

Research Article

On the Connected Safe Number of Some Classes of Graphs

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For a connected simple graph \mathcal{G} , a nonempty subset \mathcal{S} of $V(\mathcal{G})$ is a connected safe set if the induced subgraph $\mathcal{G}[\mathcal{S}]$ is connected and the inequality $|\mathcal{S}| \geq |\mathcal{D}|$ satisfies for each connected component \mathcal{D} of $\mathcal{G} \setminus \mathcal{S}$ whenever an edge of \mathcal{G} exists between \mathcal{S} and \mathcal{D} . A connected safe set of a connected graph \mathcal{G} with minimum cardinality is called the minimum connected safe set and that minimum cardinality is called the connected safe numbers. We study connected safe sets with minimal cardinality of the ladder, sunlet, and wheel graphs.

1. Introduction

A facility location problem (FLP) means to place and manage a certain facility in such a way as to get or achieve the maximum objective with minimizing cost. For further study of FLPs, we refer to the literature of combinatorial optimization [1]. Fujita et al. [2] studied the FLP and proposed a notion of a safe set of graphs.

We refer to [3] for terminology and notation not explained here. Throughout the paper, we will consider only simple and connected graphs. Let \mathcal{G} be a graph with $V(\mathcal{G})$ and $E(\mathcal{G})$ as its vertex and edge set, respectively. The number of vertices in a graph \mathcal{G} is the order of \mathcal{G} . For $v \in V(\mathcal{G})$, $N(v) = \{u \in V(\mathcal{G}) : u \text{ is adjacent to } v\}$ and $N[v] = N(v) \cup \{v\}$ is called open and closed neighborhood of v in \mathcal{G} , respectively. For $v \in V(\mathcal{G})$, the degree of vertex v is defined as $\deg(v) = |N(v)|$. For subset $\mathcal{S} \subset V(\mathcal{G})$, $\mathcal{G}[\mathcal{S}]$ denotes the subgraph induced by \mathcal{S} . For subset $X \subset V(\mathcal{G})$, if $\mathcal{G} \setminus X$ is disconnected, then X is known as vertex cut. The vertex connectivity denoted by $\kappa(\mathcal{G})$ is defined as $\min\{|X| : X \text{ is vertex cut}\}$. Let $C(\mathcal{G})$ denote the set of all connected components of \mathcal{G} .

Suppose A and B are disjoint subgraphs of \mathcal{G} ; then, the set of edges of $E(\mathcal{G})$ that joins some vertices of A and B is

denoted by $E(A, B)$. A nonempty subset \mathcal{S} of $V(\mathcal{G})$ is a safe set if, for each $X \in C(\mathcal{G} \setminus \mathcal{S})$ and each $Y \in C(\mathcal{G}[\mathcal{S}])$, the inequality $|Y| \geq |X|$ holds whenever $E(X, Y) \neq \emptyset$. If $\mathcal{G}[\mathcal{S}]$ is connected, then \mathcal{S} is known as a connected safe set. For any graph \mathcal{G} , $s(\mathcal{G}) = \min\{|\mathcal{S}| : \mathcal{S} \text{ is a safe set of } \mathcal{G}\}$ and $cs(\mathcal{G}) = \min\{|\mathcal{S}| : \mathcal{S} \text{ is a connected safe set of } \mathcal{G}\}$ are known as the safe number and connected safe number, respectively. It is clear from the definition that $s(P_n) = cs(P_n) = \lceil n/3 \rceil$ and $s(C_n) = cs(C_n) = \lceil n/2 \rceil$, where P_n and C_n are the path and cycle of order n , respectively. Fujita et al. [2] proved that for any graph \mathcal{G} , $s(\mathcal{G}) \leq cs(\mathcal{G}) \leq 2s(\mathcal{G}) - 1$.

In general, there is no algorithm available to compute a safe number and a connected safe number of a graph \mathcal{G} . It was shown in [2] that the computation of safe number and connected safe number is an NP-complete problem. But, on the contrary, Fujita et al. [2] show that $cs(\mathcal{G})$ can be computed in linear time in case of trees. Also, Árueda et al. [4] show that $s(\mathcal{G})$ can be computed in $O(n^5)$ time on trees. Any tree T with one vertex of degree not more than 3 holds that $s(T) = cs(T)$. Motivated by this equality, for the Cartesian product of two complete graphs, say \mathcal{G} , Kang et al. [5] proved that the safe number $s(\mathcal{G})$ and connected safe number $cs(\mathcal{G})$ are also the same.

For a vertex-weighted graph, Bapat et al. [6] presented the weighted safe set problem by considering the graph as a community network. For further study about the weighted safe set, we refer to [7–9]. Furthermore, the study on safe set and weighted safe set was conducted by several authors. The parameterized complexity of safe set problems was investigated by Belmonte et al. [10]. Macambiraa et al. [11] presented a mixed integer linear programming formulation and an algorithm for both the weighted safe set and the safe set problems. Fujita and Furuya [12] investigated the comparison between integrity and the safe number of graphs.

In this paper, we describe the connected safe set and compute the connected safe number of ladder, wheel, and sunlet graphs, respectively. The paper is concluded in Section 5.

2. Safe Set of the Ladder Graph

For convenience, we consider the ladder graph \mathcal{G} of order $n \geq 4$ with vertex set $V(\mathcal{G}) = \{v_1, v_2, \dots, v_n\}$ labeled as shown in Figure 1 [13].

Theorem 1. *Let \mathcal{G} be a ladder graph; then, the connected safe set is*

$$\mathcal{S} = \begin{cases} \{v_{\lfloor n/8 \rfloor + 2}, v_{\lfloor n/8 \rfloor + 3}, \dots, v_{\lfloor n/8 \rfloor + \lfloor n/4 \rfloor + 1}, v_{n - \lfloor n/8 \rfloor - \lfloor n/4 \rfloor}, v_{n - \lfloor n/8 \rfloor - 1}\}, & \text{if } n \equiv 0, 2, 6 \pmod{8}, \\ \{v_{\lfloor n/8 \rfloor + 2}, v_{\lfloor n/8 \rfloor + 3}, \dots, v_{\lfloor n/8 \rfloor + \lfloor n/4 \rfloor}, v_{n - \lfloor n/8 \rfloor - \lfloor n/4 \rfloor + 1}, v_{n - \lfloor n/8 \rfloor - 1}\}, & \text{otherwise.} \end{cases} \quad (1)$$

Proof 1. The proof is divided into two cases:

Case 1: assume that $n \equiv 0, 2, 6 \pmod{8}$. Let $\mathcal{S} = \{v_{\lfloor n/8 \rfloor + 2}, v_{\lfloor n/8 \rfloor + 3}, \dots, v_{\lfloor n/8 \rfloor + \lfloor n/4 \rfloor + 1}, v_{n - \lfloor n/8 \rfloor - \lfloor n/4 \rfloor}, v_{n - \lfloor n/8 \rfloor - 1}\}$ be a subset of $V(\mathcal{G})$. Clearly, $v_1 v_n, v_2 v_{n-1}, \dots, v_{n/2} v_{(n/2)-1} \in E(\mathcal{G})$. Hence, $v_{\lfloor n/8 \rfloor + 2}$ and $v_{n - \lfloor n/8 \rfloor - 1}$ as well as $v_{\lfloor n/8 \rfloor + \lfloor n/4 \rfloor + 1}$ and $v_{n - \lfloor n/8 \rfloor - \lfloor n/4 \rfloor}$ are adjacent. Therefore, $\mathcal{G}[\mathcal{S}]$ is connected. Now, $C(\mathcal{G} \setminus \mathcal{S}) = \{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\}$, where $\mathcal{D}_1 = \{v_{n - \lfloor n/8 \rfloor - \lfloor n/4 \rfloor + 1}, v_{n - \lfloor n/8 \rfloor - \lfloor n/4 \rfloor + 2}, \dots, v_{n - \lfloor n/8 \rfloor - 2}\}$, $\mathcal{D}_2 = \{v_{\lfloor n/8 \rfloor + \lfloor n/4 \rfloor + 2}, v_{\lfloor n/8 \rfloor + \lfloor n/4 \rfloor + 3}, \dots, v_{n - \lfloor n/8 \rfloor - \lfloor n/4 \rfloor - 1}\}$, and $\mathcal{D}_3 = \{v_1, v_2, \dots, v_{\lfloor n/8 \rfloor + 1}, v_{n - \lfloor n/8 \rfloor}, \dots, v_{n-1}, v_n\}$. It is easy to see that $|\mathcal{S}| = \lfloor n/4 \rfloor + 2$, $|\mathcal{D}_1| = \lfloor n/4 \rfloor - 2$, $|\mathcal{D}_2| = n - 2\lfloor n/8 \rfloor - 2\lfloor n/4 \rfloor - 2$, and $|\mathcal{D}_3| = 2\lfloor n/8 \rfloor + 2$. This implies $|\mathcal{D}_i| \leq |\mathcal{S}|$ and $E(\mathcal{D}_i, \mathcal{S}) \neq \emptyset$ for $1 \leq i \leq 3$. Hence, both the conditions of connected safe set are satisfied.

Case 2: the similar arguments might work for the other case. \square

Theorem 2. *For a ladder graph \mathcal{G} of order n , the following holds:*

$$cs(\mathcal{G}) = \begin{cases} \lfloor \frac{n}{3} \rfloor, & \text{if } 4 \leq n \leq 8, \\ \lfloor \frac{n}{4} \rfloor + 2, & \text{if } n \geq 10 \text{ and } n \equiv 0, 2, 6 \pmod{8}, \\ \lfloor \frac{n}{4} \rfloor + 1, & \text{otherwise.} \end{cases} \quad (2)$$

Proof 2. Assume that \mathcal{S} is a connected safe set of cardinality $cs(\mathcal{G})$ and $C(\mathcal{G} \setminus \mathcal{S}) = \{\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_t\}$, ordered so that $|\mathcal{D}_1| \leq |\mathcal{D}_2| \leq \dots \leq |\mathcal{D}_t|$.

If $t = 1$, then either $v_1, v_n \in \mathcal{S}$ or $v_{n/2}, v_{n/2+1} \in \mathcal{S}$. We assume that $v_1, v_n \in \mathcal{S}$. Let $u \in \mathcal{D}_1$ such that $E(\{u\}, \mathcal{S}) \neq \emptyset$. Then, by removing vertex v_1 from \mathcal{S} and adding vertex u in

\mathcal{S} , we get a connected safe set \mathcal{S}^* and for \mathcal{S}^* , $|\mathcal{D}_1| < \max\{|\mathcal{D}| \mid \mathcal{D} \in C(\mathcal{G} \setminus \mathcal{S}^*)\}$, which is a contradiction. Hence, $t \geq 2$.

Consider that $n \geq 10$ and $n \equiv 0, 2, 6 \pmod{8}$.

We want to show that $|\mathcal{S}| = \lfloor n/4 \rfloor + 2$. Suppose on the contrary $|\mathcal{S}| = \lfloor n/4 \rfloor + 1$ since $\kappa(\mathcal{G}) = 2$.

Let $\mathcal{X} = \{\{v_i, v_j\} \mid v_i, v_j \in V(\mathcal{G}) \text{ and } i + j = n + 1\}$ be the vertex cut sets, where $1 < i < n/2$ and $n/2 < j < n$ and clearly $|\mathcal{X}| = n/2 - 2$. We assume that \mathcal{S} contains X_l , where $l \geq 2$. Suppose, for contradictions $l = 1$, clearly $t = 2$. Let w be a vertex of \mathcal{D}_2 such that $E(X_1, \{w\}) = \emptyset$ and $E(\mathcal{S}, \{w\}) \neq \emptyset$. Then, there exists a vertex x of \mathcal{S} such that we have the following:

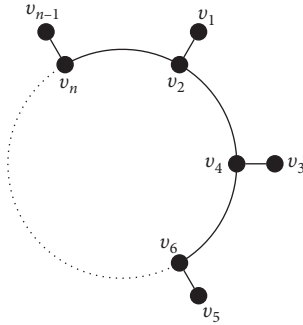
- (i) $\mathcal{S} - \{x\}$ is connected
- (ii) $X_1 \cap \{x\} = \emptyset$
- (iii) $E(\{w\}, \{x\}) = \emptyset$

Then, by removing vertex x from \mathcal{S} and adding vertex w in \mathcal{S} , we get a connected safe set \mathcal{S}^* and for \mathcal{S}^* , we have $|\mathcal{D}_1| < \max\{|\mathcal{D}| \mid \mathcal{D} \in C(\mathcal{G} \setminus \mathcal{S}^*)\}$, which is a contradiction. Hence, $l \geq 2$. It is sufficient to show for X_1 and X_2 that $|\mathcal{S}| = cs(\mathcal{G})$. Note that if $E(X_1, X_2) \neq \emptyset$, then $t = 2$. Hence, we assume that $E(X_1, X_2) = \emptyset$. Suppose that $v_{i_1}, v_{i_2} \in \mathcal{S}$ such that $E(\{v_{i_1}\}, X_1) \neq \emptyset$ and $E(\{v_{i_2}\}, X_2) \neq \emptyset$ since \mathcal{S} is connected. Therefore, half vertices between X_1 and X_2 belong to \mathcal{S} and the other half belongs to a component of $\mathcal{G} \setminus \mathcal{S}$ (say \mathcal{D}_1). This implies $|\mathcal{D}_1| \leq |\mathcal{S}| - 4 = \lfloor n/4 \rfloor - 3$. Note that $|\mathcal{D}_2| \leq |\mathcal{D}_3|$. Thus, $|\mathcal{D}_3| \geq (n - |\mathcal{S}| - |\mathcal{D}_1|)/2 = (n - 2\lfloor n/4 \rfloor + 2)/2 \geq |\mathcal{S}|$, a contradiction. Hence, $|\mathcal{S}| = \lfloor n/4 \rfloor + 2$.

The similar arguments might work for other cases. \square

3. Safe Set of the Wheel Graph

For convenience, we consider the wheel graph \mathcal{G} of order $n \geq 4$ with vertex set $V(\mathcal{G}) = \{v_1, v_2, \dots, v_n\}$ labeled as shown in Figure 2 [14].

FIGURE 3: The sunlet graph \mathcal{G} .

$$cs(\mathcal{G}) = \lfloor \frac{n}{3} \rfloor. \quad (7)$$

Proof 6. Assume that \mathcal{S} is a connected safe set of cardinality $cs(\mathcal{G})$; then, $C(\mathcal{G} \setminus \mathcal{S}) = \{\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_t\}$, ordered such that $|\mathcal{D}_1| \leq |\mathcal{D}_2| \leq \dots \leq |\mathcal{D}_t|$.

If $t = 1$, then $\mathcal{S} = X \cup Y$, where $X = \{v | v \in \mathcal{S} \text{ and } \deg(v) = 3\}$ and $Y = \{u | u \in \mathcal{S} \text{ and } \deg(v) = 1\}$. Note that $\mathcal{S} - \{v\}$ is disconnected for all $v \in X$. Let $u \in Y$ and $w \in \mathcal{D}_1$ such that $E(\{w\}, \mathcal{S}) \neq \emptyset$; then, $\mathcal{S}^* = (\mathcal{S} - \{u\}) \cup \{w\}$ is a connected safe set and for \mathcal{S}^* , $|\mathcal{D}_1| < \max\{|\mathcal{D}| | \mathcal{D} \in C(\mathcal{G} \setminus \mathcal{S}^*)\}$, a contradiction. Thus, $t \geq 2$.

Let $Z \subset V(\mathcal{G})$ such that $Z \cap \mathcal{S} = \emptyset$, $E(\mathcal{S}, Z) \neq \emptyset$, and $\mathcal{G}[Z]$ is a path. Note that $\mathcal{S}' = (\mathcal{S} - Y) \cup Z$ is a connected safe set; then, obviously $\mathcal{G}[\mathcal{S}']$ is the path, and for that choice of the connected safe set, we have $|C(\mathcal{G} \setminus \mathcal{S}')| = \text{Max}\{|C(\mathcal{G} \setminus \mathcal{S})| | \mathcal{S} \text{ is a connected safe set}\}$. As a result, $2 \leq t \leq |\mathcal{S}'| + 1$.

Suppose on the contrary that $cs(\mathcal{G}) = \lfloor n/3 \rfloor - 1$. Since $2 \leq t \leq |\mathcal{S}'| + 1$, $d = \sum_{i=1}^{k-1} |\mathcal{D}_i| \leq |\mathcal{S}'|$ and $n - 2\lfloor n/3 \rfloor + 2 \geq |V(\mathcal{G}) - \mathcal{S} - d| = |\mathcal{D}_k| > |\mathcal{S}'|$, a contradiction. Hence, $cs(\mathcal{G}) = \lfloor n/3 \rfloor$. \square

5. Conclusion

In this article, the connected safe set and connected safe number of ladder, wheel, and sunlet graphs are studied. The computation of connected safe number is an NP-complete problem and is known for few classes of graphs. A nontrivial graph of order n whose degree set consists of $n - 1$ elements is called an antiregular graph. Hence, in the future, it is interesting to study the safe set number for some other standard classes of graphs such as regular graphs, antiregular graphs, and well-known computer networks.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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