d, \quad 0 \leq t \leq 1. \tag{6}$$

Also, the additional condition that $g(t, x(t))$ satisfies a Lipschitz condition is needed to prove the uniqueness of the solution:

$$|g(t, z_1(t)) - g(t, z_2(t))| \leq M|z_1(t) - z_2(t)|, \quad 0 \leq t \leq 1, \tag{7}$$

where M is the Lipschitz constant. In [31], the following general form is considered:

$$D^\alpha y(t) = g(t, y(t), D^\beta y(t)), \quad 0 < t \leq 1, \tag{8}$$

$$y^{(k)}(0) = \mu_k, \quad 0 \leq k \leq n-1, k \in \mathbf{Z}, \tag{9}$$

where $n-1 < \beta < \alpha < n, (n \in \mathbf{N})$ and D^α and D^β are the fractional derivative orders in the Caputo definition and $g \in C([0, 1] \times \mathbf{R})$. We have used an index fixed point theorem and obtained some new results about existence and also multiplicity of solutions. In this paper, we consider the same general form (8) and (9) and try to use new fixed point theorems proved for operators having mixed monotone property to establish new results with a wide range of applications for existence and uniqueness.

2. Preliminaries

For convenience of readers, we provide some useful definitions and previous results which we use throughout the paper.

Definition 1 (see [1, 3]). Let $\alpha > 0$. The operator defined by

$$I^\alpha y(t) = \int_0^t \frac{1}{(t-r)^{\alpha-1} \Gamma(\alpha)} y(r) dr, \tag{10}$$

where $y \in C[0, \infty) \cap L^1_{loc}[0, \infty), \alpha \in (m-1, m)$ and $m \in \mathbf{N}$, is the fractional integral of y of order α in Riemann–Liouville definition.

After introducing the fractional integral operator, it is time to remind the fractional differentiation definitions. There are several ways to define fractional derivative. Here, we focus on the Caputo definition.

Definition 2 (see [1, 3]). Let $n = [\alpha] + 1$ where $\alpha \geq 0$. If $y(t) \in AC^n[0, 1]$, then the fractional derivative of y in Caputo sense exists on the interval $[0, 1]$, almost everywhere, and is obtained by

$${}^c D^\alpha y(t) = \int_0^t \frac{y^{(n)}(r)}{\Gamma(n-\alpha)(t-r)^{\alpha-n+1}} dr. \tag{11}$$

For the sake of convenience, here, we mention some properties of Caputo fractional operator. For the power function $f(t) = t^q, q \geq 0$, taking the Caputo derivative yields

$${}^c D^\alpha t^q = \begin{cases} \frac{\Gamma(q+1)}{\Gamma(q-\alpha+1)} t^{q-\alpha}, & (q \geq n-1), \\ 0, & (q \leq n-1). \end{cases} \tag{12}$$

By virtue of this formula, one can deduce ${}^c D^\alpha k = k {}^c D^\alpha t^0 = 0$. Therefore, unlike Riemann–Liouville derivative, the Caputo derivative of a constant function is zero. In the following equality:

$${}^c D^\alpha (a_0 t^r + a_1 t^{r-1} + \dots + a_n) = 0, \quad \alpha \in (m-1, m), \tag{13}$$

the polynomial's degree is less than or equal to $m-1$, i. e. $r \leq m-1$. From the following relation, it is clear that similar to the case of integer order, the α th fractional integral of the Caputo fractional derivative of order α requires to know the values of the function and its integer order derivatives:

$$I^\alpha {}^c D^\alpha g(t) = g(t) - \sum_{l=0}^{m-1} g^{(l)}(t_0^+) \frac{t^l}{l!}, \quad m-1 < \alpha \leq m. \tag{14}$$

The results of the paper about the existence and uniqueness are expressed by the famous beta function given as follows.

Definition 3. The following integral defines the beta function:

$$B(x, y) = \int_0^1 s^{x-1} (1-s)^{y-1} ds, \quad x, y > 0. \quad (15)$$

It is known that

$$B(r, s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}, \quad (16)$$

where Γ denotes the Gamma function.

The following lemma transforms problems (8) and (9) to an integral equation [29, 30], at which the nonlinear term g needs to satisfy the conditions (C1) and (C2).

Lemma 1 (see [29]). *Let $n \in \mathbf{N}, \beta < \alpha$ and $\alpha, \beta \in (n-1, n)$. If (C1) and (C2) hold, then $y \in C^n[0, 1]$ is a solution of the following problem:*

$$D^\alpha y(x) = g(x, y(x), D^\beta y(x)), \quad 0 < x \leq 1, \quad (17)$$

$$y^{(k)}(0) = \mu_k, \quad 0 \leq k \leq n-1, k \in \mathbf{Z}, \quad (18)$$

if and only if

$$y(x) = \sum_{k=0}^{n-1} \frac{x^k}{k!} \eta_k + \frac{1}{\Gamma(\beta)} \int_0^x (x-r)^{\beta-1} w(r) dr, \quad 0 \leq x \leq 1, \quad (19)$$

where w is a continuous function and satisfies the following integral equation:

$$w(x) = \int_0^1 K(x, r) g(r, w(r)) dr, \quad 0 \leq x \leq 1, \quad (20)$$

where

$$K(x, r) = \frac{1}{\Gamma(\alpha-\beta)} (x-r)_+^{\alpha-\beta-1}, \quad (21)$$

with

$$(x-r)_+^{\alpha-\beta-1} = \begin{cases} (x-r)^{\alpha-\beta-1}, & r \leq x, r, x \in [0, 1] \\ 0, & r \geq x. \end{cases} \quad (22)$$

Here and anywhere below, the function $K(x, r)$ is the Green function.

Lemma 2. *Let $n, m \in \mathbf{N}$ and $n-1 < \beta \leq m-1 < \alpha < m$. If (C1) and (C2) hold, then $y \in C^m[0, 1]$ is a solution of (17) and (18) if and only if*

$$y(x) = \sum_{l=0}^{n-1} \frac{x^l}{l!} \eta_l + \int_0^x \frac{(x-r)^{n-1} w(r) dr}{(n-1)!}, \quad 0 \leq x \leq 1, \quad (23)$$

where $w \in C[0, 1]$ satisfies the following integral equation:

$$w(x) = \sum_{l=0}^{m-n-1} \frac{x^l}{l!} \eta_{m+l} + \frac{1}{\Gamma(\alpha-n)} \int_0^x (x-r)^{\alpha-n-1} \times f(r, w(r), \frac{1}{\Gamma(n-\beta)} \int_0^r (r-h)^{n-\beta-1} w(h) dh) dr, \quad 0 \leq x \leq 1. \quad (24)$$

Now, we present the following lemma which will be useful in proving an existence and uniqueness theorem of the solution of (8) and (9).

Lemma 3. *For the Green function $G(t, s)$, the following relations hold:*

$$\int_0^1 K(x, r) dr \leq B(\alpha-\beta, \alpha-\beta), \quad 0 \leq x \leq 1, \quad (25)$$

$$\int_0^1 K(x, r) dr \geq \frac{\Gamma(\alpha-\beta)}{\Gamma 2(\alpha-\beta)}, \quad 0 \leq x \leq 1.$$

Proof. By making use of (14), it is easy and straightforward to prove the properties for $K(x, r)$.

Now, we provide some definitions and a fixed point theorem involving mixed monotone operators. We suggest references [19, 20] for more details. \square

Definition 4. Let $(E, \|\cdot\|)$ be a Banach space and $P \subseteq E$ be a nonempty closed convex subset. P is a cone in E if

- (I) For any $x \in P$ and $\lambda \geq 0$, then $\lambda x \in P$
- (II) For any $x \in P$, if $-x \in P$ then $x = \theta$

where θ denotes the zero element of the Banach space E .

If there exists a constant $C > 0$ such that, for any $x, y \in P$ with $\theta \leq x \leq y$, we have $\|x\| \leq C\|y\|$, then P is called a normal cone. Then, we give the definition of a partially ordered Banach space. We say that E is partially ordered by P if $x \leq y$ iff $y-x \in P$, for any $x, y \in P$. For arbitrary $x_1, x_2 \in E$, the ordered interval is defined by $[x_1, x_2] = \{x \in E \mid x_1 \leq x \leq x_2\}$. Let $T: E \rightarrow E$ be an operator. If $u \leq v$ implies $Tu \leq Tv$, then T is called increasing.

Let us briefly recall that the operator $T: P \times P \rightarrow P$ is increasing in its first variable and decreasing in its second variable, if from $x_i, y_i \in P$ ($i = 1, 2$), $x_1 \leq x_2, y_1 \geq y_2$, one can deduce $T(x_1, y_1) \leq T(x_2, y_2)$. If $x \geq \theta$ implies that $T(x) \geq \theta$, then the operator T is called positive.

In the following definition, P is a cone on the Banach space $(E, \|\cdot\|)$.

Definition 5 (see [19, 20]). Suppose $T: P \times P \rightarrow P$ is increasing in its first variable and decreasing in its second variable, then T is called a mixed monotone operator.

If for an element $x \in P, T(x, x) = x$, then x is called a fixed point of T .

Theorem 1 (see [4]). *Suppose $T: P \times P \rightarrow P$ is a mixed monotone operator and let the following conditions hold:*

- (i) If $c \in (0, 1), x, y \in P$, then there exists $\alpha(c, x, y) \in (1, \infty)$ such that

$$T(cx, y) \leq c^{\alpha(c, x, y)} T(x, y). \quad (26)$$

- (ii) There exist two elements $u_0, v_0 \in P, r \in (0, 1)$ such that

$$\begin{aligned} u_0 &\leq rv_0, \\ T(u_0, v_0) &\geq u_0, \\ T(v_0, u_0) &\leq v_0. \end{aligned} \tag{27}$$

Then, T has a unique fixed point u^* in $[u_0, rv_0]$ such that $T(u^*, u^*) = u^*$. Moreover, one can construct successively the following iterates:

$$\begin{aligned} u_n &= T(u_{n-1}, v_{n-1}), \\ v_n &= T(v_{n-1}, u_{n-1}), \quad n = 1, 2, \dots, \end{aligned} \tag{28}$$

starting from $x_0, y_0 \in [u_0, rv_0]$ to get the fixed point of T . Also, $\|u_n - u^*\| \rightarrow 0, \|v_n - u^*\| \rightarrow 0$ as $n \rightarrow \infty$.

3. Main Results

In this section, we prove new existence and uniqueness results for nonlinear fractional differential equations with two terms of fractional derivatives. Compared with the results of the previous studies, our results are simple and straightforward. They are expressed with the values of beta and gamma functions. For these reasons, these results can be applied for a wide range of problems. To make effective our obtained result, some examples provided at the end of section cannot be considered by techniques of [16, 30]. In order to apply the fixed point results concerning the mixed monotone operators for the study of (8) and (9), we first consider $E = C([0, 1])$ as a suitable Banach space, and let the norm on E be defined by

$$\|y\| = \max\{|y(t)|, \quad 0 \leq t \leq 1\}. \tag{29}$$

Let us consider

$$P = \{y \in E | \min y(t) \geq 0, \quad 0 \leq t \leq 1, {}^C D^\alpha \text{ is positive}\}. \tag{30}$$

It is obvious that P is a normal cone in E .

In the following theorem, g is the function appeared in the right side of equation (8).

Theorem 2. Assume that

(C6) $g(t, u, v)$ is a continuous function which is increasing in its first variable u and decreasing in its second variable v .

(C7) For any $\eta \in (0, 1)$, let u and v be two arbitrary elements of P so that there exists $\lambda(\eta, u, v) \in (1, \infty)$ such that

$$g(t, \eta u, v) \leq \eta^{\lambda(\eta, u, v)} g(t, u, v). \tag{31}$$

(C8) Let u_0 and v_0 be two elements of P and $\mu \in (0, 1)$ such that

$$\begin{aligned} u_0(t) &\leq \mu v_0(t), \\ g(t, u_0, D^\beta v_0) &\geq \frac{u_0 \Gamma[2(\alpha - \beta)]}{\Gamma(\alpha - \beta)}, \end{aligned} \tag{32}$$

$$g(t, u_0, D^\beta v_0) \leq \frac{v_0}{B(\alpha - \beta, \alpha - \beta)}.$$

Then, for equation (8) with initial conditions (9), we have a unique solution $u^* \in [u_0, \mu v_0]$. Furthermore, by computing the iterates,

$$\begin{aligned} u_{n+1}(t) &= \int_0^t K(t, r) g\left(r, u_n(r), \frac{\partial}{\partial r} v_n(r)\right) dr, \\ v_{n+1}(t) &= \int_0^t K(t, r) g\left(r, v_n(r), \frac{\partial}{\partial r} u_n(r)\right) dr. \end{aligned} \tag{33}$$

For $n = 0, 1, \dots$, one can find the unique solution. In other words, $\|u_n - u^*\| \rightarrow 0, \|v_n - u^*\| \rightarrow 0$ as $n \rightarrow \infty$.

Proof. First, we define the operator T as follows:

$$T(u, v)(t) = \int_0^t K(t, r) g(r, u(r), D^\beta v(r)) dr, \tag{34}$$

where $K(t, r)$ is the Green function (21). Taking the definition of the operator T into account and Lemma 3, it is clear that a certain function $u(t)$ is a solution of (2) and (3) iff $u(t)$ is a fixed point of T , that is, $T(u, u) = u$. Hence, it will be sufficient to show that T satisfies the conditions of Theorem 1. By virtue of (C6), it immediately follows that T is increasing in its first variable u and decreasing in its second variable v . On the other hand, under the hypothesis of (C7), if $\eta \in (0, 1)$ and u, v are two elements of P , then there exists $\lambda(\eta, u, v) \in (1, \infty)$ such that

$$\begin{aligned} T(\eta u(t), v(t)) &= \int_0^t K(t, r) g\left(r, \eta u(r), \frac{\partial}{\partial r} v(r)\right) dr \\ &\leq \eta^\alpha \int_0^t K(t, r) g\left(r, u(r), \frac{\partial}{\partial r} v(r)\right) dr \\ &= \eta^\alpha T(u(t), v(t)). \end{aligned} \tag{35}$$

Also, with the help of (C8), and using Lemma 3, we can choose u_0 and v_0 such that

$$\begin{aligned} T(u_0(t), v_0(t)) &= \int_0^t K(t, r) g\left(r, u_0(r), \frac{\partial}{\partial r} v_0(r)\right) dr \leq u_0, \\ T(v_0(t), u_0(t)) &= \int_0^t K(t, r) g\left(r, v_0(r), \frac{\partial}{\partial r} u_0(r)\right) dr \geq v_0. \end{aligned} \tag{36}$$

Consequently, it is obvious that all essentials of Theorem 1 are provided by (C6)–(C8), and therefore, there is a unique

positive solution (u^*, u^*) so that $T(u^*, u^*) = u^*$. It is the unique solution of (8) and (9). The proof is complete. \square

Example 1. Let us consider the following set of nonlinear differential equations of fractional order

$${}^c D^{m+(3/4)} y(t) = \psi(t) - \left[{}^c D^{m+(1/4)} y(t) \right]^2, \quad 0 < t \leq 1, \quad (37)$$

subject to

$$y^l(0) = 0, \quad l = 0, 1, 2, \dots, m, \quad (38)$$

where m is a positive integer, $\psi(t)$ is a continuous function such that (HTML translation failed), and $\eta_1 = \sqrt{\pi} \leq \psi(t) \leq m\pi = \eta_2$. In connection with Theorem 2, set $\alpha = m + (3/4)$ and $\beta = m + (1/4)$. By virtue of this theorem and Lemma 3, the existence of a unique solution for problems (37) and (38) is equivalent to a unique solution for the following integral equation:

$$y(t) = \int_0^t K(t, r) \left(\psi(r) - \left[{}^c D^{m+(1/4)} y(r) \right]^2 \right) dr, \quad (39)$$

where with the help of Lemma 3, one can deduce $L_1 = 1/\sqrt{\pi} \leq \int_0^1 K(t, r) dr \leq 2/\pi = L_2$. We note that if $u, v \in P$, defined by (30), then $g(t, u, v) = \psi(t) - [{}^c D^{m+(1/4)} v(t)]^2$ is increasing in u and decreasing in v . Now, let us define $A(u, v) = \int_0^t K(t, r) (\psi(r) - [{}^c D^{m+(1/4)} v(r)]^2) dr$ and choose $u_0 = 1, v_0 = 2m$. Then, considering that u_0, v_0 are constant, we can show that

$$u_0 \leq \frac{1}{2} v_0, \quad (40)$$

$$A(u_0, v_0) \geq L_1 \eta_1 \geq 1 = u_0,$$

$$A(v_0, u_0) \leq \frac{2}{\pi} \eta_2 \leq 2m = v_0.$$

Therefore, using Theorem 2, we conclude that problems (37) and (38) possess a unique positive solution such that $1 \leq y(t) \leq m$.

Example 2. In this example, we find a unique positive solution for the nonlinear fractional initial value problems of the following form:

$${}^c D^{n+(2/3)} y(t) = \chi(t) - \frac{1}{y(t)} - \sqrt{{}^c D^{n+(2/3)} y(t)}, \quad 0 < t \leq 1, \quad (41)$$

$$y^l(0) = 0, \quad l = 0, 1, 2, \dots, n, \quad (42)$$

where $n \in \mathbb{Z}^+$, $\chi(t)$ is a continuous function such that $\chi(0) \neq 0$, and $\tau_1 = 315/100 \leq \chi(t) \leq 945/100 = \tau_2$. In connection with Theorem 2, set $\alpha = n + (2/3)$ and $\beta = n + (1/3)$. By virtue of this theorem and Lemma 3, the existence of a unique solution for problems (41) and (42) is equivalent to a unique solution for the following integral equation:

$$y(t) = \int_0^t K(t, r) \left[\chi(r) - \frac{1}{y(r)} - \sqrt{{}^c D^{n+(1/3)} y(r)} \right] dr, \quad (43)$$

where with the help of Lemma 3, one deduces $2 \leq \int_0^1 K(t, r) dr \leq 53/10$. We note that if $u, v \in P$ are defined by (30), then $g(t, u, v) = \chi(s) - 1/u(t) - \sqrt{{}^c D^{n+(1/3)} v(t)}$ is increasing in u and decreasing in v .

For $c \in (0, 1)$, consider $1 < \alpha(c, u, v) < 2$; then, we have

$$g(t, cu(t), v(t)) = \chi(t) - \frac{1}{cu(t)} - \sqrt{{}^c D^{n+(1/3)} v(t)} \leq c^\alpha \left[\chi(t) - \frac{1}{u(t)} - \sqrt{{}^c D^{n+(1/3)} v(t)} \right] = c^\alpha g(t, u(t), v(t)). \quad (44)$$

Now, let us define $A(u, v) = \int_0^t K(t, r) [\chi(r) - 1/u(r) - \sqrt{{}^c D^{n+(1/3)} v(r)}] dr$ and choose $u_0 = 53/10, v_0 = 51$. Then, regarding that u_0, v_0 are constant functions, we get

$$A(u_0, v_0) = \int_0^1 K(t, r) \left[\chi(r) - \frac{1}{u_0} - \sqrt{{}^c D^{n+(1/3)} v_0} \right] dr \geq 2\gamma_1 - 1 \geq \frac{53}{10} = u_0, \quad (45)$$

$$A(v_0, u_0) = \int_0^1 K(t, r) \left[\chi(r) - \frac{1}{v_0} - \sqrt{{}^c D^{n+(1/3)} u_0} \right] dr \leq \frac{53}{10} \gamma - \frac{2}{v_0} \leq 51 = v_0.$$

Therefore, using Theorem 2, we conclude that for problems (38)–(41), there is a unique positive solution such that $y(t) \in [53/10, t51]$ for each m .

4. Conclusion

In this research, we considered nonlinear fractional differential equations of arbitrary order with two terms of fractional orders. By virtue of recent fixed point theorems on normal cones for mixed monotone operators, we obtained some new results for existence and uniqueness of solutions of these equations involving initial conditions, which are more applicable compared with previous existing results. The previous results can be proved based on very strict conditions on the function g . While, in our paper, the function $g(t, u, D^\alpha u) \equiv g(t, u, v)$ needs to be increasing in its first variable u and decreasing in its second variable v . Also, establishing the conditions (C7) and (C8) is not difficult. On the other hand, our results provided a constructive approach, based on an iterative relation, to find the solution. Indeed, the interval $(0, 1]$ can be generalized to $(0, T]$. The presented examples, at the end of the paper, cannot be considered by the previous works.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article.

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