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Research Article

New Perspectives on Classical Meanness of Some Ladder Graphs

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In this study, we investigate a new kind of mean labeling of graph. The ladder graph plays an important role in the area of communication networks, coding theory, and transportation engineering. Also, we found interesting new results corresponding to classical mean labeling for some ladder-related graphs and corona of ladder graphs with suitable examples.

1. Introduction and Preliminaries

All through this paper, by a graph, we mean an undirected, simple, and finite graph. For documentations and phrasing, we follow [1–6]. For a point-by-point review on graph labeling, refer [7].

Let P_n be a path on n nodes denoted by $u_{1,\gamma}$, where $1 \leq \gamma \leq n$, and with $n-1$ lines denoted by $e_{1,\delta}$, where $1 \leq \delta \leq n-1$, where e_γ is the line joining the vertices $u_{1,\gamma}$ and $u_{1,\gamma+1}$. On each edge e_δ, erect a ladder with $n-(\gamma-1)$ steps including the edge e_γ, for $\gamma = 1, 2, 3, \ldots, n-1$. The resulting graph is called the one-sided step graph, and it is denoted by ST_n. Let P_{2n} be a path on $2n$ vertices $u_{1,\gamma}$, where $1 \leq \gamma \leq 2n$ and with $2n-1$ edges $e_1, e_2, \ldots, e_{2n-1}$, where e_γ is the line joining the vertices $u_{1,\gamma}$ and $u_{1,\gamma+1}$. On each edge e_γ, we erect a ladder with '$\gamma+1$' steps including the edge e_γ, for $\gamma = 1, 2, 3, \ldots, n$, and on each e_γ, we erect a ladder with $2n+1-\gamma$ steps including e_γ, for $\gamma = n+1, n+2, \ldots, 2n-1$. The graph thus obtained is called the double-sided step graph, and it is denoted by $2\mathrm{ST}_{2n}$. Let G_1 and G_2 be any two graphs with p_1 and p_2 vertices, respectively. Then, $G_1 \times G_2$ is the cartesian product of two graphs. A ladder graph L_n is the graph $P_2 \times P_n$. The graph $G \circ S_m$ is obtained from G by attaching m pendant vertices to each vertex of G. The triangular ladder TL_n, for $n \geq 2$, is a graph obtained from two paths by $u_1, u_2, \ldots u_n$ and $v_1, v_2, \ldots v_n$ by adding the edges $u_\gamma v_\gamma$, $1 \leq \gamma \leq n$ and $u_\gamma v_{\gamma+1}$, $1 \leq \gamma \leq n-1$. The slanting ladder SL_n is a graph obtained from two paths $u_1, u_2, \ldots u_n$ and $v_1, v_2, \ldots v_n$ by joining each v_γ, with $u_{\gamma+1}$, $1 \leq \gamma \leq n-1$. The graph D_n^* having the vertices $\{a_{\gamma,\delta}\colon 1 \leq \gamma \leq n, \delta = 1, 2, 3, 4\}$, and its edge set is $\{a_{\gamma,1}a_{\gamma+1,1}, a_{\gamma,3}a_{\gamma+1,3}\colon 1 \leq \gamma \leq n-1\} \cup \{a_{\gamma,1}a_{\gamma,2}, a_{\gamma,2}a_{\gamma,3}, a_{\gamma,3}a_{\gamma,4}, a_{\gamma,4}a_{\gamma,1}\colon 1 \leq \gamma \leq n\}$.

2. Literature Survey

The origin of graph labeling called graceful labeling was characterized by Rosa in [8] and the mean labeling of graphs was introduced by Somasundram et al. in [9]. In [10], Arockiaraj et al. presented the idea of F-root square mean labeling of the graphs and examined its meanness [11]. Durai Baskar and Arockiaraj talked about the C-geometric meanness of some ladders in [12]. Dafik et al. researched the antimagicness of the graphs including the graph D_n^* in [13]. Durai Baskar considered the logarithmic meanness in [14] and Rajesh Kannan et al. characterized idea of exponential mean graphs in [15]. In addition, more concepts of ladder graphs and related concepts have been studied in [16–24]. Recently, Muhiuddin et al. have applied various related concepts on graphs in different aspects (see, e.g., [25–31]).

3. Methodology

A labeling χ on a graph $G(V, E)$ with p vertices and q edges is called a Smarandache (m, k) mean labeling, for an integer $m \geq 1$ and $k \geq 2$, if $\chi\colon V(G) \longrightarrow \{1, 2, 3, \ldots, q+1\}$ is injective

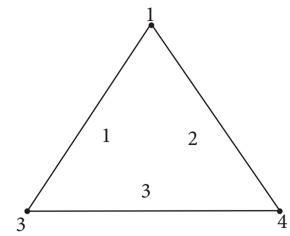

Figure 1: A classical mean labeling of C_3.

and the induced function $\chi^*: E(G) \longrightarrow \{1, 2, 3, \ldots, q\}$ defined by

$$\chi^*(uv) = \left\lfloor \frac{1}{4}\left(\sqrt[m]{\frac{\chi(u)^m + \chi(v)^m}{2}} + \sqrt[k]{\chi(u)^m \chi(v)^m} + \frac{2\chi(u)^m \chi(v)^m}{\chi(u)^m + \chi(v)^m} + \sqrt[k]{\frac{\chi(u)^k + \chi(v)^k}{2}} \right) \right\rfloor, \quad (1)$$

for all $uv \in E(G)$, is bijective.

Particularly, if $m = 1$ and $k = 2$, such a Smarandache (m, k) mean labeling is the classical mean labeling on the graph. A function χ is known as a classical mean labeling of a graph $G(V, E)$ with p nodes and q edges if $\chi: V(G) \longrightarrow \{1, 2, 3, \ldots, q + 1\}$ is injective and the incited edge assignment function $\chi^*: E(G) \longrightarrow \{1, 2, 3, \ldots, q\}$ characterized as

$$\chi^*(uv) = \left\lfloor \frac{1}{4}\left(\frac{\chi(u) + \chi(v)}{2} + \sqrt{\chi(u)\chi(v)} + \frac{2\chi(u)\chi(v)}{\chi(u) + \chi(v)} + \sqrt{\frac{\chi(u)^2 + \chi(v)^2}{2}} \right) \right\rfloor, \quad (2)$$

for all $uv \in E(G)$, is bijective. A graph that concedes a classical mean labeling is said to be classical mean graph.

A classical mean labeling of C_3 is shown in Figure 1.

Here, we found interesting new results corresponding to classical mean labeling for some ladder-related graphs and corona of ladder graphs.

4. Main Results

Theorem 1. *The one-sided step graph ST_n is a classical mean graph, for $n \geq 2$.*

Proof. Let P_n be a path on n nodes denoted by $u_{1,\gamma}$, where $1 \leq \gamma \leq n$ and with $n - 1$ lines denoted by $e_{1,\delta}$, where $1 \leq \delta \leq n - 1$, where e_γ is the line joining the vertices $u_{1,\gamma}$ and $u_{1,\gamma+1}$. On each edge e_δ, erect a ladder with $n - (\gamma - 1)$ steps including the edge e_γ, for $\gamma = 1, 2, 3, \ldots, n - 1$. The resulting graph is called the one-sided step graph, and it is denoted by ST_n. Let $u_{1,a}, u_{2,a}, u_{3,b}, u_{4,c}, \ldots, u_{n,1}, \quad u_{n,2}$, for $1 \leq a \leq n$, $1 \leq b \leq n - 1$, and $1 \leq c \leq n - 2$, be the nodes of ST_n.

Construct a mapping χ from $V(G)$ to $\{1, 2, 3, \ldots, -1 + n + n^2\}$:

$$\begin{aligned} &\chi(u_{\gamma,\delta}) = (1 - \gamma + n)^2 + \delta - 1, \quad \text{for } 2 \leq \gamma \leq n, 1 \leq \delta \leq n + 2 - \gamma, \\ &\chi(u_{1,\delta}) = n^2 + \delta - 1, \quad \text{for } 2 \leq \delta \leq n. \end{aligned} \quad (3)$$

Therefore,

$$\begin{aligned} &\chi^*(u_{\gamma,\delta} u_{\gamma,\delta+1}) = (1 - \gamma + n)^2 + \delta - 1, \quad \text{for } 2 \leq \gamma \leq n \text{ and } 1 \leq \delta \leq 1 - \gamma + n, \\ &\chi^*(u_{1,\delta} u_{1,\delta+1}) = -1 + n^2 + \delta, \quad 1 \leq \delta \leq n - 1, \\ &\chi^*(u_{\gamma,\delta} u_{\gamma+1,\delta}) = (1 - \gamma + n)(-\gamma + n) - 1 + \delta, \quad \text{for } 1 \leq \gamma \leq -1 + n \text{ and } 1 \leq \delta \leq n + 1 - \gamma. \end{aligned} \quad (4)$$

Hence, one-sided step graph ST_n is a classical mean graph, for $n \geq 2$.

A classical mean labeling of ST_7 is shown in Figure 2. □

Theorem 2. *The double-sided step graph $2ST_{2n}$ is a classical mean graph, for $n \geq 1$.*

Proof. Let P_{2n} be a path on $2n$ vertices $u_{1,\gamma}$, where $1 \leq \gamma \leq 2n$ and with $2n-1$ edges $e_1, e_2, \ldots, e_{2n-1}$, where e_γ is the line joining the vertices $u_{1,\gamma}$ and $u_{1,\gamma+1}$. On each edge e_γ, we erect a ladder with '$\gamma+1$' steps including the edge e_γ, for $\gamma = 1, 2, 3, \ldots, n$, and on each e_γ, we erect a ladder with $2n+1-\gamma$ steps including e_γ, for $\gamma = n+1, n+2, \ldots, 2n-1$. The graph thus obtained is called the double-sided step graph, and it is denoted by $2ST_{2n}$.

Let $u_{1,a}, u_{2,a}, u_{3,b}, u_{4,c}, \ldots, u_{n+1,1}, u_{n+1,2}$, for $1 \leq a \leq 2n$, $1 \leq b \leq 2n-2$, and $1 \leq c \leq 2n-4$ be the nodes of $2ST_{2n}$.

Construct a mapping χ from $V(G)$ to $\{1, 2, 3, \ldots, 3n+2n^2\}$:

$$\chi(u_{1,\delta}) = \begin{cases} 1+n+2n^2+2(-1+\delta), & 1 \leq \delta \leq n, \\ 3n+2n^2-2\delta+2n+2, & 1+n \leq \delta \leq 2n, \end{cases} \quad (5)$$

for $2 \leq \delta \leq 2+n-\gamma$ and $2 \leq \gamma \leq n$,

$$\chi(u_{\gamma,\delta}) = 2(1-\gamma+n)^2 + (n+2-\gamma) + 2(\delta-2), \quad (6)$$

for $2 \leq \gamma \leq n$ and $n+3-\gamma \leq \delta \leq 2n+3-2\gamma$,

$$\begin{aligned} \chi(u_{\gamma,\delta}) &= 2(1+n-\gamma)^2 + 3(n+1-\gamma) - 2(\gamma+\delta-n-3), \\ \chi(u_{2,1}) &= 2n^2+n-2, \\ \chi(u_{\gamma,1}) &= 2(2+n-\gamma)^2 - \gamma + n, \quad 3 \leq \gamma \leq n+1, \\ \chi(u_{\gamma,2n+4-2\gamma}) &= 2(n+2-\gamma)^2 + 1 + n - \gamma, \quad 2 \leq \gamma \leq n+1. \end{aligned} \quad (7)$$

Therefore,

$$\chi^*(u_{1,\delta}u_{1,\delta+1}) = \begin{cases} 1+n+2n^2+2(-1+\delta), & 1 \leq \delta \leq n, \\ 3n+2n^2-2\delta+2n, & n+1 \leq \delta \leq 2n-1, \end{cases} \quad (8)$$

for $2 \leq \delta \leq 2+n-\gamma$ and $2 \leq \gamma \leq n-1$,

$$\chi^*(u_{\gamma,\delta}u_{\gamma,\delta+1}) = 2(-\gamma+1+n)^2 + (-\gamma+n+2) + 2(-2+\delta), \quad (9)$$

for $-\gamma+3+n \leq \delta \leq 2+2n-2\gamma$ and $2 \leq \gamma \leq n-1$,

$$\begin{aligned} \chi^*(u_{\gamma,\delta}u_{\gamma,\delta+1}) &= 2(-\gamma+n+1)^2 + 3(-\gamma+n+1) - 2(-2+\gamma+\delta-n), \\ \chi^*(u_{\gamma,2n+3-2\gamma}u_{\gamma+1,2n+2-2\gamma}) &= 2(-\gamma+n+1)^2 + (-\gamma+n+1), \quad \text{for } 2 \leq \gamma \leq n, \\ \chi^*(u_{n,2}u_{n,3}) &= 4, \\ \chi^*(u_{n+1,1}u_{n+1,2}) &= 1, \\ \chi^*(u_{1,1}u_{2,1}) &= 2n^2+n-1, \\ \chi^*(u_{1,2n}u_{2,2n}) &= 2n^2+n, \\ \chi^*(u_{\gamma,2}u_{\gamma+1,1}) &= 2(-\gamma+n)^2 - \gamma + n, \quad 2 \leq \gamma \leq n, \\ \chi^*(u_{1,\delta}u_{2,\delta}) &= \begin{cases} 1-n+2n^2+2\delta-4, & 2 \leq \delta \leq n, \\ -2+n+2n^2-2(-1-n+\delta), & 1+n \leq \delta \leq -1+2n, \end{cases} \end{aligned} \quad (10)$$

for $3 \leq \delta \leq n+2-\gamma$ and $2 \leq \gamma \leq n-1$,

$$\chi^*(u_{\gamma,\delta}u_{\gamma+1,\delta-1}) = 2(-\gamma+1+n)^2 - (-\gamma+2+n) + 2\delta - 4, \quad (11)$$

for $3+n-\gamma \leq \delta \leq 2+2n-2\gamma$ and $2 \leq \gamma \leq n-1$,

$$\begin{aligned} \chi^*(u_{\gamma,\delta}u_{\gamma+1,\delta-1}) &= 2(-\gamma+1+n)^2 + (-\gamma+3+n) - 2\gamma - 2\delta + 2n + 2, \\ \chi^*(u_{\gamma,1}u_{\gamma,2}) &= 2(-\gamma+1+n+1)^2 + 3(-\gamma+1+n), \quad \text{for } 2 \leq \gamma \leq n, \\ \chi^*(u_{\gamma,2n+3-2\gamma}u_{\gamma,2n+4-2\gamma}) &= 2(-\gamma+1+n)^2 + 3(-\gamma+1+n) + 1, \quad \text{for } 2 \leq \gamma \leq n. \end{aligned} \quad (12)$$

Hence, the double-sided step graph $2ST_{2n}$ is a classical mean graph, for $n \geq 1$.

A classical mean labeling of $2ST_{10}$ is shown in Figure 3. □

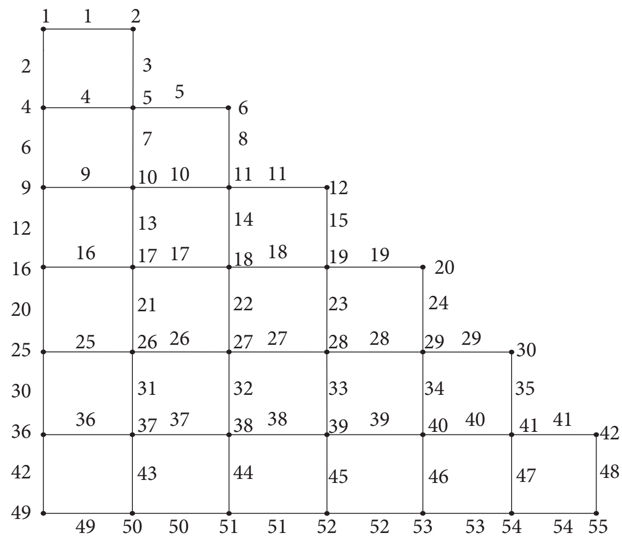

Figure 2: A classical mean labeling of ST_7.

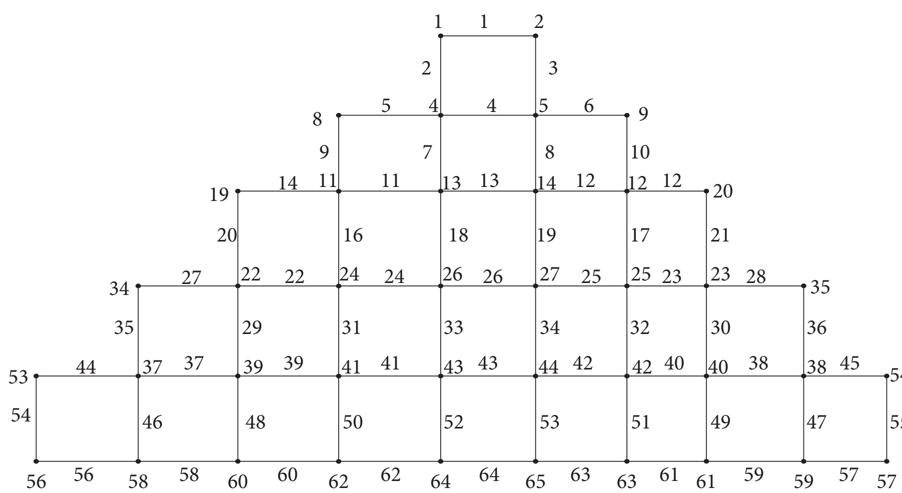

Figure 3: A classical mean labeling of $2ST_{10}$.

Theorem 3. *For $n \geq 2$ and $m \leq 4$, the graph $P_m \times P_n$ is a classical mean graph.*

Proof. Take $V(P_m \times P_n) = \{v_{\gamma\delta}\colon 1 \leq \delta \leq n, 1 \leq \gamma \leq m\}$ and $E(P_m \times P_n) = \{v_{\gamma\delta}v_{(\gamma+1)\delta}\colon\ 1 \leq \delta \leq n, 1 \leq \gamma \leq m-1\} \cup \{v_{\gamma\delta} v_{\gamma(\delta+1)}\colon 1 \leq \delta \leq n-1, 1 \leq \gamma \leq m\}$.

Case (i). $m = 2$.

Construct a mapping χ from $V(P_2 \times P_n))$ to $\{1, 2, 3, \ldots, 3n-1\}$:

$$\chi\left(v_{\gamma\delta}\right) = \gamma + 3(\delta - 1), \quad \text{for } 1 \leq \gamma \leq 2 \text{ and } 1 \leq \delta \leq n. \tag{13}$$

Therefore,

$$\chi^*\left(v_{1\delta}v_{2\delta}\right) = 3\delta - 2, \quad \text{for } 1 \leq \delta \leq n,$$
$$\chi^*\left(v_{\gamma\delta}v_{\gamma(\delta+1)}\right) = \gamma + 3\delta - 2, \quad \text{for } 1 \leq \gamma \leq 2 \text{ and } 1 \leq \delta \leq n-1. \tag{14}$$

Case (ii). $m = 3$.

Construct a mapping χ from $V(P_3 \times P_n)$ to $\{1, 2, 3, \ldots, 5n-2\}$:

$$\chi\left(v_{\gamma 1}\right) = \begin{cases} \gamma, & 1 \leq \gamma \leq 2, \\ 4, & \gamma = 3, \end{cases}$$
$$\chi\left(v_{\gamma\delta}\right) = \gamma + 5(\delta - 1), \quad \text{for } 1 \leq \gamma \leq 3 \text{ and } 2 \leq \delta \leq n. \tag{15}$$

Therefore,

$$\chi^*\left(v_{\gamma 1}v_{(\gamma+1)1}\right) = \gamma, \quad \text{for } 1 \le \gamma \le 2,$$
$$\chi^*\left(v_{\gamma 1}v_{\gamma 2}\right) = \gamma + 2, \quad \text{for } 1 \le \gamma \le 3,$$
$$\chi^*\left(v_{\gamma\delta}v_{(\gamma+1)\delta}\right) = \gamma + 5\delta - 5, \quad \text{for } 1 \le \gamma \le 2 \text{ and } 2 \le \delta \le n,$$
$$\chi^*\left(v_{\gamma\delta}v_{\gamma(\delta+1)}\right) = \gamma + 5\delta - 3, \quad \text{for } 1 \le \gamma \le 3 \text{ and } 2 \le \delta \le n-1. \tag{16}$$

Case (iii). $m = 4$.

Consider the graph $P_4 \times P_n$, for $n \ge 3$.

Construct a mapping χ from $V(P_4 \times P_n)$ to $\{1, 2, 3, \ldots, 7n-3\}$:

$$\chi\left(v_{\gamma 1}\right) = \begin{cases} \gamma, & 1 \le \gamma \le 2, \\ \gamma + 1, & 3 \le \gamma \le 4, \end{cases}$$
$$\chi\left(v_{\gamma 2}\right) = \gamma + 7, \quad \text{for } 1 \le \gamma \le 4, \tag{17}$$
$$\chi\left(v_{\gamma\delta}\right) = \gamma + 7(\delta - 1), \quad \text{for } 1 \le \gamma \le 4 \text{ and } 3 \le \delta \le n.$$

Therefore,

$$\chi^*\left(v_{1\delta}v_{1(\delta+1)}\right) = \begin{cases} 8\delta - 5, & 1 \le \delta \le 2, \\ 7\delta - 3, & 3 \le \delta \le n-1, \end{cases}$$
$$\chi^*\left(v_{\gamma\delta}v_{\gamma(\delta+1)}\right) = \gamma + 7\delta - 4, \quad \text{for } 2 \le \gamma \le 4 \text{ and } 1 \le \delta \le n-1,$$
$$\chi^*\left(v_{\gamma 1}v_{(\gamma+1)1}\right) = \begin{cases} \gamma, & 1 \le \gamma \le 2, \\ 4, & \gamma = 3, \end{cases}$$
$$\chi^*\left(v_{\gamma 2}v_{(\gamma+1)2}\right) = \gamma + 7, \quad \text{for } 1 \le \gamma \le 3,$$
$$\chi^*\left(v_{\gamma\delta}v_{(\gamma+1)\delta}\right) = \gamma + 7\delta - 7, \quad \text{for } 1 \le \gamma \le 3 \text{ and } 3 \le \delta \le n. \tag{18}$$

Hence, for $n \ge 2$ and $m \le 4$, the graph $P_m \times P_n$ is a classical mean graph.

A classical mean labeling of $P_2 \times P_4$, $P_3 \times P_8$, and $P_4 \times P_6$ are shown in Figure 4. □

Corollary 1. *Every Ladder graph $L_n = P_2 \times P_n$ is a classical mean graph.*

A classical mean labeling of $P_2 \times P_5$ is shown in Figure 5.

Theorem 4. *For $m \le 2$ and $n \ge 2$, the graph $L_n \circ S_m$ is a classical mean graph.*

Proof. Case (i). $m = 1$.

Construct a mapping χ from $V(L_n \circ S_1)$ to $\{1, 2, 3, \ldots, 5n-1\}$:

$$\chi\left(u_\gamma\right) = \begin{cases} 3, & \gamma = 1, \\ 5\gamma - 3, & 2 \le \gamma \le n, \end{cases}$$
$$\chi\left(v_\gamma\right) = \begin{cases} 4, & \gamma = 1, \\ 5\gamma - 2, & 2 \le \gamma \le n, \end{cases} \tag{19}$$
$$\chi\left(u_1^{(\gamma)}\right) = 5\gamma - 4, \quad \text{for } 1 \le \gamma \le n,$$
$$\chi\left(x_1^{(\gamma)}\right) = \begin{cases} 2, & \gamma = 1, \\ 5\gamma - 1, & 2 \le \gamma \le n. \end{cases}$$

Therefore,

$$\chi^*\left(u_\gamma u_{\gamma+1}\right) = 5\gamma - 1, \quad \text{for } 1 \le \gamma \le n-1,$$
$$\chi^*\left(v_\gamma v_{\gamma+1}\right) = 5\gamma, \quad \text{for } 1 \le \gamma \le n-1,$$
$$\chi^*\left(u_\gamma v_\gamma\right) = \begin{cases} 3, & \gamma = 1, \\ 5\gamma - 3, & 2 \le \gamma \le n, \end{cases} \tag{20}$$
$$\chi^*\left(u_\gamma w_1^{(\gamma)}\right) = 5\gamma - 4, \quad \text{for } 1 \le \gamma \le n,$$
$$\chi^*\left(v_\gamma x_1^{(\gamma)}\right) = \begin{cases} 2, & \gamma = 1, \\ 5\gamma - 2, & 2 \le \gamma \le n. \end{cases}$$

Case (ii). $m = 2$.

Construct a mapping χ from $V(L_n \circ S_2)$ to $\{1, 2, 3, \ldots, 7n-1\}$:

$$\chi\left(u_\gamma\right) = \begin{cases} 3, & \gamma = 1, \\ -2 + 7\gamma, & \gamma \text{ is even and } 2 \le \gamma \le n, \\ 7\gamma - 5, & \gamma \text{ is odd and } 2 \le \gamma \le n, \end{cases}$$
$$\chi\left(v_\gamma\right) = \begin{cases} 4, & \gamma = 1, \\ 7\gamma - 4, & \gamma \text{ is even and } 2 \le \gamma \le n, \\ 7\gamma - 1, & \gamma \text{ is odd and } 2 \le \gamma \le n, \end{cases}$$
$$\chi\left(w_1^{(\gamma)}\right) = \begin{cases} 1, & \gamma = 1, \\ -3 + 7\gamma, & \gamma \text{ is even and } 2 \le \gamma \le n, \\ -6 + 7\gamma, & \gamma \text{ is odd and } 2 \le \gamma \le n, \end{cases}$$
$$\chi\left(w_2^{(\gamma)}\right) = \begin{cases} 2, & \gamma = 1, \\ -1 + 7\gamma, & \gamma \text{ is even and } 2 \le \gamma \le n, \\ -4 + 7\gamma, & \gamma \text{ is odd and } 2 \le \gamma \le n, \end{cases} \tag{21}$$
$$\chi\left(x_1^{(\gamma)}\right) = \begin{cases} 2\gamma + 3, & 1 \le \gamma \le 2, \\ -6 + 7\gamma, & \gamma \text{ is even and } 2 \le \gamma \le n, \\ -3 + 7\gamma, & \gamma \text{ is odd and } 2 \le \gamma \le n, \end{cases}$$
$$\chi\left(x_2^{(\gamma)}\right) = \begin{cases} 8, & \gamma = 1, \\ -5 + 7\gamma, & \gamma \text{ is even and } 2 \le \gamma \le n, \\ -2 + 7\gamma & \gamma \text{ is odd and } 2 \le \gamma \le n. \end{cases}$$

Therefore,

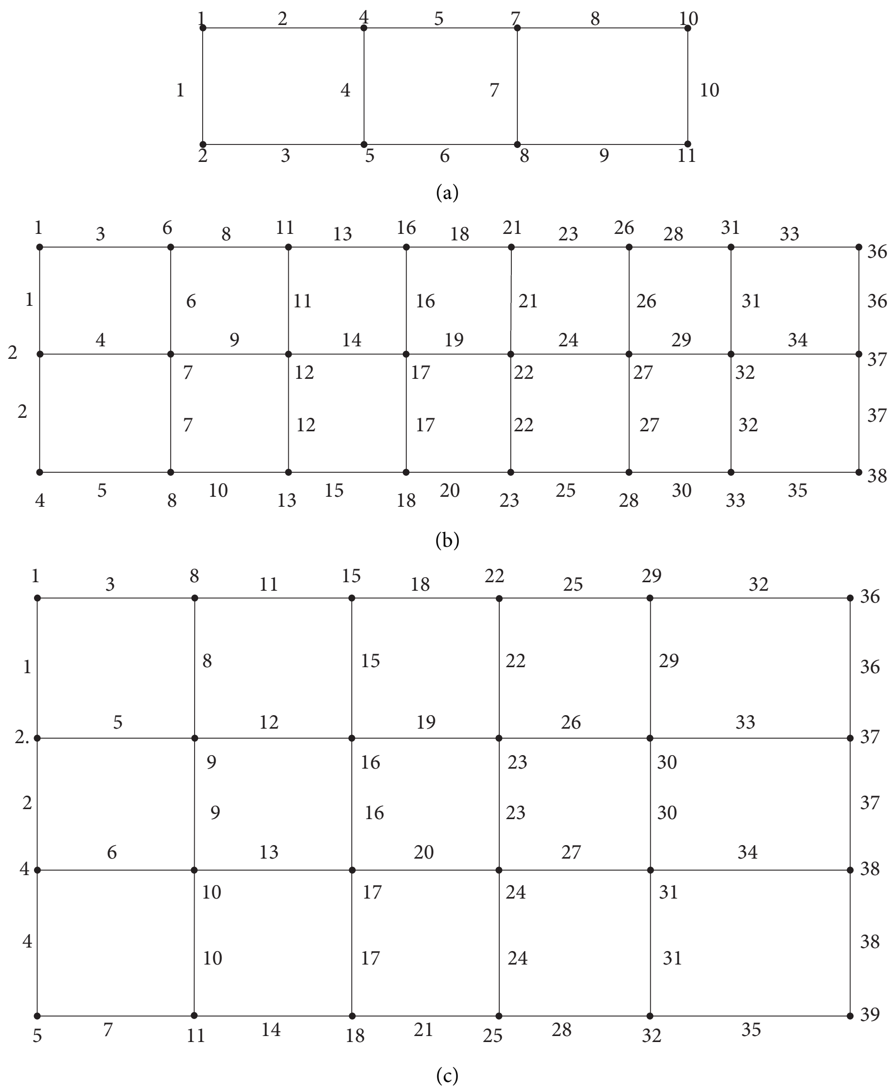

Figure 4: A classical mean labeling of $P_2 \times P_4$, $P_3 \times P_8$, and $P_4 \times P_6$.

$$\chi^*(u_\gamma u_{\gamma+1}) = \begin{cases} 5, & \gamma = 1, \\ 7\gamma - 1, & 2 \le \gamma \le n-1, \end{cases}$$

$$\chi^*(v_\gamma v_{\gamma+1}) = 7\gamma, \quad \text{for } 1 \le \gamma \le n-1,$$

$$\chi^*(u_\gamma v_\gamma) = 7\gamma - 4, \quad \text{for } 1 \le \gamma \le n,$$

$$\chi^*(u_\gamma w_1^{(\gamma)}) = \begin{cases} 1, & \gamma = 1, \\ 7\gamma - 3, & 2 \le \gamma \le n \text{ and } \gamma \text{ is even}, \\ 7\gamma - 6, & 1 \le \gamma \le n \text{ and } \gamma \text{ isodd}, \end{cases} \tag{22}$$

$$\chi^*(u_\gamma w_2^{(\gamma)}) = \begin{cases} 7\gamma - 2, & 1 \le \gamma \le n \text{ and } \gamma \text{ is even}, \\ 7\gamma - 5, & 1 \le \gamma \le n \text{ and } \gamma \text{ isodd}, \end{cases}$$

$$\chi^*(v_\gamma x_1^{(\gamma)}) = \begin{cases} 7\gamma - 6, & 1 \le \gamma \le n \text{ and } \gamma \text{ is even}, \\ 7\gamma - 3, & 4 \le \gamma \le n \text{ and } \gamma \text{ isodd}, \end{cases}$$

$$\chi^*(v_\gamma x_2^{(\gamma)}) = \begin{cases} 7\gamma - 5, & 2 \le \gamma \le n \text{ and } \gamma \text{ is even}, \\ 7\gamma - 2, & 1 \le \gamma \le n \text{ and } \gamma \text{ isodd}. \end{cases}$$

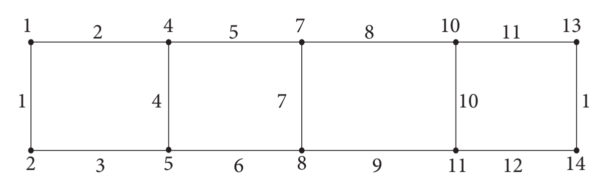

Figure 5: A classical mean labeling of $P_2 \times P_5$.

Hence, for $m \le 2$ and $n \ge 2$, the graph $L_n \circ S_m$ is a classical mean graph.

A classical mean labeling of $L_5 \circ S_1$ is shown in Figure 6.

A classical mean labeling of $L_5 \circ S_2$ is shown in Figure 7. □

Theorem 5. *The triangular ladder graph TL_n is a classical mean graph, for $n \ge 2$.*

Proof. Construct a mapping χ from $V(\mathrm{TL}_n)$ to $\{1, 2, 3, \ldots, 4n-2\}$:

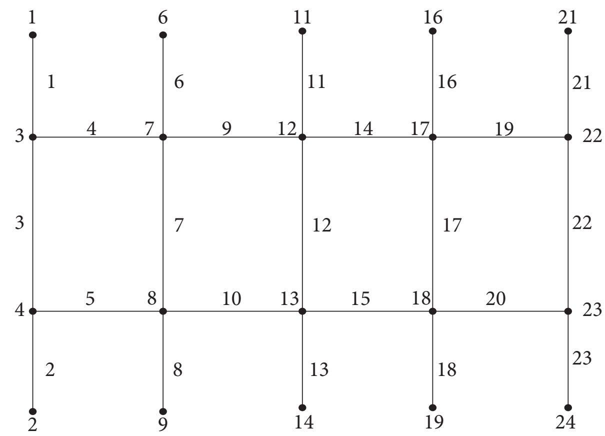

FIGURE 6: A classical mean labeling of $L_5 \circ S_1$.

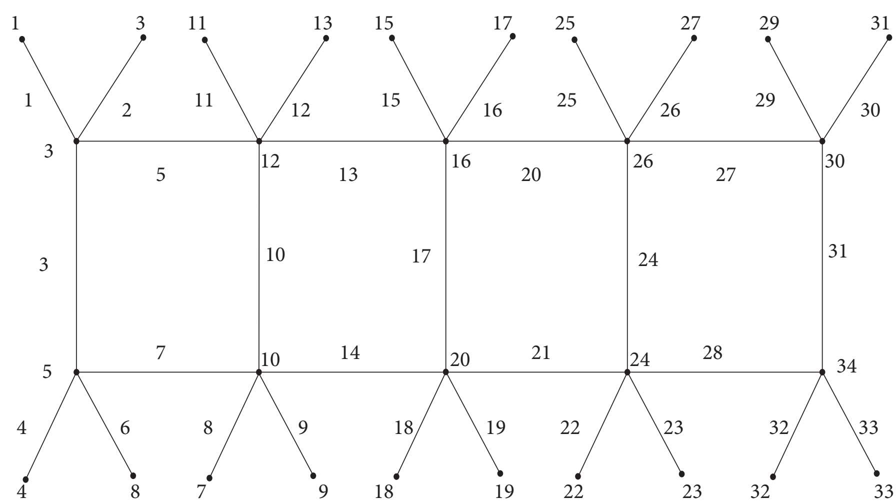

FIGURE 7: A classical mean labeling of $L_5 \circ S_2$.

$$\chi(u_\gamma) = 4\gamma - 3, \quad \text{for } 1 \le \gamma \le n,$$
$$\chi(v_\gamma) = \begin{cases} 4\gamma - 1, & 1 \le \gamma \le n-1, \\ 4n-2, & \gamma = n. \end{cases} \tag{23}$$

Therefore,

$$\begin{aligned} \chi^*(u_\gamma u_{\gamma+1}) &= 4\gamma - 2, \quad \text{for } 1 \le \gamma \le n-1, \\ \chi^*(u_\gamma v_\gamma) &= 4\gamma - 3, \quad \text{for } 1 \le \gamma \le n, \\ \chi^*(v_\gamma v_{\gamma+1}) &= 4\gamma, \quad \text{for } 1 \le \gamma \le n-1, \\ \chi^*(u_\gamma v_{\gamma+1}) &= -1 + 4\gamma, \quad \text{for } 1 \le \gamma \le n-1. \end{aligned} \tag{24}$$

Hence, the triangular ladder graph TL_n is a classical mean graph, for $n \ge 2$.

A classical mean labeling of TL_8 is shown in Figure 8. □

Theorem 6. *For $m \le 2$ and $n \ge 2$, the graph $TL_n \circ S_m$ is a classical mean graph.*

Proof. Case (i). $m = 1$.

Construct a mapping χ from $V(L_n \circ S_1)$ to $\{1, 2, 3, \ldots, 6n-2\}$:

$$\begin{aligned} \chi(u_\gamma) &= \begin{cases} 5\gamma - 3, & 1 \le \gamma \le 2, \\ 6\gamma - 4, & 3 \le \gamma \le n, \end{cases} \\ \chi(v_\gamma) &= 6\gamma - 2, \quad \text{for } 1 \le \gamma \le n, \\ \chi(w_1^{(\gamma)}) &= \begin{cases} 7\gamma - 6, & 1 \le \gamma \le 2, \\ 6\gamma - 5, & 3 \le \gamma \le n, \end{cases} \\ \chi(x_1^{(\gamma)}) &= \begin{cases} 3, & \gamma = 1, \\ 6\gamma - 3, & 2 \le \gamma \le n. \end{cases} \end{aligned} \tag{25}$$

Therefore,

$$\chi^*(u_\gamma u_{\gamma+1}) = -2 + 6\gamma, \quad \text{for } 1 \le \gamma \le n-1,$$
$$\chi^*(v_\gamma v_{\gamma+1}) = 6\gamma, \quad \text{for } 1 \le \gamma \le n-1,$$
$$\chi^*(u_\gamma v_{\gamma+1}) = 6\gamma - 1, \quad \text{for } 1 \le \gamma \le n-1,$$
$$\chi^*(u_\gamma v_\gamma) = \begin{cases} 2, & \gamma = 1, \\ 6\gamma - 4, & 2 \le \gamma \le n, \end{cases} \tag{26}$$
$$\chi^*(u_\gamma w_1^{(\gamma)}) = 6\gamma - 5, \quad \text{for } 1 \le \gamma \le n,$$
$$\chi^*(v_\gamma x_1^{(\gamma)}) = \begin{cases} 3, & \gamma = 1, \\ 6\gamma - 3, & 2 \le \gamma \le n. \end{cases}$$

Case (ii). $m = 2$.

Construct a mapping χ from $V(L_n \circ S_2)$ to $\{1, 2, 3, \ldots, 8n-2\}$:

$$\chi(u_\gamma) = \begin{cases} 2, & \gamma = 1, \\ 8\gamma - 3, & 2 \le \gamma \le n, \end{cases}$$
$$\chi(v_\gamma) = \begin{cases} 6, & \gamma = 1, \\ 8\gamma - 5, & 2 \le \gamma \le n, \end{cases}$$
$$\chi(w_1^{(\gamma)}) = \begin{cases} 1, & \gamma = 1, \\ 8\gamma - 4, & 2 \le \gamma \le n, \end{cases}$$
$$\chi(w_2^{(\gamma)}) = \begin{cases} 3, & \gamma = 1, \\ 8\gamma - 2, & 2 \le \gamma \le n, \end{cases} \tag{27}$$
$$\chi(x_1^{(\gamma)}) = \begin{cases} 4, & \gamma = 1 \\ 8\gamma - 7, & 2 \le \gamma \le n, \end{cases}$$
$$\chi(x_2^{(\gamma)}) = \begin{cases} 9, & \gamma = 1, \\ 8\gamma - 6, & 2 \le \gamma \le n. \end{cases}$$

Therefore,

$$\chi^*(u_\gamma u_{\gamma+1}) = \begin{cases} 6, & \gamma = 1, \\ 8\gamma, & 2 \le \gamma \le n-1, \end{cases}$$
$$\chi^*(v_\gamma v_{\gamma+1}) = \begin{cases} 8, & \gamma = 1, \\ 8\gamma - 2, & 2 \le \gamma \le n-1, \end{cases}$$
$$\chi^*(u_\gamma v_\gamma) = 8\gamma - 5, \quad \text{for } 1 \le \gamma \le n,$$
$$\chi^*(u_\gamma v_{\gamma+1}) = \begin{cases} 5, & \gamma = 1, \\ 8\gamma - 1, & 2 \le \gamma \le n-1, \end{cases}$$
$$\chi^*(u_\gamma w_1^{(\gamma)}) = \begin{cases} 1, & \gamma = 1, \\ 8\gamma - 4, & 2 \le \gamma \le n, \end{cases} \tag{28}$$
$$\chi^*(u_\gamma w_2^{(\gamma)}) = \begin{cases} 2, & \gamma = 1, \\ 8\gamma - 3, & 2 \le \gamma \le n, \end{cases}$$
$$\chi^*(v_\gamma x_1^{(\gamma)}) = \begin{cases} 4, & \gamma = 1, \\ 8\gamma - 5, & 2 \le \gamma \le n, \end{cases}$$
$$\chi^*(v_\gamma x_2^{(\gamma)}) = \begin{cases} 7, & \gamma = 1, \\ 8\gamma - 6, & 2 \le \gamma \le n. \end{cases}$$

Hence, for $m \le 2$ and $n \ge 2$, the graph $TL_n \circ S_m$ is a classical mean graph.

A classical mean labeling of $TL_5 \circ S_1$ is shown in Figure 9.

A classical mean labeling of $TL_4 \circ S_2$ is shown in Figure 10. □

Theorem 7. *For $n \ge 2$, the slanting ladder graph SL_n is a classical mean graph.*

Proof. Construct a mapping χ from $V(SL_n)$ to $\{1, 2, 3, \ldots, 3n-2\}$:

$$\chi(u_1) = 1,$$
$$\chi(u_\gamma) = -4 + 3\gamma, \quad \text{for } 2 \le \gamma \le n,$$
$$\chi(v_\gamma) = 3\gamma, \quad \text{for } 1 \le \gamma \le -1 + n, \tag{29}$$
$$\chi(v_n) = -2 + 3n.$$

Therefore,

$$\chi^*(u_\gamma u_{\gamma+1}) = \begin{cases} 1, & \gamma = 1, \\ 3\gamma - 3, & 2 \le \gamma \le n-1, \end{cases}$$
$$\chi^*(v_\gamma v_{\gamma+1}) = 3\gamma + 1, \quad \text{for } 1 \le \gamma \le n-2, \tag{30}$$
$$\chi^*(v_{n-1} v_n) = 3n - 3,$$
$$\chi^*(v_\gamma u_{\gamma+1}) = 3\gamma - 1, \quad \text{for } 1 \le \gamma \le n-1.$$

Hence, for $n \ge 2$, the slanting ladder graph SL_n is a classical mean graph.

A classical mean labeling of SL_8 is shown in Figure 11. □

Theorem 8. *For $m \le 2$ and $n \ge 2$, the graph $SL_n \circ S_m$ is a classical mean graph.*

Proof. Let $E(SL_n \circ S_m) = \{u_\gamma u_{\gamma+1}, \ v_\gamma v_{\gamma+1}: 1 \le \gamma \le n-1\} \cup \{u_\gamma v_{\gamma-1}: 2 \le \gamma \le n\} \cup \{u_\gamma u_\delta^{(\gamma)}: 1 \le \gamma \le n, 1 \le \delta \le m\} \cup \{v_\gamma v_\delta^{(\gamma)}: 1 \le \gamma \le n, 1 \le \delta \le m\}$.

Case (i). $n \ge 3$ and $m = 1$.

Construct a mapping χ from $V(SL_n \circ S_1)$ to $\{1, 2, 3, \ldots, 5n-2\}$:

$$\chi(u_\gamma) = \begin{cases} 1 + \gamma, & 1 \le \gamma \le 2, \\ -6 + 5\gamma, & 3 \le \gamma \le n, \end{cases}$$
$$\chi(v_\gamma) = \begin{cases} 6, & \gamma = 1, \\ 5\gamma, & 2 \le \gamma \le n-1, \\ -2 + 5n, & \gamma = n, \end{cases}$$
$$\chi(w_1^{(\gamma)}) = \begin{cases} -2 + 3\gamma, & 1 \le \gamma \le 2, \\ -7 + 5\gamma, & 3 \le \gamma \le n, \end{cases} \tag{31}$$
$$\chi(x_1^{(\gamma)}) = \begin{cases} 7, & \gamma = 1, \\ 1 + 5\gamma, & 2 \le \gamma \le n-1, \\ -3 + 5n, & \gamma = n. \end{cases}$$

Therefore,

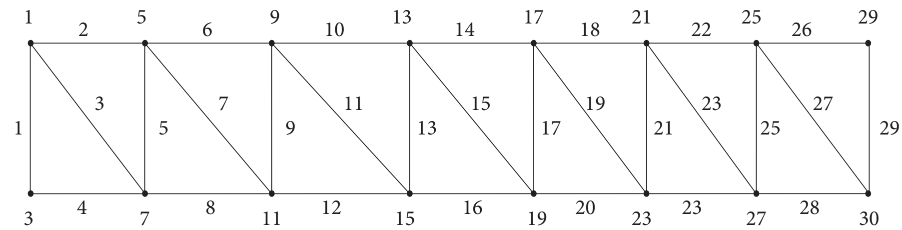

FIGURE 8: A classical mean labeling of TL_8.

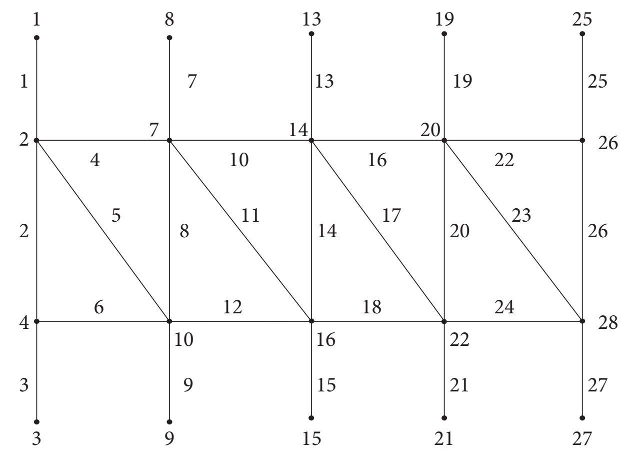

FIGURE 9: A classical mean labeling of $TL_5 \circ S_1$.

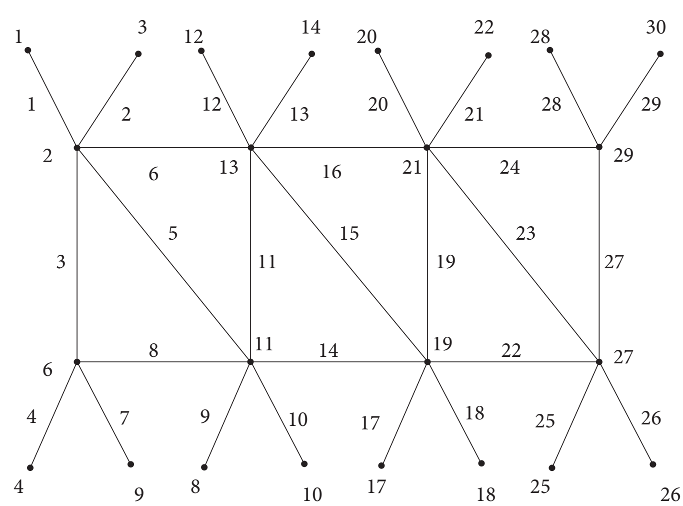

FIGURE 10: A classical mean labeling of $TL_4 \circ S_2$.

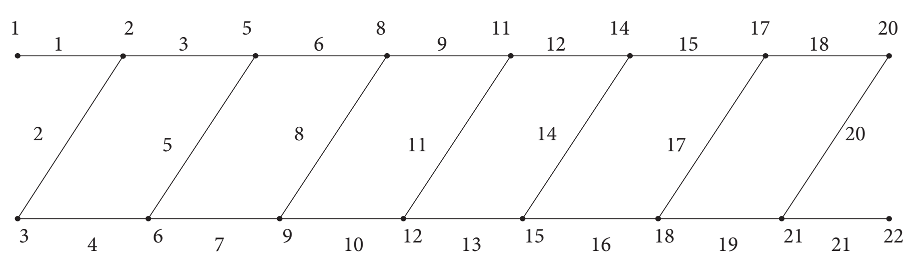

FIGURE 11: A classical mean labeling of SL_8.

$$\chi^*\left(u_\gamma u_{\gamma+1}\right) = \begin{cases} -1 + 3\gamma, & 1 \le \gamma \le 2, \\ -4 + 5\gamma, & 3 \le \gamma \le n-1, \end{cases}$$

$$\chi^*\left(v_\gamma v_{\gamma+1}\right) = \begin{cases} 2 + 5\gamma, & 1 \le \gamma \le n-2, \\ -4 + 5n, & \gamma = n-1, \end{cases}$$

$$\chi^*\left(v_\gamma u_{\gamma+1}\right) = -1 + 5\gamma, \quad \text{for } 1 \le \gamma \le n-1, \tag{32}$$

$$\chi^*\left(u_\gamma w_1^{(\gamma)}\right) = \begin{cases} 1, & \gamma = 1, \\ -7 + 5\gamma, & 2 \le \gamma \le n, \end{cases}$$

$$\chi^*\left(v_\gamma x_1^{(\gamma)}\right) = \begin{cases} 6, & \gamma = 1, \\ 5\gamma, & 2 \le \gamma \le n-1, \\ -3 + 5n, & \gamma = n. \end{cases}$$

Case (ii). $n \ge 3$ and $m = 2$.

Construct a mapping χ from $V(SL_n \circ S_2)$ to $\{1, 2, 3, \ldots, 7n-2\}$

$$\chi\left(u_\gamma\right) = \begin{cases} 2 + \gamma, & 1 \le \gamma \le 2, \\ -9 + 7\gamma, & \gamma \text{ is odd and } 3 \le \gamma \le n-1, \\ -6 + 7\gamma, & \gamma \text{ is even and } 3 \le \gamma \le n-1, \\ -10 + 7n, & n \text{ is even and } \gamma = n, \\ -9 + 7n, & n \text{ is odd and } \gamma = n, \end{cases}$$

$$\chi\left(v_\gamma\right) = \begin{cases} 8, & \gamma = 1, \\ -1 + 7\gamma, & \gamma \text{ is odd and } 2 \le \gamma \le n-3 \\ 2 + 7\gamma, & \gamma \text{ is even and } 2 \le \gamma \le n-3 \\ -13 + 7n, & n \text{ is even and } \gamma = n-2, \\ -15 + 7n, & n \text{ is odd and } \gamma = n-2, \\ 7n - 5, & \gamma = n-1, \\ 7n - 3, & \gamma = n, \end{cases}$$

$$\chi\left(w_1^{(\gamma)}\right) = \begin{cases} 1, & \gamma = 1 \\ -5 + 5\gamma, & 2 \le \gamma \le 3, \\ -7 + 7\gamma, & \gamma \text{ is even and } 4 \le \gamma \le n-1, \\ -10 + 7\gamma, & \gamma \text{ is odd and } 4 \le \gamma \le n-1, \\ -11 + 7n, & n \text{ is even and } \gamma = n, \\ -10 + 7n, & n \text{ is odd and } \gamma = n, \end{cases}$$

$$\chi\left(w_2^{(\gamma)}\right) = \begin{cases} -5 + 7\gamma, & 1 \le \gamma \le 2, \\ -5 + 7\gamma, & \gamma \text{ is even and } 3 \le \gamma \le n-1, \\ -8 + 7\gamma, & \gamma \text{ is odd and } 3 \le \gamma \le n-1, \\ -7 + 7n, & n \text{ is even and } \gamma = n, \\ -8 + 7n, & n \text{ is odd and } \gamma = n, \end{cases}$$

$$\chi\left(x_1^{(\gamma)}\right) = \begin{cases} 9, & \gamma = 1, \\ 7\gamma, & 2 \le \gamma \le n-3 \text{ and } \gamma \text{ iseven}, \\ -3 + 7\gamma, & 2 \le \gamma \le n-3 \text{ and } \gamma \text{ isodd}, \\ -12 + 7n, & n \text{ is even and } \gamma = n-2, \\ -17 + 7n, & n \text{ is odd and } \gamma = n-2, \\ -8 + 7n, & n \text{ is even and } \gamma = n-1, \\ -7 + 7n, & n \text{ is odd and } \gamma = n-1, \\ -4 + 7n, & \gamma = n, \end{cases} \tag{33}$$

$$\chi\left(x_2^{(\gamma)}\right) = \begin{cases} 11, & \gamma = 1, \\ 7\gamma + 1, & \gamma \text{ is even and } 2 \le \gamma \le n-3, \\ -2 + 7\gamma, & \gamma \text{ is odd and } 2 \le \gamma \le n-3, \\ -9 + 7n, & n \text{ is even and } \gamma = n-2, \\ -16 + 7n, & n \text{ is odd and } \gamma = n-2, \\ -6 + 7n, & \gamma = n-1, \\ -2 + 7n, & \gamma = n. \end{cases}$$

Therefore,

$$\chi^*\left(u_\gamma u_{\gamma+1}\right) = \begin{cases} -1 + 4\gamma, & 1 \le \gamma \le 2, \\ -5 + 7\gamma, & 3 \le \gamma \le n-2, \\ -14 + 7n, & n \text{ is even and } \gamma = n-1, \\ -12 + 7n, & n \text{ is odd and } \gamma = n-1, \end{cases}$$

$$\chi^*\left(v_\gamma v_{\gamma+1}\right) = \begin{cases} 11, & \gamma = 1, \\ 3 + 7\gamma, & 2 \le \gamma \le n-3, \\ -10 + 7n, & n \text{ is even and } \gamma = n-2, \\ -11 + 7n, & n \text{ is odd and } \gamma = n-2, \\ -5 + 7n, & \gamma = n-1, \end{cases}$$

$$\chi^*\left(v_\gamma u_{\gamma+1}\right) = \begin{cases} 5, & \gamma = 1, \\ -1 + 7\gamma, & 2 \le \gamma \le n-1, \end{cases}$$

$$\chi^*\left(u_\gamma w_1^{(\gamma)}\right) = \begin{cases} 1, & \gamma = 1, \\ -8 + 6\gamma, & 2 \le \gamma \le 3, \\ -7 + 7\gamma, & \gamma \text{ is even and } 4 \le \gamma \le n-1, \\ -10 + 7\gamma, & \gamma \text{ is odd and } 4 \le \gamma \le n-1, \\ -11 + 7n, & n \text{ is even and } \gamma = n, \\ -10 + 7n, & n \text{ is odd and } \gamma = n, \end{cases}$$

$$\chi^*\left(u_\gamma w_2^{(\gamma)}\right) = \begin{cases} -2 + 4\gamma, & 1 \le \gamma \le 2, \\ -6 + 7\gamma, & \gamma \text{ is even and } 3 \le \gamma \le n-1, \\ -9 + 7\gamma, & \gamma \text{ is odd and } 3 \le \gamma \le n-1, \\ -9 + 7n, & \gamma = n, \end{cases}$$

$$\chi^*\left(v_\gamma x_1^{(\gamma)}\right) = \begin{cases} 8, & \gamma = 1, \\ 7\gamma, & \gamma \text{ is even and } 2 \le \gamma \le n-3, \\ -3+7\gamma, & \gamma \text{ is odd and } 2 \le \gamma \le n-3, \\ -13+7n, & n \text{ is even and } \gamma = n-2, \\ -17+7n, & n \text{ is odd and } \gamma = n-2, \\ -7+7n, & \gamma = n-1, \\ -4+7n, & \gamma = n, \end{cases}$$
$$\chi^*\left(v_\gamma x_2^{(\gamma)}\right) = \begin{cases} 9, & \gamma = 1, \\ 1+7\gamma, & \gamma \text{ is even and } 2 \le \gamma \le n-3, \\ -2+7\gamma, & \gamma \text{ is odd and } 2 \le \gamma \le n-3, \\ -12+7n, & n \text{ is even and } \gamma = n-2, \\ -16+7n, & n \text{ is odd and } \gamma = n-2, \\ -6+7\gamma, & \gamma = n-1, \\ -3+7n, & \gamma = n. \end{cases} \tag{34}$$

Case (iii). $n = 2$ and $m = 1, 2$.

A classical mean labeling of $SL_2 \circ S_1$ and $SL_2 \circ S_2$ are shown in Figure 12.

A classical mean labeling of $SL_6 \circ S_1$ is shown in Figure 13.

A classical mean labeling of $SL_5 \circ S_2$ is shown in Figure 14. □

Theorem 9. *For $n \ge 2$, the graph D_n^* is a classical mean graph.*

Proof. Take $V(D_n^*) = \{a_{\gamma,\delta}\colon 1 \le \gamma \le n, \delta = 1,2,3,4\}$ and $E(D_n^*) = \{a_{\gamma,1}a_{\gamma+1,1}, a_{\gamma,3}a_{\gamma+1,3}\colon 1 \le \gamma \le n-1\} \cup \{a_{\gamma,1}a_{\gamma,2}, a_{\gamma,2}a_{\gamma,3}, a_{\gamma,3}a_{\gamma,4}, a_{\gamma,4}a_{\gamma,1}\colon 1 \le \gamma \le n\}$.

Construct a mapping χ from $V(D_n^*)$ to $\{1, 2, 3, \ldots, 6n-1\}$:

$$\begin{aligned} \chi\left(a_{\gamma,1}\right) &= -2+6\gamma, \quad \text{for } 1 \le \gamma \le n, \\ \chi\left(a_{\gamma,2}\right) &= -5+6\gamma, \quad \text{for } 1 \le \gamma \le n, \\ \chi\left(a_{\gamma,3}\right) &= -3+6\gamma, \quad \text{for } 1 \le \gamma \le n, \\ \chi\left(a_{\gamma,4}\right) &= -1+6\gamma, \quad \text{for } 1 \le \gamma \le n. \end{aligned} \tag{35}$$

Therefore,

$$\begin{aligned} \chi^*\left(a_{\gamma,1}a_{\gamma+1,1}\right) &= 6\gamma, \quad \text{for } 1 \le \gamma \le n-1, \\ \chi^*\left(a_{\gamma,3}a_{\gamma+1,3}\right) &= -1+6\gamma, \quad \text{for } 1 \le \gamma \le n-1, \\ \chi^*\left(a_{\gamma,1}a_{\gamma,2}\right) &= -4+6\gamma, \quad \text{for } 1 \le \gamma \le n, \\ \chi^*\left(a_{\gamma,2}a_{\gamma,3}\right) &= -5+6\gamma, \quad \text{for } 1 \le \gamma \le n, \\ \chi^*\left(a_{\gamma,3}a_{\gamma,4}\right) &= -3+6\gamma, \quad \text{for } 1 \le \gamma \le n, \\ \chi^*\left(a_{\gamma,4}a_{\gamma,1}\right) &= -2+6\gamma, \quad \text{for } 1 \le \gamma \le n. \end{aligned} \tag{36}$$

Hence, for $n \ge 2$, the graph D_n^* is a classical mean graph.

A classical mean labeling of D_4^* is shown in Figure 15. □

Theorem 10. *For $n \ge 1$, the diamond ladder graph Dl_n is a classical mean graph.*

Proof. $V(Dl_n) = \{x_\gamma, y_\gamma\colon 1 \le \gamma \le n\} \cup \{z_\gamma\colon 1 \le \gamma \le 2n\}$ and $E(Dl_n) = \{x_\gamma y_\gamma\colon 1 \le \gamma \le n\} \cup \{x_\gamma x_{\gamma+1}, y_\gamma y_{\gamma+1}\colon 1 \le \gamma \le n-1\} \cup \{x_\gamma z_{2\gamma-1}, x_\gamma z_{2\gamma}, y_\gamma z_{2\gamma-1}, y_\gamma z_{2\gamma}\colon 1 \le \gamma \le n\} \cup \{z_{2\gamma}z_{2\gamma+1}\colon 1 \le \gamma \le n-1\}$. Thus, $|V(Dl_n)| = 4n$ and $|E(Dl_n)| = 8n-3$.

Construct a mapping χ from $V(Dl_n)$ to $\{1, 2, 3, \ldots, 8n-2\}$:

$$\chi\left(x_\gamma\right) = -5+8\gamma, \quad \text{for } 1 \le \gamma \le n,$$

$$\chi\left(y_\gamma\right) = -3+8\gamma, \quad \text{for } 1 \le \gamma \le n,$$

$$\chi\left(z_\gamma\right) = \begin{cases} 1, & \gamma = 1, \\ 4\gamma - \left(\dfrac{(-1)^{\gamma+1}+1}{2}\right) - 2, & \gamma \text{ is even and } 2 \le \gamma \le 2n, \\ 4\gamma - \left(\dfrac{(-1)^{\gamma+1}+1}{2}\right) - 2, & \gamma \text{ is odd and } 3 \le \gamma \le 2n. \end{cases} \tag{37}$$

Therefore,

$$\begin{aligned} \chi^*\left(x_\gamma x_{\gamma+1}\right) &= -2+8\gamma, \quad \text{for } 1 \le \gamma \le n-1, \\ \chi^*\left(y_\gamma y_{\gamma+1}\right) &= 8\gamma, \quad \text{for } 1 \le \gamma \le n-1, \\ \chi^*\left(x_\gamma y_\gamma\right) &= -5+8\gamma, \quad \text{for } 1 \le \gamma \le n, \\ \chi^*\left(z_{2\gamma}z_{2\gamma+1}\right) &= -1+8\gamma, \quad \text{for } 1 \le \gamma \le n-1, \\ \chi^*\left(x_\gamma z_{2\gamma-1}\right) &= -7+8\gamma, \quad \text{for } 1 \le \gamma \le n, \\ \chi^*\left(x_\gamma z_{2\gamma}\right) &= -4+8\gamma, \quad \text{for } 1 \le \gamma \le n, \\ \chi^*\left(y_\gamma z_{2\gamma-1}\right) &= -6+8\gamma, \quad \text{for } 1 \le \gamma \le n, \\ \chi^*\left(y_\gamma z_{2\gamma}\right) &= -3+8\gamma, \quad \text{for } 1 \le \gamma \le n. \end{aligned} \tag{38}$$

Hence, for $n \ge 1$, the diamond ladder graph Dl_n is a classical mean graph.

A classical mean labeling of Dl_4 is shown in Figure 16. □

Theorem 11. *For $n \ge 4$, the latitude ladder graph LL_n is a classical mean graph.*

Proof. Here, $V(LL_n) = \{u_\gamma\colon 1 \le \gamma \le n\}$ and $E(LL_n) = \{u_\gamma u_{\gamma+1}\colon 1 \le \gamma \le n-1\} \cup \{u_n u_1\} \cup \{u_\gamma u_{n+2-\gamma}\colon 2 \le \gamma \le (n/2)\}$.

Construct a mapping χ from $V(LL_n)$ to $\{1, 2, 3, \ldots, (3n/2)\}$:

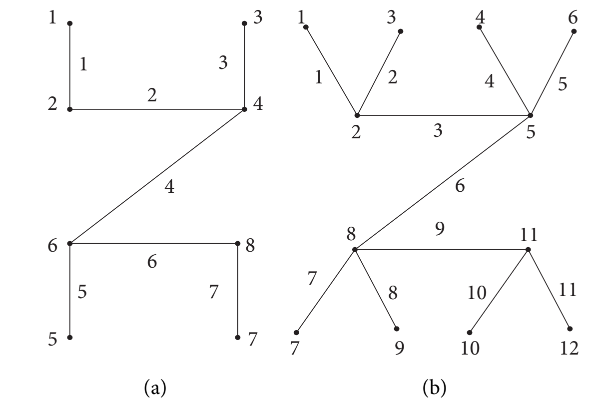

FIGURE 12: A classical mean labeling of $SL_2 \circ S_1$ and $SL_2 \circ S_2$.

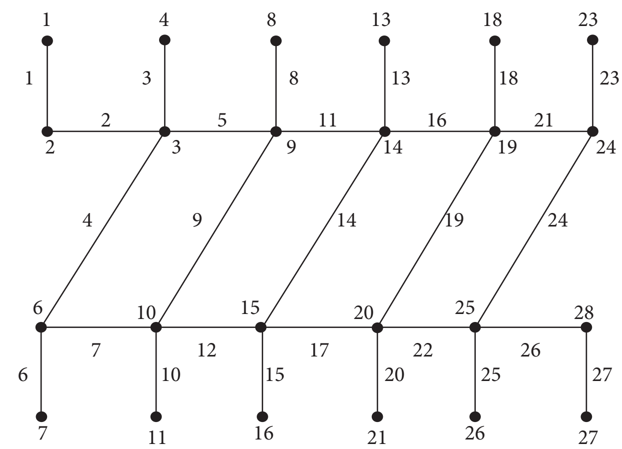

FIGURE 13: A classical mean labeling of $SL_6 \circ S_1$.

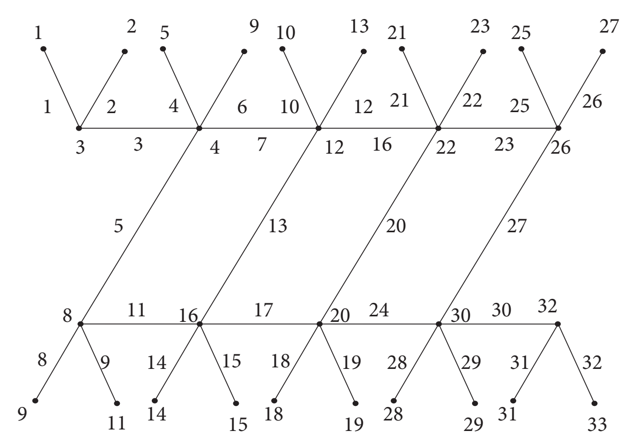

FIGURE 14: A classical mean labeling of $SL_5 \circ S_2$.

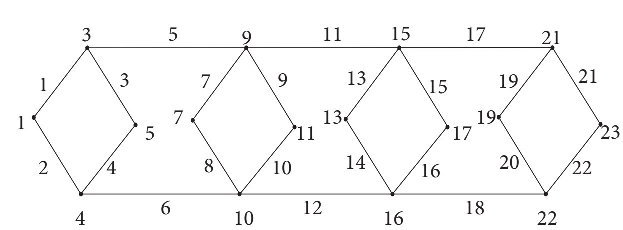

FIGURE 15: A classical mean labeling of D_4^*.

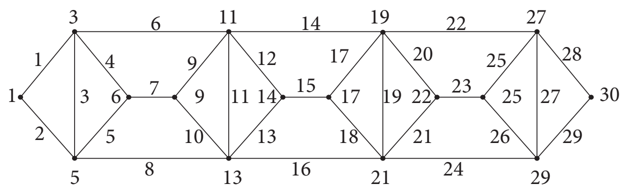

Figure 16: A classical mean labeling of Dl_4.

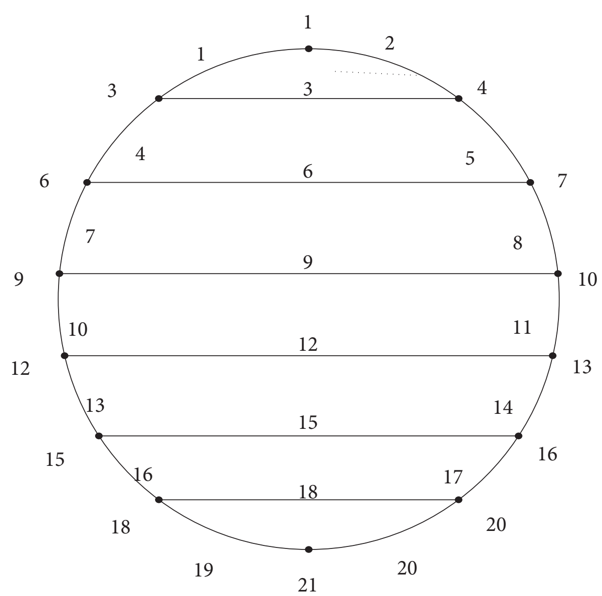

Figure 17: A classical mean labeling of latitude graph LL_{14}.

$$\chi(u_\gamma) = \begin{cases} 3\gamma - 2, & 1 \leq \gamma \leq \dfrac{n}{2} - 1, \\ 3\gamma - 1, & \gamma = \dfrac{n}{2}, \\ \dfrac{3n}{2}, & \gamma = 1 + \dfrac{n}{2}, \\ 3\gamma + 3n - +3, & 2 + \dfrac{n}{2} \leq \gamma \leq n. \end{cases} \quad (39)$$

Therefore,

$$\chi^*(u_\gamma u_{\gamma+1}) = \begin{cases} 3\gamma - 1, & 1 \leq \gamma \leq \dfrac{n}{2}, \\ 3n - 3\gamma + 1, & \dfrac{n}{2} + 1 \leq \gamma \leq n, \end{cases} \quad (40)$$

$$\chi^*(u_\gamma u_{n+2-\gamma}) = 3\gamma - 3, \quad \text{for } 2 \leq \gamma \leq \frac{n}{2}.$$

Hence, for $n \geq 4$, the latitude ladder graph LL_n is a classical mean graph.

A classical mean labeling of LL_{14} is shown in Figure 17. □

5. Conclusion

The Cartesian product is one among graph operations. Based on this operation, the classical mean labeling of various graphs such as the one-sided step graph ST_n, double-sided step graph $2ST_{2n}$, graph $P_m \times P_n$, ladder graph L_n, graph $L_n \circ S_m$, triangular ladder graph TL_n, graph $TL_n \circ S_m$, slanting ladder graph SL_n, graph $SL_n \circ S_m$, graph D_n^*, diamond ladder graph Dl_n, and latitude ladder graph LL_n are established. It would be very interesting to analyze that the classical meanness of various ladder-related graphs. Investigating classical mean labeling of other classes of graphs is still open and this is for future work. One can also explore the exclusive applications of classical mean labeling in real-life problems.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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