Hindawi
Journal of Mathematics
Volume 2021, Article ID 7519643, 10 pages
https://doi.org/10.1155/2021/7519643

Research Article

Graph of Fuzzy Topographic Topological Mapping in relation to *k*-Fibonacci Sequence

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Received 18 July 2021; Accepted 9 August 2021; Published 24 August 2021

Academic Editor: Jia-Bao Liu

A generated n-sequence of fuzzy topographic topological mapping, $FTTM_n$, is a combination of n number of FTTM's graphs. An assembly graph is a graph whereby its vertices have valency of one or four. A Hamiltonian path is a path that visits every vertex of the graph exactly once. In this paper, we prove that assembly graphs exist in $FTTM_n$ and establish their relations to the Hamiltonian polygonal paths. Finally, the relation between the Hamiltonian polygonal paths induced from $FTTM_n$ to the k-Fibonacci sequence is established and their upper and lower bounds' number of paths is determined.

1. Introduction

The fuzzy topographic topological mapping (FTTM) model is built to solve the neuromagnetic inverse problem proposed in 1999 [1]. It consists of four topological spaces, namely, magnetic contour plane (MC), base magnetic plane (BM), fuzzy magnetic field (FM), and topographic magnetic field (TM). The FTTM is developed to determine the location of a simulated neuromagnetic current source [2] as shown in Figure 1.

Later, Ahmad et al. [3] proved that the components of FTTM, namely, MC, BM, FM, and TM, were homeomorphic. The FTTM's structures and proofs of their homeomorphisms were outlined in [4].

Furthermore, FTTM can also be viewed as a sequence. The idea is possible when FTTM version 2 was successfully constructed by Rahman et al. [5] as shown in Figure 2. It was specially designed to solve the multiple current sources.

In 2006, Yun and Ahmad [4] noticed that if there are two elements of FTTM (see Figure 2), they will generate

$$\left[\binom{2}{1}\times\binom{2}{1}\times\binom{2}{1}\times\binom{2}{1}\right]-2=14 \text{ new elements of FTTM.} \quad (1)$$

These 14 elements of FTTM are (MC, BM, FM, TMI), (MC, BM, FMI, TM), (MC, BMI, FM, TM), (MI, BM, FM, TM), (MC, BM, FMI, TMI), (MC, BMI, FMI, TM), (MI, BMI, FM, TM), (MI, BM, FM, TMI), (MI, BM, FMI, TM), (MC, BMI, FM, TMI), (MC, BMI, FMI, TMI), (MI, BM, FMI, TMI), (MI, BMI, FM, TMI), and (MI, BMI, FMI, TM).

Further, Yun [6] conjectured the following.

Conjecture 1. *If there exist n elements of FTTM that are homeomorphic to each other componentwise, the number of new elements of FTTM that can be generated from these n elements is*

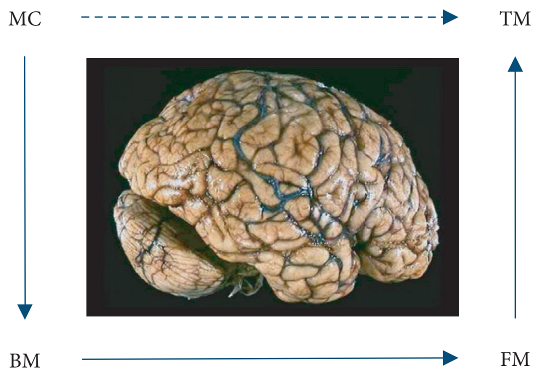


Figure 1: FTTM model.

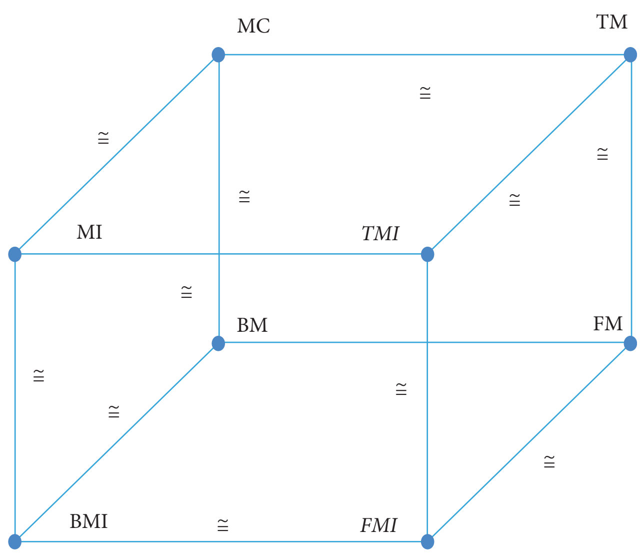


Figure 2: FTTM1 and FTTM2.

$$\begin{aligned}
&\left[\binom{n}{1}\times\binom{n}{1}\times\binom{n}{1}\times\binom{n}{1}\right]-n \\
&=\left[\left(\frac{n!}{1!\,(n-1)!}\right)\times\left(\frac{n!}{1!\,(n-1)!}\right)\times\left(\frac{n!}{1!\,(n-1)!}\right)\times\left(\frac{n!}{1!\,(n-1)!}\right)\right]-n \\
&=[n\times n\times n\times n]-n \\
&=n^4-n.
\end{aligned} \tag{2}$$

In order to prove the conjecture, Jamaian et al. [7] introduced the concept of sequence of FTTM as stated below.

Definition 1 (see [7]). Let $\text{FTTM}_i = (\text{MC}_i, \text{BM}_i, \text{FM}_i, \text{TM}_i)$ such that $\text{MC}_i, \text{BM}_i, \text{FM}_i, \text{TM}_i$ are topological spaces with $\text{MC}_i \cong \text{BM}_i \cong \text{FM}_i \cong \text{TM}_i$. Set of FTTM_i is denoted by $\text{FTTM} = \{\text{FTTM}_i\colon i = 1, 2, 3, \ldots, n\}$. Sequence of $n\text{FTTM}_i$ of FTTM is $\text{FTTM}_1, \text{FTTM}_2, \text{FTTM}_3, \text{FTTM}_4, \ldots, \text{FTTM}_n$ such that $\text{MC}_i \cong \text{MC}_{i+1}, \text{BM}_i \cong \text{BM}_{i+1}, \text{FM}_i \cong \text{FM}_{i+1}$ and $\text{TM}_i \cong \text{TM}_{i+1}$.

A sequence of $n\,\text{FTTM}_i$, without loss of generality, abbreviated as FTTM_n, is illustrated in Figure 3.

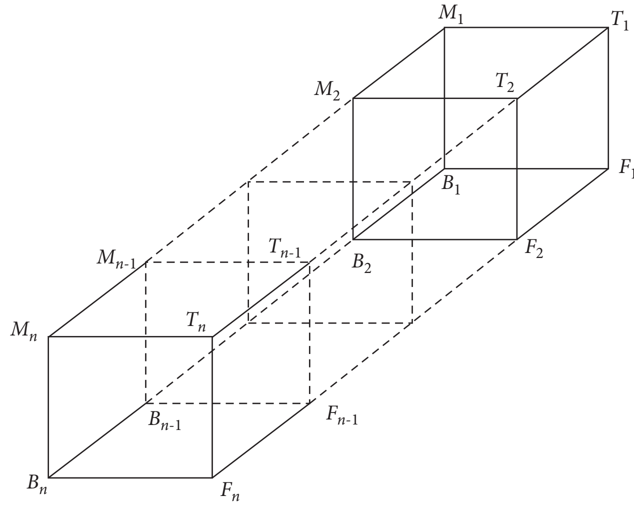


Figure 3: The sequence of FTTM_n.

Finally, the conjecture was proven [7], and surprisingly the FTTM_n is related to the Pascal triangle.

Elsafi [8] then brought the concept of sequence of FTTM to another level. The researcher viewed and furnished sequence of FTTM as a graph. The details of the concept are presented in the following section.

2. Graph of FTTM_n

Sayed and Ahmad introduced for the first time the representation of FTTM as a graph in [5]. Further, they defined the notion of order with respect to sequence of FTTM as follows.

Definition 2 (see [5]). Let $\text{FTTM}^n = \{\text{FTTM}_1, \text{FTTM}_2, \text{FTTM}_3, \ldots, \text{FTTM}_n\}$ be a sequence of n-FTTM (see Figure 3); then,

(1) $C_{i,j}\text{FTTM}_n$ are cubes of order two that can be produced from the combination of FTTM_i and FTTM_j in FTTM_n for $1 \le i, j \le n$:

$$\begin{aligned} i &= \{1, 2, 3, \ldots, n-1\}, \\ j &= \{2, 3, \ldots, n\}. \end{aligned} \tag{3}$$

(2) $|C_{i,j}\,\text{FTTM}_n|_{1\le i<j\le n}$ represent the number of cubes of order two that can be produced from the combination of FTTM_i and FTTM_j in FTTM_n, such that $i \in I, j \in J, \forall i < j \le n$.

Figure 4 shows the sequence of three terms of FTTM_3 such that $\text{FTTM}_3 = \{(M_1, B_1, F_1, T_1), (M_2, B_2, F_2, T_2), (M_3, B_3, F_3, T_3)\}$ and

$$\begin{aligned} C_{i,j}\,\text{FTTM}_3 &= \left\{C_{1,2}\,\text{FTTM}_3, C_{2,3}\,\text{FTTM}_3, C_{1,3}\,\text{FTTM}_3\right\}, \\ \left|C_{i,j}\,\text{FTTM}_3\right|_{1\le i<J\le 3} &= 3. \end{aligned} \tag{4}$$

Figure 5(a) shows that (M_1, B_1, F_3, T_3) is an element of order two since its components appear in two terms of

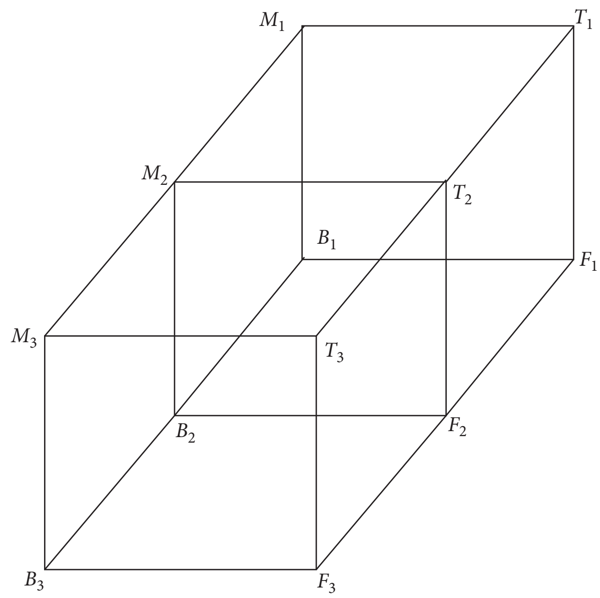


Figure 4: $FTTM_3$.

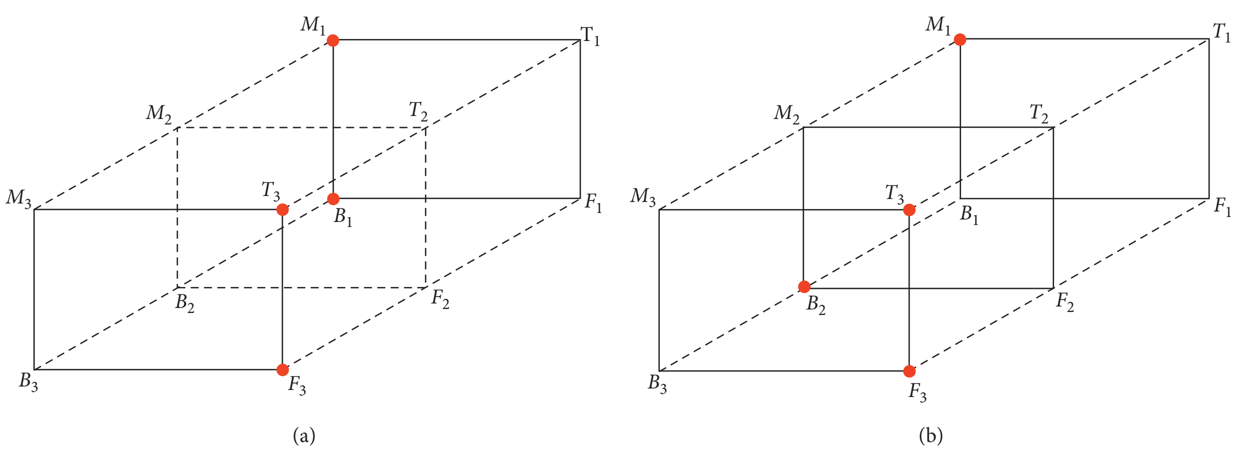


Figure 5: Example of $FTTM_3$ with elements of different orders: (a) (M_1, B_1, F_3, T_3); (b) (M_1, B_2, F_3, T_3).

FTTM, namely, in $FTTM_1$ and $FTTM_3$. By replacing B_1 with B_2, then (M_1, B_2, F_3, T_3) is an element of order three since its components appear in $FTTM_1, FTTM_2$, and $FTTM_3$ as presented in Figure 5(b).

Later, Ahmad et al. [9] established the relation of sequence of $FTTM_n$ to k-Fibonacci sequence.

Theorem 1 (see [9]). *The number of cubes produced by the combination of any three terms in $FTTM_n$; $FTTM_{3/n}$ can be presented as*

$$FTTM_{3/n} = \sum_{i=3}^{n} \left[\binom{n+3-i}{i} - \binom{n+2-i}{i+1} \right] = \frac{n(n-1)(n-2)}{3!}, \quad \text{for } n \geq 3. \tag{5}$$

For examples, $FTTM_{3/1} = 0$ for $FTTM_1$ (see Figure 1), $FTTM_{3/2} = 0$ since $FTTM_2$ is made of two terms FTTM only and $FTTM_{3/3} = 1$ (see Figure 3). The numbers of $FTTM_{3/n}$ for $n = 1, 2, 3, \ldots, 10$ are summarized in Table 1.

3. Assembly Graph and Hamiltonian Path

The concept of an assembly graph was first introduced by Angeleska et al. [10] for DNA structure through recombination process. The formal definition of an assembly graph is as follows.

Definition 3 (see [10]). An assembly graph is a finite connected graph, where all vertices are rigid vertices of valency 1 or 4. A vertex of valency 1 is called an end point. Let $\Gamma = (V, E)$ be a finite graph with a set of vertices V and a set of edges E. The number of 4-valent vertices in Γ is denoted with

Table 1: $\mathrm{FTTM}_{3/n}$ for $n = 1, 2, 3, \ldots, 10$

n	$\mathrm{FTTM}_{3/n}$
$n = 1$	0
$n = 2$	0
$n = 3$	1
$n = 4$	4
$n = 5$	10
$n = 6$	20
$n = 7$	35
$n = 8$	56
$n = 9$	84
$n = 10$	120

$|\Gamma|$. The assembly graph is called trivial if $|\Gamma| = 0$ (see Figure 6).

Angeleska et al. [10] also defined isomorphism between two assembly graphs. Basically, their isomorphism is a special case of the ordinary graph isomorphism.

Definition 4 (see [10]). Two assembly graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ are isomorphic if there is a graph isomorphism Φ that preserves the cyclic order of each rigid vertex. More specifically, for a graph isomorphism $\Phi = (\Phi_v, \Phi_e\colon \Gamma_1 \longrightarrow \Gamma_2)$ with $\Phi_v = V_1 \longrightarrow V_2$ and $\Phi_e = E_1 \longrightarrow E_2$, for every rigid vertex $(v, (e_1, e_2, e_3, e_4)^{\mathrm{cyc}})$ in Γ_1, we have

$$\begin{aligned}&(\Phi_v(v), (\Phi_v(e_1), \Phi_v(e_2), \Phi_v(e_3), \Phi_v(e_4))^{\mathrm{cyc}})\\ &= (\Phi_v(v), E^{\mathrm{cyc}}(\Phi_v(v))).\end{aligned} \quad (6)$$

Angeleska et al. [10] then defined a composition operator for two assembly graphs. In particular, the initial vertex of $\Gamma_1 \circ \Gamma_2$ is the initial vertex of Γ_1 and the terminal vertex of $\Gamma_1 \circ \Gamma_2$ is the terminal vertex of Γ_2.

Definition 5 (see [10]). A composition $\Gamma_1 \circ \Gamma_2$ of two (directed simple) assembly graphs Γ_1 and Γ_2 is the directed simple assembly graph, obtained by identifying the terminal vertex of Γ_1 with the initial vertex of Γ_2.

Furthermore, the following definitions yield some immediate properties for graph FTTM_n.

Definition 6 (see [10]). Let Γ be an assembly graph. An open path in Γ is a homeomorphic image of the open interval (0, 1) in Γ. An open path is also represented by a sequence:

$$((e_1 \backslash v_0), v_1, e_2, v_2, e_3, \ldots, v_{m-1}, e_m, v_m, (e_{m+1} \backslash v_{m+1})), \quad (7)$$

where $v_i's$ are vertices in Γ for $i \in \{1, 2, \ldots, m\}$ such that $v_i \neq v_j$ when $i \neq j$ and $e_i's$ are edges in Γ for $i \in \{1, 2, \ldots, m\}$ with endpoints v_{i-1} and v_i, respectively, such that the initial vertex of e_1 (and possibly part of e_1) and the terminal vertex of e_{m+1} (and possibly part of e_{m+1}) are not included.

An open path is a cycle if $e_1 = e_{m+1}$.

Definition 7 (see [10]). A set of pairwise disjoint open paths $\{\gamma_1, \ldots, \gamma_k\}$ in Γ is called Hamiltonian if their union contains all 4-valent vertices of Γ. An open path γ is called Hamiltonian if the set $\{\gamma\}$ is Hamiltonian.

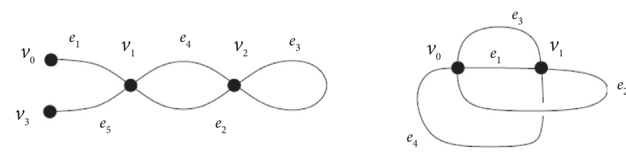


Figure 6: Examples of assembly graph [10].

Definition 8 (see [10]). Let Γ be an assembly graph. The assembly number of Γ, denoted by $\mathrm{An}(\Gamma)$, is defined by $\mathrm{An}(\Gamma) = \min\{k \mid$ there exists a Hamiltonian set of polygonal paths $\{\gamma_1, \ldots, \gamma_k\}$ in $\Gamma\}$.

Definition 9 (see [10]). For a positive integer m, we define the minimal realization number for m to be $R_{\min}(m)\min = \{|\Gamma|\colon \mathrm{An}(\Gamma) = m\}$ where $|\Gamma|$ is the number of 4-valent vertices in Γ. A graph Γ such that $R_{\min}(m) = |\Gamma|$ is called a realization of $R_{\min}(m)$.

A Hamiltonian cycle is a cycle which passes through all vertices and the path ends at the initial vertex, and a Hamiltonian path is a path that visits every vertex of the graph exactly once.

A theorem that relates the number of Hamiltonian polygonal paths in an assembly graph is as follows.

Theorem 2 (see [11]). *If Γ is a simple assembly graph with $|\Gamma| = k$ and C is the collection of all Hamiltonian polygonal paths of Γ, then*

$$|C| \leq F_{2k+1} - 1, \quad (8)$$

where F_k is the kth Fibonacci number.

4. Assembly Graph of FTTM_n

A graph of FTTM_n as described above contains many subgraphs including assembly graphs. A new concept called maximal assembly graph for assembly subgraphs of FTTM_n is introduced.

Definition 10. Let $G_1, G_2, G_3, \ldots, G_n$ be subgraphs of G (V, E) whereby each G_i is an assembly graph. A maximal assembly subgraph of G_i is defined as $|\Gamma_{G_i}| = \max\{|\Gamma_{G_1}|, |\Gamma_{G_2}|, \ldots, |\Gamma_{G_n}|\}$.

Table 2 lists all assembly subgraphs for FTTM_3.

Let Γ_1 be the assembly subgraph as in Table 2 (5) and Γ_2 be the assembly subgraph as in Table 2 (7); then $|\Gamma_1 \circ \Gamma_2| = |\Gamma_1| + |\Gamma_2| = 2 + 2 = \mathrm{An}(\Gamma_1) + \mathrm{An}(\Gamma_2)$.

An FTTM_4 produced 23 assembly subgraphs [12].

Then, consider $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ as assembly graphs as depicted in Figures 7(a) and 7(b) whereby $V_1 = \{B_1, B_2, M_2, F_2, B_3, M_3, F_3, B_5\}$, $E_1 = \{(B_1, B_2), (B_2, M_2), (B_2, F_2), (B_2, B_3), (B_3, M_3), (B_3, F_3), (B_3, B_4)\}$ and $V_2 = \{T_1, T_2, M_2, F_2, T_3, M_3, F_3, T_4\}$, $E_2 = \{(T_1, T_2), (T_2, F_2), (T_2, M_2), (T_2, T_3), (T_3, M_3), (T_3, F_3), (T_3, T_4)\}$, respectively.

Now, define $\Phi = (\Phi_V, \Phi_E\colon \Gamma_1 \longrightarrow \Gamma_2)$ such that

Table 2: Assembly subgraphs of $FTTM_3$

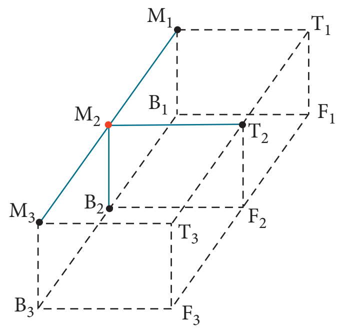
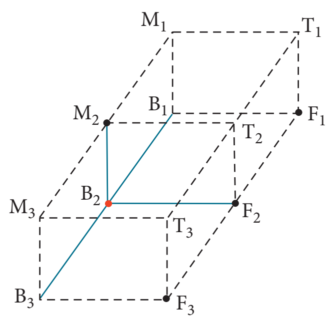
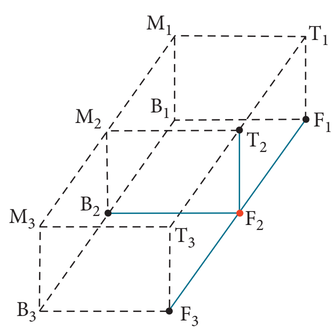
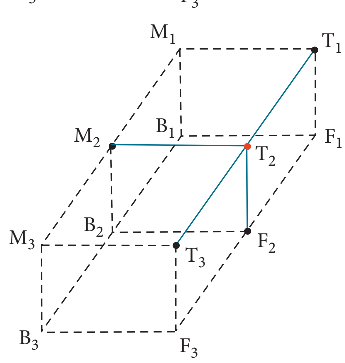
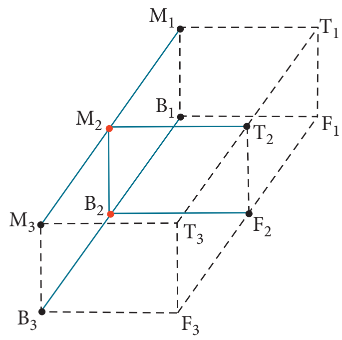
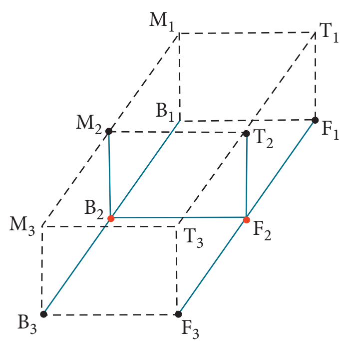
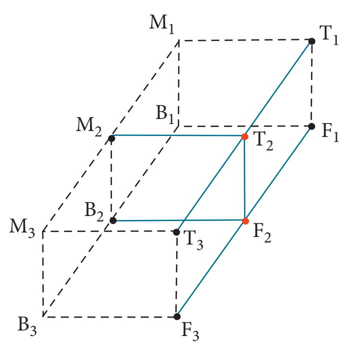
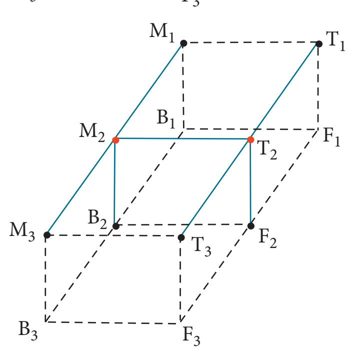
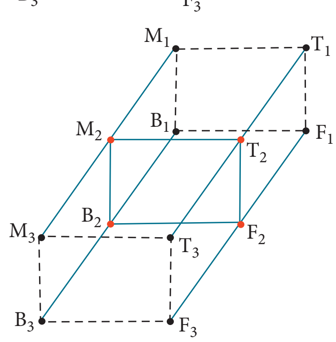
—No.	Geometrical features	The number of 4-valent vertices
1		$\lvert\Gamma_{FTTM_3}\rvert = \lvert\{M_2\}\rvert = 1$
2		$\lvert\Gamma_{FTTM_3}\rvert = \lvert\{B_2\}\rvert = 1$
3		$\lvert\Gamma_{FTTM_3}\rvert = \lvert\{F_2\}\rvert = 1$
4		$\lvert\Gamma_{FTTM_3}\rvert = \lvert\{T_2\}\rvert = 1$

Table 2: Continued.

—No.	Geometrical features	The number of 4-valent vertices
5		$\lvert\Gamma_{FTTM_3}\rvert = \lvert\{M_2, B_2\}\rvert = 2$
6		$\lvert\Gamma_{FTTM_3}\rvert = \lvert\{B_2, F_2\}\rvert = 2$
7		$\lvert\Gamma_{FTTM_3}\rvert = \lvert\{F_2, T_2\}\rvert = 2$
8		$\lvert\Gamma_{FTTM_3}\rvert = \lvert\{M_2, T_2\}\rvert = 2$
9		$\lvert\Gamma_{FTTM_3}\rvert = \lvert\{M_2, B_2, F_2, T_2\}\rvert = 4$

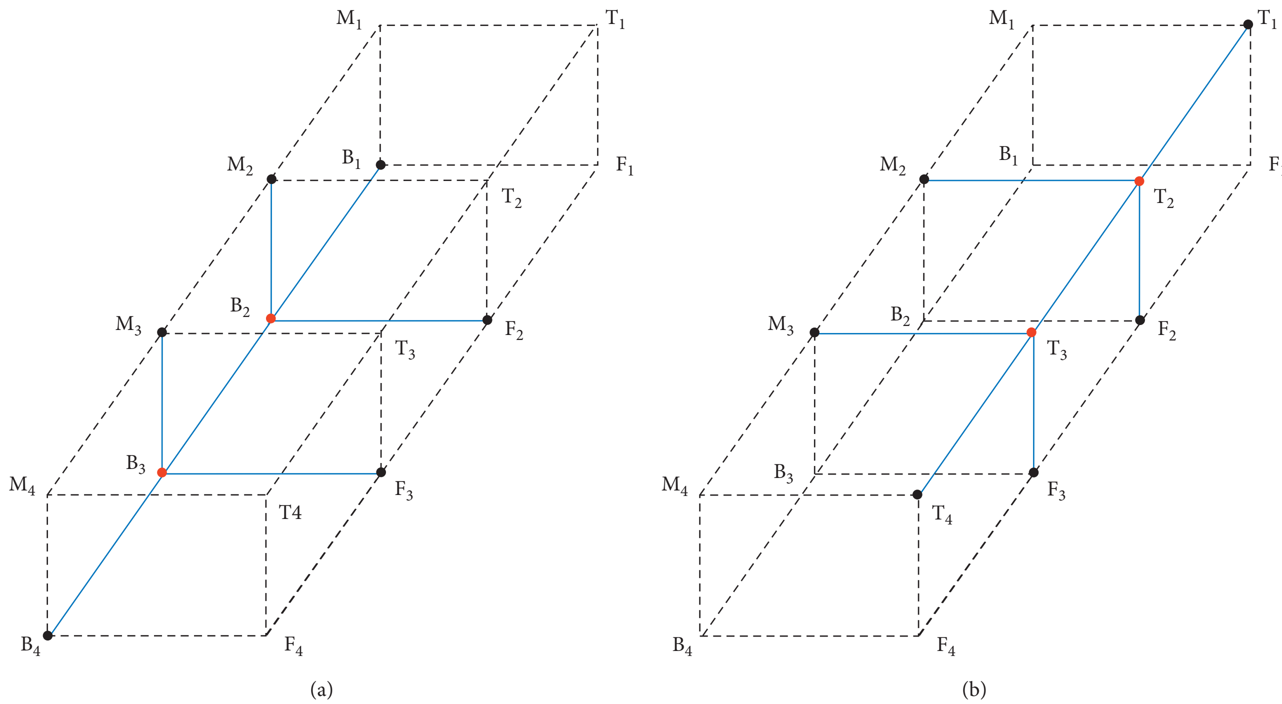


Figure 7: Example of $FTTM_4$ with different 4-valent vertices. (a) $|\Gamma_{FTTM_4}| = |\{B_2, B_3\}| = 2$. (b) $|\Gamma_{FTTM_4}| = |\{T_2, T_3\}| = 2$.

$$\begin{aligned}\Phi_v(B_1) &= T_1,\\ \Phi_E(B_1, B_2) &= (T_1, T_2),\\ \Phi_v(B_2) &= T_2,\\ \Phi_E(B_2, M_2) &= (T_2, F_2),\\ \Phi_v(M_2) &= M_2,\\ \Phi_E(B_2, F_2) &= (T_2, M_2),\\ \Phi_v(F_2) &= F_2,\\ \Phi_E(B_2, B_3) &= (T_2, T_3),\\ \Phi_v(B_3) &= T_3,\\ \Phi_E(B_3, M_3) &= (T_3, M_3),\\ \Phi_v(M_3) &= M_3,\\ \Phi_E(B_3, F_3) &= (T_3, F_3),\\ \Phi_v(F_3) &= F_3,\\ \Phi_E(B_3, B_4) &= (T_3, T_4),\\ \Phi_v(B_4) &= T_4.\end{aligned} \tag{9}$$

Hence, $\Gamma_1 \cong \Gamma_2$.

Clearly, a maximal assembly subgraph for $FTTM_n$ is the resultant graph with edges for the first and the last terms of FTTM, in particular,$FTTM_1$ and $FTTM_n$ are neglected. The formal definition of a maximal assembly graph of $FTTM_n$ is stated.

Definition 11. The maximal assembly graph of $FTTM_n$ is

$$\Gamma_{FTTM_n} = FTTM_n - [E(FTTM_1) \cup E(FTTM_n)], \quad \text{for } n \geq 3, \tag{10}$$

and $|\Gamma_{FTTM_n}|$ is the number of its 4-valent vertices.

From now on, the maximal assembly subgraph of $FTTM_n$ is referred to as an assembly graph of $FTTM_n$ until mentioned otherwise. Some properties on assembly graph of $FTTM_n$ for $n = 3$ and 4 are summarized as follows.

Theorem 3 (see [13]). *The $FTTM_3$ consists of an assembly subgraph.*

Theorem 4 (see [13]). *The $FTTM_4$ consists of an assembly subgraph.*

The previous two results can be generalized to any $FTTM_n$.

Theorem 5 (see [13]). *Every sequence of $FTTM_n$ contains an assembly subgraph for $n \geq 3$.*

Furthermore, Ahmad et al. [13] proved that the number of 4-valent vertices of the maximal assembly graph, Γ_{FTTM_n}, is as follows.

Theorem 6 (see [13]). $|\Gamma_{FTTM_{n+2}}| = 4 + (n-1)4$, *for $n \in N$.*

The following theorems are immediate.

Theorem 7. *Every sequence of $FTTM_n$ yields minimal realization, $R_{min}(m)$ number, for $n \geq 3$.*

Proof. Theorem 5 guarantees that every sequence of $FTTM_n$ contains an assembly subgraph for $n \geq 3$. An assembly graph for sequence of $FTTM_n$ is a maximal assembly graph by Definition 10 whereby $|\Gamma_{FTTM_n}|$ is the number of its 4-valent vertices. By Definition 8, every sequence of FTTM_n yields minimal realization, $R_{\min}(m)$ number, for $n \geq 3$. □

Theorem 8. *$R_{\min}(FTTM_{n+2}) = 4n$ for sequence of $FTTM_n$ and $n \in N$.*

Proof. Theorem 7 guarantees that every sequence of FTTM_n yields minimal realization, $R_{\min}(\mathrm{FTTM}_n)$ number, for $n \geq 3$. Theorem 6 states that $|\Gamma_{\mathrm{FTTM}_{n+2}}| = 4 + (n-1)4$ for $n \in N$. Hence, $R_{\min}(\mathrm{FTTM}_{n+2}) = 4 + (n-1)4 = 4 + 4n - 4 = 4n$ for sequence of FTTM_n and $n \in N$.

In fact, Γ_{FTTM_n} is a realization of $R_{\min}(\mathrm{FTTM}_n)$ since $R_{\min}(\mathrm{FTTM}_n) = |\Gamma_{\mathrm{FTTM}_n}|$ for $n \geq 3$. Consequently, the following theorem is deduced. □

Theorem 9. *$R_{\min}(FTTM_n) < R_{\min}(FTTM_{n+1})$ for sequence of $FTTM_n$ and $n \geq 3$.*

Proof.

$$\begin{aligned} R_{\min}(\mathrm{FTTM}_n) &= 4n, \text{By Theorem 8} \\ &< 4(n+1) \\ &< R_{\min}(\mathrm{FTTM}_{n+1}) \text{ for sequence of } \mathrm{FTTM}_n \text{ and } n \geq 3. \end{aligned} \tag{11}$$

□

5. Hamiltonian Paths in an Assembly Graph of FTTM_n

In previous section, we proved the existence of an assembly graph in any sequence of FTTM_n for $n \geq 3$. Hamiltonian polygonal paths exist in any assembly graph FTTM_n as well.

Theorem 10 (see [13]). *Γ_{FTTM_3} consists of a set of Hamiltonian polygonal paths.*

Theorem 11 (see [13]). *Γ_{FTTM_4} consists of a set of Hamiltonian polygonal paths.*

In fact, the existence of Hamiltonian paths in any sequence of FTTM_n for $n \geq 3$ is generalized in the following theorem.

Theorem 12 (see [13]). *Γ_{FTTM_n} consists of a set of Hamiltonian paths, for $n \geq 3$.*

A coded program in [14] is modified to calculate the number of all Hamiltonian polygonal paths in an assembly graph of FTTM_n. Table 3 summarizes the number of Hamiltonian polygonal paths in assembly graphs of FTTM_n for $n = 3, 4, 5, \ldots, 10$.

6. Graph of FTTM_n in Association to *k*-Fibonacci Sequence

The following theorem is the highlight of this paper. It links the work of Sayed and Ahmad [15] and Ahmad et al. [9], i.e., the relation of graph of FTTM and Fibonacci number (see Figure 8).

Theorem 13 (see [12]). *Let $FTTM_n$ be a sequence of n-FTTM with $|\Gamma_{FTTM_n}| = k$ and C be the collection of all sets of Hamiltonian polygonal paths of $FTTM_n$; then,*

$$|C| \leq F_{2k+1} - 1, \tag{12}$$

where F_k is the kth Fibonacci number.

Proof. Let FTTM_n be a sequence of n-FTTM. By Theorem 5, FTTM_n consists of assembly graphs, namely, Γ_{FTTM_n}. Then, Theorem 12 guarantees that Γ_{FTTM_n} consists of a set of Hamiltonian polygonal paths, say C. Using Theorem 2, for FTTM_n, $|C| \leq F_{2k+1} - 1$ as required whereby F_k is the kth Fibonacci number.

Thus, the connections illustrated in Figure 8 are completed. A refinement of Theorem 13 is given in the following corollary. □

Corollary 14 (see [12]). *Let $FTTM_n$ be a sequence of n-FTTM for $n \geq 3$ and C be the set of all Hamiltonian polygonal paths of $FTTM_n$; then,*

$$|C| \leq F_{8n+1} - 1. \tag{13}$$

Proof. Let FTTM_n be a sequence of n-FTTM for $n \geq 3$. By Theorem 5, FTTM_n consists of assembly graphs, namely, Γ_{FTTM_n}. Further, Theorem 6 reveals that $|\Gamma_{\mathrm{FTTM}_{n+2}}| = 4 + (n-1)4$, for $n \in N$. Theorem 12 guarantees that Γ_{FTTM_n} consists of a set C, that is, all its Hamiltonian polygonal paths. By replacing $k = 4 + (n-1)4$, for $n \in N$, in Theorem 13,

$$\begin{aligned} |C| &\leq F_{2k+1} - 1 \\ &= F_{2(4+(n-1)4)+1} - 1, \text{replace } k = 4 + (n-1)4 \\ &= F_{2(4+4n-4)+1} - 1 \\ &= F_{2(4n)+1} - 1 \\ &= F_{8n+1} - 1. \end{aligned} \tag{14}$$

Table 4 lists Hamiltonian polygonal paths of FTTM_n in relation to k-Fibonacci numbers for $n = 3$ to 10.

The following theorem highlights the lower and upper bounds for Hamiltonian polygonal paths of FTTM_n. □

Theorem 15. *Let $FTTM_n$ be a sequence of n-FTTM for $n \geq 3$ and C be the set of all Hamiltonian polygonal paths of $FTTM_n$; then,*

$$\frac{n(n-1)(n-2)}{3!} \leq |C| \leq F_{8n+1} - 1. \tag{15}$$

Proof. Definition 11 states that $|\Gamma_{\mathrm{FTTM}_n}|$ is the number of 4-valent vertices. These 4-valent vertices can only exist only for (at least) three terms of FTTM, i.e., $\mathrm{FTTM}_{3/n}$. But Theorem 1 guarantees that the number of cubes produced by the combination of any three terms in FTTM_n; $\mathrm{FTTM}_{3/n}$ is $(n(n-1)(n-2)/3!)$. In other words, the lower bound of Hamiltonian polygonal paths of FTTM_n, C, is obtained as $|C| \geq (n(n-1)(n-2)/3!)$. Corollary 14 states $|C| \leq F_{8n+1} - 1$ for $n \geq 3$. Hence, $(n(n-1)\ (n-2)/3!) \leq |C| \leq F_{8n+1} - 1$ as required.

TABLE 3: Hamiltonian polygonal paths in assembly graphs of $FTTM_n$ for $n = 3, 4, 5, \ldots, 10$

FTTM$_n$	No. of vertices	No. of 4-valent vertices	Hamiltonian polygonal paths
$FTTM_3$	12	4	8
$FTTM_4$	16	8	144
$FTTM_5$	20	12	1,168
$FTTM_6$	24	16	8,032
$FTTM_7$	28	20	49,312
$FTTM_8$	32	24	281,248
$FTTM_9$	36	28	1,523,920
$FTTM_{10}$	40	32	7,953,408

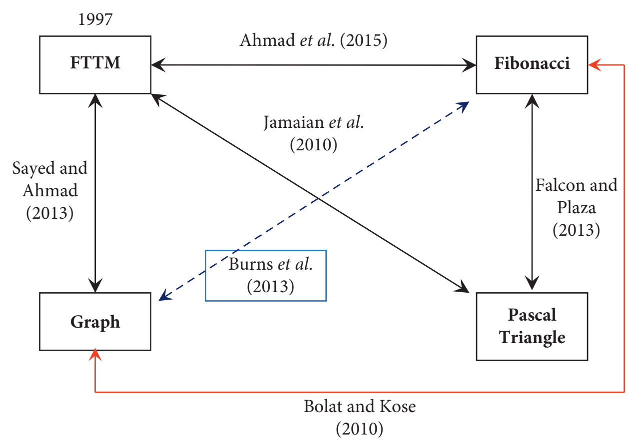


FIGURE 8: Three mathematical concepts with respect to FTTM.

TABLE 4: Hamiltonian polygonal paths and F_{2k+1} for $n = 3$ to 10.

$\|\Gamma_{FTTM_n}\| = \mathbf{k}$	$\mathbf{F_{2k+1}}$	$\|\mathbf{C}\|$	$\|\mathbf{C}\| \leq \mathbf{F_{2k+1}} - 1$
$\|\Gamma_{FTTM_3}\| = 4$	$F_{2(4)+1} = F_9 = 34$	8	$8 \leq 33$
$\|\Gamma_{FTTM_4}\| = 8$	$F_{2(8)+1} = F_{17} = 1,597$	144	$144 \leq 1,596$
$\|\Gamma_{FTTM_5}\| = 12$	$F_{2(12)+1} = F_{25} = 75,025$	1,168	$1,168 \leq 75,024$
$\|\Gamma_{FTTM_6}\| = 16$	$F_{2(16)+1} = F_{33} = 3,524,578$	8,032	$8,032 \leq 3,524,577$
$\|\Gamma_{FTTM_7}\| = 20$	$F_{2(20)+1} = F_{41} = 156,580,141$	49,312	$49,312 \leq 165,580,140$
$\|\Gamma_{FTTM_8}\| = 24$	$F_{2(24)+1} = F_{49} = 7,778,742,049$	281,248	$281,248 \leq 7,778,742,048$
$\|\Gamma_{FTTM_9}\| = 28$	$F_{2(28)+1} = F_{57} = 365,435,296,162$	1,523,920	$1,523,920 \leq 365,435,296,161$
$\|\Gamma_{FTTM_{10}}\| = 32$	$F_{2(32)+1} = F_{65} = 17,167,680,177,565$	7,953,408	$7,953,408 \leq 17,167,680,177,564$

TABLE 5: Lower and upper bounds for Hamiltonian polygonal paths of $FTTM_n$ for $n = 3$ to 10.

FTTM$_n$	**FTTM$_{3/n}$**	$\|\mathbf{C}\|$	$\mathbf{F_{8n+1}} - 1$
$FTTM_3$	1	8	33
$FTTM_4$	4	144	1,596
$FTTM_5$	10	1,168	75,024
$FTTM_6$	20	8,032	3,524,577
$FTTM_7$	35	49,312	165,580,140
$FTTM_8$	56	281,248	7,778,742,048
$FTTM_9$	84	1,523,920	365,435,296,161
$FTTM_{10}$	120	7,953,408	17,167,680,177,564

The following table (see Table 5) lists the lower and upper bounds for Hamiltonian polygonal paths in $FTTM_n$ for $n = 3$ to 10. □

7. Conclusions

The aim of this paper is to prove that there exists a relationship between $FTTM_n$ (sequence of n-FTTM) and k-Fibonacci sequence. We have established the lower and upper bounds for the established relation.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors acknowledge the support of Universiti Teknologi Malaysia (UTM) and Ministry of Higher Education Malaysia (MOHE). This research was also supported by Fundamental Research Grant Scheme (FRGS) (FRGS/1/2020/STG06/UTM/01/1), Ministry of Education, Malaysia.

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