Hindawi
Journal of Mathematics
Volume 2022, Article ID 3945178, 6 pages
https://doi.org/10.1155/2022/3945178

Research Article

Global Existence for the 3D Tropical Climate Model with Small Initial Data in $\dot{H}^{1/2}(\mathbb{R}^3)$*

Hui Zhang and Juan Xu

College of Mathematical and Physical Sciences, Anqing Normal University, Anqing 246133, China

Correspondence should be addressed to Juan Xu; 750771486@qq.com

Received 7 March 2022; Accepted 21 April 2022; Published 21 May 2022

Academic Editor: Qingkai Zhao

The well-posedness problem is an important but challenging research topic in nonlinear partial differential equations. In this paper, we establish a global-in-time existence result of strong solutions for small initial data in terms of the $\dot{H}^{1/2}(\mathbb{R}^3)$ norm on three-dimensional tropical climate model with viscosities by derive a blow-up criterion combine with energy estimates. This result can be regard as a generalization of the famous Fujita–Kato result to 3D Navier–Stokes equations.

1. Introduction

In this paper, we investigate the global existence of smooth solutions to the tropical climate model (TCM) in $\mathbb{R}^3$:

$$\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p + \nabla \cdot (v \otimes v) = 0, \\ \partial_t v - \Delta v + v \cdot \nabla u + u \cdot \nabla v + \nabla \theta = 0, \\ \partial_t \theta - \Delta \theta + u \cdot \nabla \theta + \nabla \cdot v = 0, \\ \nabla \cdot u = 0, \end{cases} \tag{1}$$

with initial data

$$u(x,0) = u_0(x), v(x,0) = v_0(x), \theta(x,0) = \theta_0(x), \tag{2}$$

where $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ denote the barotropic mode and the first baroclinic mode of the velocity, while the scalar functions θ, p denote the temperature and pressure, respectively.

It should be pointed out that the original system derived in [1] has no viscous terms because it is derived from the inviscid primitive equations. In many research studies of tropical atmospheric dynamics, Gill and Matsuno [2, 3] first used the baroclinic mode models. Majda-Biello [4] pointed out that the transport of momentum between the barotropic and baroclinic model is an important effect, it is necessary to retain both the barotropic and baroclinic modes of the velocity. There are geophysical circumstances in which the Laplacian may arise. Fundamental issues such as the global existence and regularity of solutions have attracted considerable attention. Some important results in terms of the global existence and uniqueness of classical solutions in 2D have been obtained, see [5–7] etc. In addition, some results on the global well-posedness issue for the 3D tropical climate model with fractional dissipation and damping terms can be referred to [8, 9].

This paper focuses on the global existence of the classical solutions for TCM in $\mathbb{R}^3$. System (1) contains the 3D incompressible Navier–Stokes (NS) equations as a special case ($v = 0, \theta = 0$), for which the issue of global well-posedness with small initial data in the Sobolev spaces $\dot{H}^{1/2}(\mathbb{R}^3)$ has been solved by Fujita and Kato [10]. Our goal of this paper is to extend similar results to TCM. We should point out the $\dot{H}^{1/2}(\mathbb{R}^3)$ space is a critical space with respect to the scaling invariance in terms of Navier–Stokes equations, but to system (1) which is not scaling-invariant. System (1) contains the Navier–Stokes equations as a subsystem, so in general, we cannot expect any better results than those for the Navier-Stokes equations. Therefore, a natural problem is whether we can establish the global existence and regularity of (1) under the same assumption. It should be noted that the result for (1) is not completely parallel to that for the

Navier–Stokes equations. The difficulty is coming from without $\nabla \cdot v = 0$. Thus, it seems difficult to obtain the desired results by using the same process. Here, we use the trick from a series of works on the Hall-magnetohydrodynamcis equations by Dongho Chae and his collaborators [12, 13] etc. and by utilizing the structure of the system to overcome this difficulty. The main results of this paper are the following:

Theorem 1. *Let $s > 5/2$, and $(u_0, v_0, \theta_0) \in H^s(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$. Then, there exists a constant $c_0 > 0$, such that*

$$\|u_0\|_{\dot{H}^{1/2}} + \|v_0\|_{\dot{H}^{1/2+\epsilon}} + \|\theta_0\|_{\dot{H}^{1/2}} < c_0, \quad 0 < \epsilon < 1. \tag{3}$$

Then, there exists a unique global classical solution $(u, v, \theta) \in L^\infty([0, +\infty); H^s(\mathbb{R}^3))$.

Remark 1. In the case of $v = 0, \theta = 0$, system (1) can be read as the incompressible Navier–Stokes equations, and what proved in [10] is a straightforward consequence of Theorem 1. An interesting point is that the parameter ϵ goes to zero. The detailed discussion of this case will be given in a separate paper.

2. Proof of the Theorem

The homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^3)$ is defined as follows:

$$\|f\|_{\dot{H}^s} = \|\Lambda^s f\|_{L^2} = \left(\int_{\mathbb{R}^3} |\xi|^{2s} |\hat{f}(\xi)|^2 \mathrm{d}\xi\right)^{1/2}, \tag{4}$$

where $\Lambda = (-\Delta)^{1/2}$.

The BMO is the space of bounded mean oscillation defined by

$$f \in L^1_{loc}(\mathbb{R}^3), \sup_{x,R} \frac{1}{|B_R|} \int_{B_R(x)} \left| f(y) - \overline{f}_{B_R}(x) \right| \mathrm{d}y < +\infty, \tag{5}$$

where $\overline{f}_{B_R}(x)$ is the average of f over $B_R(x)\{y \in \mathbb{R}^3 \| x - y| < R\}$.

The following lemma is essential in the process of proof [11].

Lemma 1. *Let $1 < r < \infty$, then we have the following:*

$$\left\|\partial^\alpha f \cdot \partial^\beta g\right\|_{L^r} \leq C\left(\|f\|_{BMO} \left\|(-\Delta)^{|\alpha|+|\beta|/2} g\right\|_{L^r} + \|g\|_{BMO} \left\|(-\Delta)^{|\alpha|+|\beta|/2} f\right\|_{L^r}\right), \tag{6}$$

for all $f, g \in BMO \cap W^{|\alpha|+|\beta|, r}$, when $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ are multi-indices with $|\alpha|; |\beta| \geq 1$.

We begin with the local existence and uniqueness theorem of strong solutions.

Lemma 2. *Let $(u_0, v_0, \theta_0) \in H^s(\mathbb{R}^3)$; $s > 5/2$, there exists a positive time T^* and a unique strong solution (u, v, θ) on $[0, T^*)$ to system (1) satisfying*

$$(u, v, \theta) \in L^\infty\left([0, T^*); H^s\left(\mathbb{R}^3\right)\right) \cap L^2\left(0, T^*; H^{s+1}\left(\mathbb{R}^3\right)\right). \tag{7}$$

Proof. The main part of the proof consists of a priori estimates. To make the proof rigorous requires some approximation procedure such as that employed in [12], to which we refer for details. Nevertheless, we would like to point out two key observations in the proof for the convenience of the reader. We can establish priori estimates (the process is similar as a part of the proof of Theorem 1).

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s v(t)\|_{L^2}^2 + \|\Lambda^s \theta(t)\|_{L^2}^2\right) \\ &\quad + \|\Lambda^{s+1} u(t)\|_{L^2}^2 + \|\Lambda^{s+1} v(t)\|_{L^2}^2 + \|\Lambda^{s+1} \theta(t)\|_{L^2}^2 \\ &\leq C\left(\|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s v\|_{L^2}^2 + \|\Lambda^s \theta\|_{L^2}^2 + 1\right) \\ &\qquad \left(\|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s v(t)\|_{L^2}^2 + \|\Lambda^s \theta(t)\|_{L^2}^2\right). \end{aligned} \tag{8}$$

We set

$$X(t) = \|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s v(t)\|_{L^2}^2 + \|\Lambda^s \theta(t)\|_{L^2}^2 + 1. \tag{9}$$

Then, from the above inequality, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} X \leq C X^2. \tag{10}$$

Then, by applying the Gronwall inequality, we have

$$X(t) \leq \frac{X(0)}{1 - C_0 X(0) t}, \tag{11}$$

Now, choose $T = 1/2C_0 X(0)$. Then,

$$X(t) \leq 2X(0), \quad \forall t \in [0, T]. \tag{12}$$

By applying the same argument as [12], we can obtain the desired result. □

The proof of Theorem 1 will be divided into two steps.

Proof

Step 1. We will derive a blow-up criterion for the strong solutions to (1).

Applying ∇^3 to system (1), multiplying the resultant by $\nabla^3 u, \nabla^3 v$ and $\nabla^3 \theta$, respectively, and integrating over $\mathbb{R}^3$, we obtain the following:

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla^3 u(t)\|_{L^2}^2 + \|\nabla^3 v(t)\|_{L^2}^2 + \|\nabla^3 \theta(t)\|_{L^2}^2\right) \\ &\quad + \|\nabla^4 u(t)\|_{L^2}^2 + \|\nabla^4 v(t)\|_{L^2}^2 + \|\nabla^4 \theta(t)\|_{L^2}^2 \\ &= -\int_{\mathbb{R}^3} \nabla^3 (u \cdot \nabla u) \nabla^3 u \mathrm{d}x - \int_{\mathbb{R}^3} \nabla^3 \nabla \cdot (v \otimes v) \nabla^3 u \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^3} \nabla^3 (v \cdot \nabla u) \nabla^3 v \mathrm{d}x - \int_{\mathbb{R}^3} \nabla^3 (u \cdot \nabla v) \nabla^3 v \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^3} \nabla^3 \nabla \theta \nabla^3 v \mathrm{d}x - \int_{\mathbb{R}^3} \nabla^3 (u \cdot \nabla \theta) \nabla^3 \theta \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^3} \nabla^3 \nabla \cdot v \nabla^3 \theta \mathrm{d}x = I_1 + I_2 + \cdots + I_7. \end{aligned} \tag{13}$$

□

Remark 2. The operator $\nabla^3 = D^3/\partial x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}, \alpha_1 + \alpha_2 + \alpha_3 = 3$.

By using the cancellation property,

$$\int (u\cdot\nabla)\nabla^3 u\cdot\nabla^3 u \mathrm{d}x = 0. \tag{14}$$

By virtue of integrating by part and H ö lder inequality and Lemma 2, we have the following:

$$\begin{aligned} |I_1| &= \left|-\int_{\mathbb{R}^3}\left[\nabla^3(u\cdot\nabla u)\nabla^3 u - (u\cdot\nabla)\nabla^3 u\cdot\nabla^3 u\right]\mathrm{d}x\right|, \\ &= \left|-\sum_{|\alpha|+|\beta|=3;|\alpha|\geq 1}\int_{\mathbb{R}^3}\nabla^\alpha u\cdot\nabla\nabla^\beta u\nabla^3 u\mathrm{d}x\right| \\ &\leq C\sum_{|\alpha|+|\beta|=3;|\alpha|\geq 1}\left\|\nabla^\alpha u\cdot\nabla\nabla^\beta u\right\|_{L^2}\left\|\nabla^3 u\right\|_{L^2} \\ &\leq C\|u\|_{BMO}\left\|\nabla^3 u\right\|_{L^2}\left\|\nabla^4 u\right\|_{L^2} \\ &\leq C\|u\|_{BMO}^2\left\|\nabla^3 u\right\|_{L^2}^2 + C_1\left\|\nabla^4 u\right\|_{L^2}^2. \end{aligned} \tag{15}$$

We rewrite it as follows:

$$\nabla\cdot(v\otimes v) = v\cdot\nabla v + v\nabla\cdot v. \tag{16}$$

We choose $|\alpha| = 3, \alpha = (\alpha_1, \alpha_2, \alpha_3)$,

$$\begin{aligned} \int_{\mathbb{R}^3}(v\cdot\nabla)\partial^\alpha u\cdot\partial^\alpha v\mathrm{d}x &= \int_{\mathbb{R}^3} v_i\partial_i\partial^\alpha u_j\partial^\alpha v_j\mathrm{d}x \\ &= -\int_{\mathbb{R}^3}\partial_i v_i\partial^\alpha u_j\cdot\partial^\alpha v_j\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^3} v_i\partial^\alpha u_j\cdot\partial_i\partial^\alpha v_j\mathrm{d}x. \end{aligned} \tag{17}$$

Then, we have the following equality:

$$\begin{aligned} &\int_{\mathbb{R}^3}(v\cdot\nabla)\nabla^3 u\cdot\nabla^3 v\mathrm{d}x + \int_{\mathbb{R}^3}(v\cdot\nabla)\nabla^3 v\cdot\nabla^3 u\mathrm{d}x \\ &\quad = -\int_{\mathbb{R}^3}\nabla\cdot v\nabla^3 u\nabla^3 v\mathrm{d}x. \end{aligned} \tag{18}$$

Now, we split I_2 as follows:

$$\begin{aligned} -\int_{\mathbb{R}^3}\nabla^3\nabla\cdot(v\otimes v)\nabla^3 u\mathrm{d}x &= -\int_{\mathbb{R}^3}\nabla^3[v\cdot\nabla v]\nabla^3 u\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^3}\nabla^3[v\nabla\cdot v]\nabla^3 u\mathrm{d}x, \\ &= I_{21} + I_{22}. \end{aligned} \tag{19}$$

Then we can estimate $I_{21} + I_3$ by using Lemma 2,

$$\begin{aligned} |I_{21} + I_3| &\leq \left|\int_{\mathbb{R}^3}\left[\nabla^3(v\cdot\nabla v)\nabla^3 u + \nabla^3(v\cdot\nabla u)\nabla^3 v\mathrm{d}x\right.\right. \\ &\qquad \left.\left. - v\nabla^3\nabla\cdot v\cdot\nabla^3 u - (v\cdot\nabla)\nabla^3 v\cdot\nabla^3 u\right]\mathrm{d}x\right| \\ &\qquad + \left|\int_{\mathbb{R}^3}\nabla\cdot v\nabla^3 v\nabla^3 u\mathrm{d}x\right|, \\ &\leq C\left(\|u\|_{BMO}\left\|\nabla^3 v\right\|_{L^2} + \|v\|_{BMO}\left\|\nabla^3 u\right\|_{L^2}\right) \\ &\qquad \left(\left\|\nabla^4 u\right\|_{L^2} + \left\|\nabla^4 v\right\|_{L^2}\right) \\ &\leq C\left(\|u\|_{BMO}^2 + \|v\|_{BMO}^2\right)\left(\left\|\nabla^3 v\right\|_{L^2}^2 + \left\|\nabla^3 u\right\|_{L^2}^2\right) \\ &\qquad + C_2\left(\left\|\nabla^4 v\right\|_{L^2}^2 + \left\|\nabla^4 u\right\|_{L^2}^2\right). \end{aligned} \tag{20}$$

Next, we consider the term I_{22}

$$\begin{aligned} |I_{22}| &= \left|\int_{\mathbb{R}^3}\nabla^3[v\nabla\cdot v]\nabla^3 u\mathrm{d}x\right| \\ &\leq \left|\sum_{|\alpha|+|\beta|=3;|\alpha|\geq 1}\int_{\mathbb{R}^3}\nabla^\alpha u\cdot\nabla^\beta(\nabla\cdot v)\nabla^3 u\mathrm{d}x\right| \\ &\quad + \left|\int_{\mathbb{R}^3} v\nabla^3(\nabla\cdot v)\nabla^3 u\mathrm{d}x\right| = I_{221} + I_{222}. \end{aligned} \tag{21}$$

We can estimate I_{221} by using Lemma 2

$$\begin{aligned} |I_{221}| &\leq C\left(\|u\|_{BMO}\left\|\nabla^3 v\right\|_{L^2} + \|v\|_{BMO}\left\|\nabla^3 u\right\|_{L^2}\right)\left(\left\|\nabla^3 u\right\|_{L^2} + \left\|\nabla^3 v\right\|_{L^2}\right), \\ &\leq C\left(\|u\|_{BMO} + \|v\|_{BMO}\right)\left(\left\|\nabla^3 u\right\|_{L^2}^2 + \left\|\nabla^3 v\right\|_{L^2}^2\right). \end{aligned} \tag{22}$$

It is necessary to point out we cannot use Lemma 1 to estimate I_{222} due to we have not $\nabla\cdot v = 0$. Thus, we can estimate it by H ö lder inequality only,

$$\begin{aligned} |I_{222}| &\leq C\|v\|_{L^\infty}\left\|\nabla^3 u\right\|_{L^2}\left\|\nabla^4 v\right\|_{L^2}, \\ &\leq C\|v\|_{L^\infty}^2\left\|\nabla^3 u\right\|_{L^2}^2 + C_3\left\|\nabla^4 v\right\|_{L^2}^2, \end{aligned} \tag{23}$$

By using the cancellation property,

$$\begin{aligned} &\int (u\cdot\nabla)\nabla^3 v\cdot\nabla^3 v\mathrm{d}x = 0; \\ &\int (u\cdot\nabla)\nabla^3\theta\cdot\nabla^3\theta\mathrm{d}x = 0. \end{aligned} \tag{24}$$

We can estimate I_4 and I_6 by using Lemma 2.

$$\begin{aligned}|I_4| &= \left|\int_{\mathbb{R}^3}\left[\nabla^3(u\cdot\nabla v)-(u\cdot\nabla)\nabla^3 v\cdot\nabla^3 v\right]\nabla^3 v\mathrm{d}x\right|,\\ &\le C\left(\|u\|_{\mathrm{BMO}}\left\|\nabla^3 v\right\|_{L^2}+\|v\|_{\mathrm{BMO}}\left\|\nabla^3 u\right\|_{L^2}\right)\\ &\quad\cdot\left(\left\|\nabla^4 u\right\|_{L^2}+\left\|\nabla^4 v\right\|_{L^2}\right),\\ &\le C\left(\|u\|_{\mathrm{BMO}}^2+\|v\|_{\mathrm{BMO}}^2\right)\left(\left\|\nabla^3 v\right\|_{L^2}^2+\left\|\nabla^3 u\right\|_{L^2}^2\right)\\ &\quad+C_4\left(\left\|\nabla^4 v\right\|_{L^2}^2+\left\|\nabla^4 u\right\|_{L^2}^2\right).\end{aligned} \tag{25}$$

$$\begin{aligned}|I_6| &= \left|\int_{\mathbb{R}^3}\left[\nabla^3(u\cdot\nabla\theta)-(u\cdot\nabla)\nabla^3\theta\cdot\nabla^3\theta\right]\nabla^3 v\mathrm{d}x\right|,\\ &\le C\left(\|u\|_{\mathrm{BMO}}\left\|\nabla^3\theta\right\|_{L^2}+\|\theta\|_{\mathrm{BMO}}\left\|\nabla^3 u\right\|_{L^2}\right)\\ &\quad\cdot\left(\left\|\nabla^4 u\right\|_{L^2}+\left\|\nabla^4\theta\right\|_{L^2}\right)\\ &\le C\left(\|u\|_{\mathrm{BMO}}^2+\|\theta\|_{\mathrm{BMO}}^2\right)\left(\left\|\nabla^3\theta\right\|_{L^2}^2+\left\|\nabla^3 u\right\|_{L^2}^2\right)\\ &\quad+C_5\left(\left\|\nabla^4\theta\right\|_{L^2}^2+\left\|\nabla^4 u\right\|_{L^2}^2\right).\end{aligned} \tag{26}$$

By using the H ö lder inequality, we have the following:

$$\begin{aligned}|I_5+I_7| &= \left|\int_{\mathbb{R}^3}\nabla^3\nabla\theta\nabla^3 v\mathrm{d}x+\int_{\mathbb{R}^3}\nabla^3\nabla\cdot v\nabla^3\theta\mathrm{d}x\right|,\\ &\le C\left(\left\|\nabla^3\theta\right\|_{L^2}+\left\|\nabla^3 v\right\|_{L^2}\right)\left(\left\|\nabla^4 v\right\|_{L^2}+\left\|\nabla^4\theta\right\|_{L^2}\right)\\ &\le C\left(\left\|\nabla^3\theta\right\|_{L^2}^2+\left\|\nabla^3 v\right\|_{L^2}^2\right)+C_6\left(\left\|\nabla^4\theta\right\|_{L^2}^2+\left\|\nabla^4 u\right\|_{L^2}^2\right).\end{aligned} \tag{27}$$

Thus, combining (15) to (27), we obtain the following:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\nabla^3 u(t)\right\|_{L^2}^2+\left\|\nabla^3 v(t)\right\|_{L^2}^2+\left\|\nabla^3\theta(t)\right\|_{L^2}^2\right)\\ &\quad+\left\|\nabla^4 u(t)\right\|_{L^2}^2+\left\|\nabla^4 v(t)\right\|_{L^2}^2+\left\|\nabla^4\theta(t)\right\|_{L^2}^2,\\ &\le C\left(\|u\|_{\mathrm{BMO}}^2+\|v\|_{L^\infty}^2+\|\theta\|_{\mathrm{BMO}}^2+1\right)\\ &\quad\cdot\left(\left\|\nabla^3 u(t)\right\|_{L^2}^2+\left\|\nabla^3 v(t)\right\|_{L^2}^2+\left\|\nabla^3\theta(t)\right\|_{L^2}^2\right).\end{aligned} \tag{28}$$

Here, we used the inclusion relation: $L^\infty(\mathbb{R}^3)\subsetneq \mathrm{BMO}(\mathbb{R}^3)$.

Now, we set

$$H(t)=\left\|\nabla^3 u(t)\right\|_{L^2}^2+\left\|\nabla^3 v(t)\right\|_{L^2}^2+\left\|\nabla^3\theta(t)\right\|_{L^2}^2. \tag{29}$$

We can obtain the following inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}H(t)\le C\left(\|u\|_{\mathrm{BMO}}^2+\|v\|_{L^\infty}^2+\|\theta\|_{\mathrm{BMO}}^2+1\right)H(t), \tag{30}$$

By using Gronwall type inequality, we have the following:

$$\sup_{0\le t\le T}H(t)\le CH(0)\exp\left(\int_0^T\left(\|u\|_{\mathrm{BMO}}^2+\|v\|_{L^\infty}^2+\|\theta\|_{\mathrm{BMO}}^2\right)\mathrm{d}t.\right. \tag{31}$$

From the above inequalities, we obtain that if

$$\int_0^T\left(\|u\|_{\mathrm{BMO}}^2+\|v\|_{L^\infty}^2+\|\theta\|_{\mathrm{BMO}}^2\right)\mathrm{d}t<+\infty, \tag{32}$$

then

$$\begin{aligned}&\sup_{0\le t\le T}\left(\left\|\nabla^3 u(t)\right\|_{L^2}^2+\left\|\nabla^3 v(t)\right\|_{L^2}^2+\left\|\nabla^3\theta(t)\right\|_{L^2}^2\right),\\ &\quad+\int_0^T\left\|\nabla^4 u(\tau)\right\|_{L^2}^2+\left\|\nabla^4 v(\tau)\right\|_{L^2}^2+\left\|\nabla^4\theta(\tau)\right\|_{L^2}^2\mathrm{d}\tau<+\infty.\end{aligned} \tag{33}$$

Let (u,v,θ) be a unique local strong solutions and T^* be the first blow-up time, then

$$T^*<+\infty\Leftrightarrow\int_0^{T^*}\left(\|u\|_{\mathrm{BMO}}^2+\|v\|_{L^\infty}^2+\|\theta\|_{\mathrm{BMO}}^2\right)\mathrm{d}t=+\infty. \tag{34}$$

Step 2. We take the operator $\Lambda^{1/2}$ on both sides of u-equation and θ-equation of (1); taking the scalar product with $\Lambda^{1/2}u,\Lambda^{1/2}\theta$, then

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\Lambda^{1/2}u(t)\right\|_{L^2}^2+\left\|\Lambda^{1/2}\theta(t)\right\|_{L^2}^2\right)+\left\|\Lambda^{3/2}u(t)\right\|_{L^2}^2\\ &\quad+\left\|\Lambda^{3/2}\theta(t)\right\|_{L^2}^2=-\int_{\mathbb{R}^3}\Lambda^{1/2}(u\cdot\nabla u)\Lambda^{1/2}u\mathrm{d}x,\\ &\quad-\int_{\mathbb{R}^3}\Lambda^{1/2}\nabla\cdot(v\otimes v)\Lambda^{1/2}u\mathrm{d}x\\ &\quad-\int_{\mathbb{R}^3}\Lambda^{1/2}(u\cdot\nabla\theta)\Lambda^{1/2}\theta\mathrm{d}x-\int_{\mathbb{R}^3}\Lambda^{1/2}\nabla\cdot v\Lambda^{1/2}\theta\mathrm{d}x.\end{aligned} \tag{35}$$

We take operator $\Lambda^{1/2+\epsilon}$ on both sides of v-equation of (1); taking a scalar product with $\Lambda^{1/2+\epsilon}v$, then

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Lambda^{1/2+\epsilon}v(t)\right\|_{L^2}^2+\left\|\Lambda^{3/2+\epsilon}v(t)\right\|_{L^2}^2&=-\int_{\mathbb{R}^3}\Lambda^{1/2+\epsilon}(v\cdot\nabla u)\Lambda^{1/2+\epsilon}v\mathrm{d}x\\ &\quad-\int_{\mathbb{R}^3}\Lambda^{1/2+\epsilon}(u\cdot\nabla v)\Lambda^{1/2+\epsilon}v\mathrm{d}x\\ &\quad-\int_{\mathbb{R}^3}\Lambda^{1/2+\epsilon}\nabla\theta\Lambda^{1/2+\epsilon}v\mathrm{d}x,\end{aligned} \tag{36}$$

By using H ö lder inequality and Gagliardo–Nirenberg type interpolation inequality, we can obtain the following estimates:

$$\begin{aligned}\left|\int_{\mathbb{R}^3}\Lambda^{1/2}(u\cdot\nabla u)\Lambda^{1/2}u\mathrm{d}x\right|&\le C\|u\cdot\nabla u\|_{L^{3/2}}\|\Lambda u\|_{L^3}\\ &\le C\left\|\Lambda^{1/2}u\right\|_{L^2}\left\|\Lambda^{3/2}u\right\|_{L^2}^2.\end{aligned} \tag{37}$$

$$\left|\int_{\mathbb{R}^3}\Lambda^{1/2}(u\cdot\nabla\theta)\Lambda^{1/2}\theta\mathrm{d}x\right|\le C\left\|\Lambda^{1/2}u\right\|_{L^2}\left\|\Lambda^{3/2}\theta\right\|_{L^2}^2, \tag{38}$$

$$\begin{aligned}&\left|\int_{\mathbb{R}^3}\Lambda^{1/2}\nabla\cdot(v\otimes v)\Lambda^{1/2}u\mathrm{d}x\right|\\&\quad\le\|\nabla\cdot(v\otimes v)\|_{L^{3/2}}\|\Lambda u\|_{L^3},\\&\quad\le C\left\|\Lambda^{1/2}v\right\|_{L^2}\left\|\Lambda^{3/2}v\right\|_{L^2}\left\|\Lambda^{3/2}u\right\|_{L^2},\\&\quad\le C\left\|\Lambda^{1/2+\epsilon}v\right\|_{L^2}\left(\left\|\Lambda^{3/2}u\right\|_{L^2}^2+\left\|\Lambda^{3/2}v\right\|_{L^2}^2\right),\end{aligned}\tag{39}$$

$$\begin{aligned}&\left|\int_{\mathbb{R}^3}\Lambda^{1/2}\nabla\cdot v\Lambda^{v}\theta\mathrm{d}x\right|\\&\quad\le C\left\|\Lambda^{1/2}v\right\|_{L^2}\left\|\Lambda^{3/2}\theta\right\|_{L^2},\end{aligned}\tag{40}$$

$$\begin{aligned}&\left|\int_{\mathbb{R}^3}\Lambda^{1/2+\epsilon}(v\cdot\nabla u)\Lambda^{1/2+\epsilon}v\mathrm{d}x\right|,\\&\quad\le C\left\|\Lambda^{1/2}v\right\|_{L^6}\left\|\Lambda^{3/2}u\right\|_{L^2}\left\|\Lambda^{1/2+2\epsilon}v\right\|_{L^3}\\&\quad\le C\left\|\Lambda^{3/2}v\right\|_{L^2}\left\|\Lambda^{3/2}u\right\|_{L^2}\left\|\Lambda^{1/2+2\epsilon}v\right\|_{L^3}\\&\quad\le C\left\|\Lambda^{1/2+\epsilon}v\right\|_{L^2}\left\|\Lambda^{3/2}u\right\|_{L^2}\left\|\Lambda^{3/2+\epsilon}v\right\|_{L^3}.\end{aligned}\tag{41}$$

Here, we used the condition $\epsilon<1$.

$$\begin{aligned}&\left|-\int_{\mathbb{R}^3}\Lambda^{1/2+\epsilon}(u\cdot\nabla v)\Lambda^{1/2+\epsilon}v\mathrm{d}x\right|,\\&\quad\le C\left\|\Lambda^{1/2}u\right\|_{L^6}\left\|\Lambda^{3/2}v\right\|_{L^2}\left\|\Lambda^{1/2+2\epsilon}v\right\|_{L^3},\\&\quad\le C\left\|\Lambda^{1/2+\epsilon}v\right\|_{L^2}\left\|\Lambda^{3/2}u\right\|_{L^2}\left\|\Lambda^{3/2+\epsilon}v\right\|_{L^3}.\end{aligned}\tag{42}$$

From (35) to (42), we have the following:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\Lambda^{1/2}u(t)\right\|_{L^2}^2+\left\|\Lambda^{1/2+\epsilon}v(t)\right\|_{L^2}^2+\left\|\Lambda^{1/2}\theta(t)\right\|_{L^2}^2\right)\\&\quad+\left\|\Lambda^{3/2}u(t)\right\|_{L^2}^2+\left\|\Lambda^{3/2+\epsilon}v(t)\right\|_{L^2}^2+\left\|\Lambda^{3/2}\theta(t)\right\|_{L^2}^2,\\&\le C\left(\left\|\Lambda^{1/2}u(t)\right\|_{L^2}+\left\|\Lambda^{1/2+\epsilon}v(t)\right\|_{L^2}+\left\|\Lambda^{1/2}\theta(t)\right\|_{L^2}\right)\\&\quad\cdot\left(\left\|\Lambda^{3/2}u(t)\right\|_{L^2}^2+\left\|\Lambda^{3/2+\epsilon}v(t)\right\|_{L^2}^2+\left\|\Lambda^{3/2}\theta(t)\right\|_{L^2}^2\right).\end{aligned}\tag{43}$$

Choosing c_0 so small that

$$C\left(\left\|\Lambda^{1/2}u_0\right\|_{L^2}+\left\|\Lambda^{1/2+\epsilon}v_0\right\|_{L^2}+\left\|\Lambda^{1/2}\theta_0\right\|_{L^2}\right)\le\frac{1}{2},\tag{44}$$

Now, we have the following:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\Lambda^{1/2}u(t)\right\|_{L^2}^2+\left\|\Lambda^{1/2}v(t)\right\|_{L^2}^2+\left\|\Lambda^{1/2}\theta(t)\right\|_{L^2}^2\right)\le 0,\tag{45}$$

Integrating in time from 0 to t,

$$\begin{aligned}&\left\|\Lambda^{1/2}u(t)\right\|_{L^2}^2+\left\|\Lambda^{1/2+\epsilon}v(t)\right\|_{L^2}^2+\left\|\Lambda^{1/2}\theta(t)\right\|_{L^2}^2\\&\quad\le\left\|\Lambda^{1/2}u_0\right\|_{L^2}^2+\left\|\Lambda^{1/2+\epsilon}v_0\right\|_{L^2}^2+\left\|\Lambda^{1/2}\theta_0\right\|_{L^2}^2,\end{aligned}\tag{46}$$

Applying the following inequalities

$$\begin{aligned}&\left\|\Lambda^{1/2}u(t)\right\|_{L^2}^2+\left\|\Lambda^{1/2+\epsilon}v(t)\right\|_{L^2}^2+\left\|\Lambda^{1/2}\theta(t)\right\|_{L^2}^2\\&\quad\ge\frac{1}{3}\left(\left\|\Lambda^{1/2}u(t)\right\|_{L^2}+\left\|\Lambda^{1/2+\epsilon}v(t)\right\|_{L^2}+\left\|\Lambda^{1/2}\theta(t)\right\|_{L^2}\right)^2,\\&\left\|\Lambda^{1/2}u_0\right\|_{L^2}^2+\left\|\Lambda^{1/2+\epsilon}v_0\right\|_{L^2}^2+\left\|\Lambda^{1/2}\theta_0\right\|_{L^2}^2\\&\quad\le\left(\left\|\Lambda^{1/2}u_0\right\|_{L^2}+\left\|\Lambda^{1/2+\epsilon}v_0\right\|_{L^2}+\left\|\Lambda^{1/2}\theta_0\right\|_{L^2}\right)^2,\end{aligned}\tag{47}$$

then we yield

$$\sup_{0\le t\le T}\left(\left\|\Lambda^{1/2}u(t)\right\|_{L^2}^2+\left\|\Lambda^{1/2+\epsilon}v(t)\right\|_{L^2}^2+\left\|\Lambda^{1/2}\theta(t)\right\|_{L^2}^2\right)<\frac{\sqrt{3}}{2C}.\tag{48}$$

From (48), we further know that

$$\begin{aligned}&(u,\theta)\in L^{\infty}\left(0,T;\dot{H}^{1/2}\left(\mathbb{R}^3\right)\right)\cap L^2\left(0,T;\dot{H}^{3/2}\left(\mathbb{R}^3\right)\right),\\&\quad v\in L^{\infty}\left(0,T;\dot{H}^{1/2+\epsilon}\left(\mathbb{R}^3\right)\right)\cap L^2\left(0,T;\dot{H}^{3/2+\epsilon}\left(\mathbb{R}^3\right)\right).\end{aligned}\tag{49}$$

Applying the fact

$$\begin{aligned}\dot{H}^{3/2}\left(\mathbb{R}^3\right)&\longrightarrow\mathrm{BMO}\left(\mathbb{R}^3\right),\\\dot{H}^{3/2+\epsilon}\left(\mathbb{R}^3\right)&\longrightarrow L^{\infty}\left(\mathbb{R}^3\right),\end{aligned}\tag{50}$$

then we imply that

$$u,\theta\in L^2(0,T;\mathrm{BMO}),v\in L^2\left(0,T;L^{\infty}\left(\mathbb{R}^3\right)\right),\quad\text{for all }T\in\left(0,T^*\right).\tag{51}$$

Now, we suppose $T<T^*$ is the maximal existence time for solutions, from the blow-up criterion, the existence time can be extended after $t=T$ which contradicts the maximality of $t=T$. This completes the proof of Theorem 1.

3. Conclusion

It is deserving to point out that our method is not adapt to $\epsilon=0$. The reason is that we have not $\nabla\cdot v=0$. We need some new ideas and methods to deal with the parameter ϵ goes to zero. The detailed discussion of this case will be given in a separate paper.

Data Availability

The data used to support the working are cited within the article as references.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors contributed equally and significantly in writing this article. All the authors have read and approved the final manuscript.

Acknowledgments

Xu was supported by the Anhui Education Bureau (No. KJ2020A0505). Zhang was partially supported by the NSFC (No. 31971185).

References

[1] D. M. Frierson, A. J. Majda, and O. M. Pauluis, "Large scale dynamics of precipitation fronts in the tropical atmosphere: a novel relaxation limit," *Communications in Mathematical Sciences*, vol. 2, no. 4, pp. 591–626, 2004.
[2] A. E. Gill, *Atmosphere-Ocean Dynamics*, Academic Press, 1982.
[3] T. Matsuno, "Quasi-geostrophic motions in the equatorial area," *Journal of the Meteorological Society of Japan. Ser. II*, vol. 44, no. 1, pp. 25–43, 1966.
[4] A. J. Majda and J. A. Biello, "The nonlinear interaction of barotropic and equatorial baroclinic Rossby waves," *Journal of the Atmospheric Sciences*, vol. 60, no. 15, pp. 1809–1821, 2003.
[5] E. Titi and E. S. Titi, "Global well-posedness of strong solutions to a tropical climate model," *Discrete and Continuous Dynamical Systems*, vol. 36, no. 8, pp. 4495–4516, 2016.
[6] Z. Ye, "Global regularity for a class of 2D tropical climate model," *Journal of Mathematical Analysis and Applications*, vol. 446, no. 1, pp. 307–321, 2017.
[7] B. Dong, W. Wang, J. Wu, and H. Zhang, "Global regularity results for the climate model with fractional dissipation," *Discrete & Continuous Dynamical Systems - B*, vol. 24, no. 1, pp. 211–229, 2019.
[8] M. Zhu, "Global regularity for the tropical climate model with fractional diffusion on barotropic mode," *Applied Mathematics Letters*, vol. 81, pp. 99–104, 2018.
[9] B. Yuan and X. Chen, "Global regularity for the 3D tropical climate model with damping," *Applied Mathematics Letters*, vol. 121, Article ID 107439, 2021.
[10] H. Fujita and T. Kato, "On the Navier-Stokes initial value problem. I," *Archive for Rational Mechanics and Analysis*, vol. 16, no. 4, pp. 269–315, 1964.
[11] H. Kozono and Y. Taniuchi, "Bilinear estimates in BMO and the Navier-Stokes equations," *Mathematische Zeitschrift*, vol. 235, no. 1, pp. 173–194, 2000.
[12] D. Chae, P. Degond, and J.-G. Liu, "Well-posedness for Hall-magnetohyd-rodynamics," *Inst. H. Poincare. Anal. Non Lineaire*, vol. 31, pp. 555–565, 2014.
[13] D. Chae and J. Lee, "On the blow-up criterion and small data global existence for the Hall-magnetohydrodynamics," *Journal of Differential Equations*, vol. 256, no. 11, pp. 3835–3858, 2014.