Hindawi
Journal of Mathematics
Volume 2022, Article ID 2294627, 3 pages
https://doi.org/10.1155/2022/2294627

Research Article

Finite Groups of Order p^2qr in which the Number of Elements of Maximal Order Is p^3q*

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Received 31 December 2021; Accepted 31 January 2022; Published 18 March 2022

Academic Editor: Naihuan Jing

Let G be a finite group. We know that the order of G and the number of elements of maximal order in G are closely related to the structure of G. This topic involves Thompson's conjecture. In this paper, we classify the finite groups of order p^2qr in which the number of elements of maximal order is p^3q, where $p<q<r$ are different primes.

1. Introduction

All groups considered in the present paper are finite. Let n be an integer. We denote by $\pi(n)$ the set of all prime divisors of n. Let G be a finite group. Then $\pi(|G|)$ is denoted by $\pi(G)$. And we denote by $k(G)$ and $m(G)$ the maximal order of elements in G and the number of elements of order $k(G)$ in G, respectively. We write H char G if H is a characteristic subgroup of G. $G = N \times Q$ stands for the split extension of a normal subgroup N of G by a complement Q. By $M \lesssim G$ we denote M is isomorphic to a subgroup of G. And we denote by Z_n a cyclic group of order n. All unexplained notations are standard and can be found in [1].

As is well known, for a finite group G, $|G|$ and $m(G)$ have an important influence on the structure of G. The authors in [2–4] proved that finite groups with $m(G) = lp$ are soluble, where $l = 2, 4$, or 18. In [5] it was proved that finite groups with $m(G) = 2p^2$ are soluble. The authors in [6, 7] gave a classification of the finite groups with $m(G) = 30$ and $m(G) = 24$. These studies are closely related to the following conjecture.

1.1. Thompson's Conjecture. Let G be a finite group. For a positive integer d, define $G(d) = |\{x \in G \| x| = d\}|$. If H is a soluble group, $G(d) = H(d)$ for $d = 1, 2, \ldots,$ then G is soluble.

In this paper we classify the finite groups of order p^2qr in which the number of elements of maximal order is p^3q, where $p<q<r$ are different primes. Our result is:

1.2. Main Theorem. Suppose that G is a finite group satisfying $|G| = p^2$qr and $m(G) = p^3q$, where $p<q<r$ are different primes. Then one of the following statements holds:

(1) $G \cong M <$ imesZ_r, where M is a group of order $4q$. Moreover, $M/Z_2 \lesssim$ Aut (Z_r), $(M/Z_2) <$ imesZ_r is a Frobenius group and $r-1=8q$.

(2) $G \cong Z_{4r} \rtimes Z_q$. Moreover, $Z_q \lesssim$ Aut (Z_{4r}) and $r-1=4q$.

(3) $G \cong H < imesZ_{qr}$, where H is a group of order 4, $q = 3$ and $r = 13$ or $q = 5$ and $r = 11$. Moreover, H is isomorphic to a subgroup of Aut (Z_{qr}). Hence $H \lesssim Z_2 \times Z_{12}$ or $H \lesssim Z_4 \times Z_{10}$.

(4) $G \cong Z_{84}$.

(5) $G \cong (Z_2 \times Z_2) \times Z_{15}$.

(6) $G \cong M < imesZ_{qr}$, where M is a group of order 4, $q = 3$ and $r = 13$ or $q = 5$ and $r = 11$. Moreover, M/Z_2 is isomorphic to a subgroup of Aut (Z_{qr}). Hence $M/Z_2 \lesssim Z_2 \times Z_{12}$ or $M/Z_2 \lesssim Z_4 \times Z_{10}$.

(7) G is a Frobenius group and $G \cong Z_{4q} <$ imesZ_r. Moreover, $r-1=8q$.

(8) $G \cong A_5$.

2. Preliminaries

We need the following lemmas to show the main theorem.

Lemma 1 (see [8]). *Let G be a finite group. Then the number of elements whose orders are multiples of n is either zero, or a multiple of the greatest divisor of |G| that is prime to n.*

Lemma 2 (see [2]). *Let G be a finite group. We denote by A_i $(1 \le i \le s)$ a complete representative system of conjugate classes of cyclic subgroups of order $k(G)$, respectively. Then we have the following:*

(1) $m(G) = \varphi(k(G)) \sum n_i$, where $\varphi(k(G))$ is the Euler function, $n_i = |G\colon N_G(A_i)|$ and $1 \le i \le s$.

(2) $|G| = |G\colon N_G(A_i)||N_G(A_i)\colon C_G(A_i)||C_G(A_i)|$, where $1 \le i \le s$.

(3) $|N_G(A_i)\colon C_G(A_i)| \| \varphi(k(G))$, where $1 \le i \le s$.

(4) $\pi(C_G(A_i)) = \pi(A_i)$, where $1 \le i \le s$.

Lemma 3 (see [9]). *Let G be a soluble group of order mn, where m is prime to n. Then the number of subgroups of G of order m may be expressed as a product of factors, each of which (i) is congruent to 1 modulo some prime factor of m and (ii) is a power of a prime and divides the order of some chief factor of G.*

Lemma 4 (see [10]). *Let G be a finite simple group. If $|\pi(G)| = 3$, then we call G a simple K_3-group. If G is a simple K_3-group, then G is isomorphic to one of the following groups: A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$ or $U_4(2)$.*

3. Proof of the Theorem

We know that $\pi(G) \subseteq \pi(m(G)) \bigcup \pi(k(G))$ by Lemma 1. Then $r \in \pi(k(G))$. In the following, we discuss four cases.

Case 1. If $\pi(k(G)) = \{p, r\}$, then $k(G) = pr$ or $p^2 r$.

Suppose that $k(G) = \text{pr}$. Since $\varphi(k(G)) = (p-1)(r-1) | m(G) = p^3 q$ by Lemma 2, we have $p = 2$. Choose an arbitrary element x of order $k(G)$ in G and let $\langle x \rangle = A$. It is clear that $x^p \in Z(C_G(A))$ and so G has a Sylow r-subgroup P_r such that $P_r \le Z(C_G(A))$. Therefore P_r char $C_G(A)$ and it follows that $P_r \lhd N_G(A)$ since $C_G(A) \lhd N_G(A)$. Therefore $N_G(A) \le N_G(P_r)$ and thus $|G\colon N_G(P_r)| \| |G\colon N_G(A)|$. By Lemma 2 we get that $|G\colon N_G(A)| \| 2q$. So $|G\colon N_G(P_r)| \| 2q$.

If $P_r \ntrianglelefteq G$, then $|G\colon N_G(P_r)| = |G\colon N_G(A)| = 2q$ by Sylow's theorem. Consequently, we have $2q|n$ by Lemma 2, where n is the number of cyclic subgroups of order $k(G)$ in G. Note that $n = m(G)/\varphi(\text{pr}) = 8q/r-1$, thus $r - 1 = 4$ and so $r = 5$. It follows that $q = 3$. It is well known that a group of order 60 is isomorphic to A_5, if it is not 5-closed. Therefore G is isomorphic to A_5, which is a contradiction since A_5 has no elements of order 10.

If $P_r \lhd G$, then $C_G(P_r)$ contains all the elements of order $k(G)$ in G since $A \le C_G(A) \le C_G(P_r)$. Note that $P_r \le Z(C_G(P_r))$, thus $|C_G(P_r)| = 2^\alpha r$, where $1 \le \alpha \le 2$. Moreover, $C_G(P_r) = H \times P_r$ by Schur–Zassenhaus's theorem, where H is a group of order 2^α. If $\alpha = 2$, then H is an elementary abelian group of order 4. Thus $3(r-1) = 8q$ and it follows that $r - 1 = 8$, which is a contradiction since r is a prime. If $\alpha = 1$, then $r - 1 = 8q$. By Schur–Zassenhaus's theorem we get that $G \cong M < \text{imes}Z_r$, where M is a group of order $4q$. It is obvious that $M/Z_2 \lesssim \text{Aut}\ (Z_r)$ and $(M/Z_2) < \text{imes}Z_r$ is a Frobenius group. Hence (1) holds.

Suppose that $k(G) = p^2 r$. Since $\varphi(k(G)) = p(p-1)(r-1) | m(G) = p^3 q$ by Lemma 2, we get that $p - 1 = 1$ and so $p = 2$. Choose an arbitrary element x of order $k(G)$ in G and let $\langle x \rangle = A$. It is clear that $x^{p^2} \in Z(C_G(A))$ and so G has a Sylow r-subgroup P_r such that $P_r \le Z(C_G(A))$. Therefore P_r char $C_G(A)$ and it follows that $P_r \lhd N_G(A)$ since $C_G(A) \lhd N_G(A)$. So $N_G(A) \le N_G(P_r)$. Then $|G\colon N_G(P_r)| \| |G\colon N_G(A)|$. Note that $|G\colon N_G(A)| q$ by Lemma 2, thus $|G\colon N_G(P_r)| = 1$ by Sylow's theorem. It follows that $P_r \lhd G$ and $C_G(P_r)$ contains all the elements of order $k(G)$ in G. Furthermore, $|C_G(P_r)| = 4q^\alpha r$, where $0 \le \alpha \le 1$.

If $\alpha = 0$, then $C_G(P_r) \cong Z_{4r}$. Therefore $\varphi(4r) = 8q$ and thus $r - 1 = 4q$. Since $C_G(P_r) \lhd G$, we obtain $G \cong Z_{4r} \rtimes Z_q$ by Schur–Zassenhaus's theorem. Moreover, $Z_q \lesssim \text{Aut}\ (Z_{4r})$. Hence (2) holds.

If $\alpha = 1$, then $C_G(P_r) = G$ and so $P_r \le Z(G)$. Hence G has elements of order qr, but has no elements of order $2q$ since $k(G) = 4r$ is the maximal element order of G. Since $k(G) = 4r > \text{qr}$, we get that $q = 3$. Note that the Sylow 2-subgroup P_2 of G is cyclic, then G has a normal 2-complement H. Obviously $H \cong Z_{3r}$ and so the Sylow 3-subgroup P_3 of G is normal in G. Since G has no elements of 6, we get that P_2 acts fixed-point-freely on P_3. Therefore $|P_2| (|P_3| - 1)$, namely, $4|2$, which is a contradiction.

Case 2. If $\pi(k(G)) = \{q, r\}$, then $k(G) = \text{qr}$.

Since $\varphi(k(G)) = (q-1)(r-1) | m(G) = p^3 q$ by Lemma 2, we get that $q - 1 | p^3$. Since $2 | q - 1$, we have $2 | p^3$ and so $p = 2$. Choose an arbitrary element x of order $k(G)$ in G and let $\langle x \rangle = A$. It is clear that $Z(C_G(A))$ contains elements of order r, and so G has a Sylow r-subgroup P_r such that $P_r \le Z(C_G(A))$. Therefore P_r char $C_G(A)$ and it follows that $P_r \lhd N_G(A)$ since $C_G(A) \lhd N_G(A)$. So $N_G(A) \le N_G(P_r)$. Then $|G\colon N_G(P_r)| \| |G\colon N_G(A)|$. Note that $|G\colon N_G(A)| \| 4$ by Lemma 2, thus we get that $|G\colon N_G(P_r)| \| 4$. Since $2 < q < r$, we get that $|G\colon N_G(P_r)| = 1$ by Sylow's theorem. It follows that $P_r \lhd G$. Then G is soluble. It follows that $|G\colon N_G(A)| = 1$ or 4 by Lemma 3. If $|G\colon N_G(A)| = 4$, then $4 \equiv 1 \pmod{q}$ by Lemma 3. Therefore $q = 3$ and thus $n = 24/\varphi(k(G) = 24/(q-1)(r-1) = 4$, where n is the number of cyclic subgroups of order $k(G)$ in G. Then $r = 4$, which is a contradiction since r is a prime. If $|G\colon N_G(A)| = 1$, then from $\varphi(\text{qr}) = (q-1)(r-1) = 8q$ we get that $q = 3$ and $r = 13$ or $q = 5$ and $r = 11$. Furthermore, $G \cong H < \text{imes}Z_{\text{qr}}$ by Schur–Zassenhaus's theorem, where H is a group of order 4. It is evident that the conjugate action of H on Z_{qr} is faithful. Therefore H is isomorphic to a subgroup of $\text{Aut}\ (Z_{\text{qr}})$. Hence $H \lesssim Z_2 \times Z_{12}$ or $H \lesssim Z_4 \times Z_{10}$ and so (3) holds.

Case 3. If $\pi(k(G)) = \{p, q, r\}$, then $k(G) = p^2$qr or pqr.

If $k(G) = p^2$qr, then $\varphi(p^2\text{qr}) = p(p-1)(q-1)(r-1) = p^3q$. Therefore $p-1=1$ and so $p=2$. Consequently, $q - 1/2 \cdot r - 1/2 = q$. Since $r - 1/2 > 1$, we have $q - 1/2 = 1$ and so $q = 3$. It follows that $r = 7$. Hence $G \cong Z_{84}$ and thus (4) holds.

If $k(G) =$ pqr, then by $\varphi(k(G)) = (p-1)(q-1)(r-1)|p^3q$ we know that $p-1=1$ and so $p=2$. Choose an arbitrary element x of order $k(G)$ in G and let $\langle x\rangle = A$. It is clear that $Z(C_G(A))$ contains elements of order qr, and so G has a subgroup H of order qr such that $H \le Z(C_G(A))$. Therefore H char $C_G(A)$ and it follows that $H \lhd N_G(A)$ since $C_G(A) \lhd N_G(A)$. So $N_G(A) \le N_G(H)$. Then $|G: N_G(H)|\,\|\,|G: N_G(A)|$. Note that $\|G: N_G(A)| = 1$, thus $|G: N_G(H)| = 1$ and so $H \lhd G$. Therefore $C_G(H)$ contains all the elements of order $k(G)$ in G and so $|C_G(H)| = 2^\alpha$qr, where $1 \le \alpha \le 2$.

If $\alpha = 2$, then $H \le Z(G)$. So $G = K \times H$ by Schur–Zassenhaus's theorem. Obviously H is a non-cyclic group of order 4. Hence $3(q-1)(r-1) = 8q$ and so $q = 3$, $r = 5$. Therefore $G \cong (Z_2 \times Z_2) \times Z_{15}$. Hence (5) holds.

If $\alpha = 1$, then $C_G(H) \cong Z_2 \times H$. So $(q-1)(r-1) = 8q$. It follows that $q = 3$ and $r = 13$ or $q = 5$ and $r = 11$. Furthermore, $G \cong M <$ imesZ_{qr} by Schur–Zassenhaus's theorem, where M is a group of order 4. It is evident that the kernel of the conjugate action of M on Z_{qr} is isomorphic to Z_2. Therefore M/Z_2 is isomorphic to a subgroup of Aut (Z_{qr}). Hence $M/Z_2 \lesssim Z_2 \times Z_{12}$ or $M/Z_2 \lesssim Z_4 \times Z_{10}$. Thus (6) holds.

Case 4. If $\pi(k(G)) = \{r\}$, then $k(G) = r$.

Since $r-1|p^3q$ and $2|r-1$, we get that $p=2$. We know that the number n_r of Sylow r-subgroups of G is equal to 1, $2q$, or $4q$ by Sylow's theorem.

If $n_r = 1$, then the Sylow r-subgroup P_r is normal in G and $r - 1 = 8q$. Moreover, G has an r-complement H of order $4q$ by Schur–Zassenhaus's theorem. Note that the conjugate action of H on P_r is fixed-point-free, thus G is a Frobenius group with Frobenius kernel P_r and Frobenius complement H. Note that $P_r \cong Z_r$ and H is a cyclic group, thus $G \cong Z_{4q} <$ imesZ_r. Hence (7) holds.

If $n_r = 4q$, then $4q(r-1) = 8q$, which is impossible.

If $n_r = 2q$, then G is non-soluble by Lemma 3 and so $G \cong A_5$ by Lemma 4. Hence (8) holds.

Now the proof of the theorem is complete.

Data Availability

This paper is a theoretical study without experimental data.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by NNSF of China (11401324).

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