Research Article

Curved Nanotube Structures under Mechanical Loading

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Configuration of carbon nanotube (CNT) has been the subject of research to perform theoretical development for analyzing nanocomposites. A new theoretical solution is developed to study curved nanotube structures subjected to mechanical loadings. A curved nanotube structure is considered. A nonlocal displacement-based solution is proposed by using a displacement approach of Toroidal Elasticity based on Eringen's theory of nonlocal continuum mechanics. The governing equations of curved nanotube structures are developed in toroidal coordinate system. The method of successive approximation is used to discretize the displacement-based governing equations and find the general solution subjected to bending moment. The numerical results show that all displacement components increase with increasing the nonlocal parameter. The present theoretical study highlights the significance of the geometry and nonlocal parameter effects on mechanical behavior of nanotube structures.

1. Introduction

Due to superior properties of CNT which offer unprecedented potential to produce multifunctional nanocomposites, many investigations have been performed on different applications of these nanostructures such as nanomechanical [1, 2] and nanobiological [3, 4] applications and optic [5] and molecular engineering [6–8]. CNTs appear in different shapes. Each shape has its own special properties and potential applications. CNT with curved configuration is one of their shapes. Actually, a single nanotube naturally curves during growth if no external forces exist [9]. Continuum mechanics approach has been employed by many researchers to study mechanical behavior of curved nanostructures.

The continuum mechanics approach has been used to investigate several aspects of mechanical analysis of CNT such as the static [10–13], the buckling [14–16], free vibration [17, 18], wave propagation [19–23], and thermomechanical analysis of CNT [24].

The effect of initial stress on the vibration of an individual multiwall carbon nanotube (MWCNT) with simply supported ends was investigated based on a laminated elastic beam model [12]. Wang et al. [14] proposed nonlocal elastic beam and shell models. They applied the proposed method

to study the small-scale effect on buckling analysis of CNT under compression. Based on the continuum mechanics and the molecular dynamics simulation, the investigation on the wave propagation in a single-walled carbon nanotube (SWCNT) was performed [18]. Hu et al. [19] used nonlocal elastic beam theories to study the scale effect of the CNT on the wave dispersion. They compared the results obtained by nonlocal elastic Timoshenko beam theory with the results obtained by molecular dynamics simulations. Based on thermal elasticity mechanics and nonlocal elasticity theory, a single-elastic beam model was presented to investigate the thermal vibration of SWCNT [20]. Murmu and Pradhan [20] studied the buckling of SWCNT. They considered the effect of temperature change and surrounding elastic medium.

Compared with the classical continuum theory, the nonlocal elasticity theory uses the constitutive assumptions that the stress at a point is a function of strains at all points in the continuum [25–27]. Nonlocal elasticity theory was used by many researchers to analyze nanostructures advanced by Eringen [25, 26]. Peddieson et al. [27] used a version of nonlocal elasticity theory to propose a nonlocal Bernoulli-Euler beam model. Zhang et al. [28] presented a nonlocal double-elastic beam model for the free vibrations

of double-walled carbon nanotubes (DWCNTs) based on theory of nonlocal elasticity and by considering the effect of small length scale. The elastic buckling analysis of nanotubes was analyzed based on Eringen's nonlocal elasticity Timoshenko beam theories [29, 30]. The small-scale effect was considered through Eringen's nonlocal elasticity theory while the effect of transverse shear deformation was considered through Timoshenko beam theory. Frequency equations and modal shape functions of beam structures were derived based on a nonlocal Bernoulli-Euler beam [31].

The above review shows that little work has been done to study behavior of curved nanotube structures under mechanical loadings. A new displacement-based solution is proposed to study theoretically mechanical behavior of curved nanotube structures based on Toroidal Elasticity that includes the full three-dimensional constitutive relations. Governing equations of curved nanotube are developed by using the constitutive relations of Eringen. Then, the method of successive approximation is employed to find the general solution for mechanical behavior of curved nanotube structures subjected to pure bending moment. The effects of nonlocal parameter, nanotube length, and thickness on the mechanical behavior of nanotubes are studied.

2. Nonlocal Theory (Constitutive Relations)

According to Eringen [25, 26], the stress field at a point x in an elastic continuum depends not only on the strain field at the point but also on strains at all other points of the body. Eringen attributed this fact to the atomic theory of lattice dynamics and experimental observations on phonon dispersion. Therefore, the following equivalent differential form is represented based on Eringen [25, 26]:

$$(1 - \mu \nabla^2) \sigma_{ij} = C_{ijmn} \epsilon_{mn}, \quad (1)$$

where ∇^2 and $\mu = (e_0 a)^2$ are the Laplacian operator and nonlocal parameter, respectively. In addition, e_0 and a are a material constant and internal characteristic length, respectively.

3. Governing Equations for Curved Nanotube Structures

The nonclassical toroidal coordinate system is shown in Figure 1. The curved nanotube has a bend radius R and an annular cross section bounded by radii a and b (see Figure 1). A general point P in a constant thickness curved nanotube is represented easily by the nonclassical toroidal coordinate system r , ϕ , and θ , where r and ϕ are polar coordinates in the plane of the nanotube cross section and θ defines the position of the nanotube cross section. $\zeta = r/c$ is a nondimensional radial coordinate in the toroidal coordinate, and c is a reference length to be specified later.

FIGURE 1: Geometry of curved nanotube and coordinate system.

For convenience, nondimensional displacements u , v , and w are defined as

$$\begin{aligned} u &= \frac{u_\zeta}{c}, \\ v &= \frac{u_\phi}{c}, \\ w &= \frac{u_\theta}{c}. \end{aligned} \quad (2)$$

Nondimensional stress components are defined as

$$\begin{aligned} \tau_{\zeta\zeta} &= \frac{1 + \nu}{E} \sigma_{\zeta\zeta}, \\ \tau_{\phi\phi} &= \frac{1 + \nu}{E} \sigma_{\phi\phi}, \\ \tau_{\theta\theta} &= \frac{1 + \nu}{E} \sigma_{\theta\theta}, \\ \tau_{\zeta\phi} &= \frac{1 + \nu}{E} \sigma_{\zeta\phi}, \\ \tau_{\zeta\theta} &= \frac{1 + \nu}{E} \sigma_{\zeta\theta}, \\ \tau_{\phi\theta} &= \frac{1 + \nu}{E} \sigma_{\phi\theta}, \end{aligned} \quad (3)$$

where E and ν are modulus of elasticity and Poisson ratio for nanotubes. The kinematics relations for the case of small displacements are [34]

$$\begin{aligned} \epsilon_r &= \frac{\partial u}{\partial r}, \\ \epsilon_\phi &= \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \phi}, \\ \epsilon_\theta &= \frac{1}{\rho} \left(u \cos \phi - v \sin \phi + \frac{\partial w}{\partial \theta} \right), \end{aligned}$$

$$\begin{aligned}
\varepsilon_{r\phi} &= \frac{1}{2} \left(\frac{1}{r} \frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial r} - \frac{v}{r} \right), \\
\varepsilon_{r\theta} &= \frac{1}{2} \left(\frac{\partial w}{\partial r} + \frac{1}{\rho} \frac{\partial u}{\partial \theta} - \frac{w}{\rho} \cos \phi \right), \\
\varepsilon_{\phi\theta} &= \frac{1}{2} \left(\frac{1}{r} \frac{\partial w}{\partial \phi} + \frac{1}{\rho} \frac{\partial v}{\partial \theta} + \frac{w}{\rho} \sin \phi \right),
\end{aligned} \quad (4)$$

where u , v , and w represent the displacement components in the r , ϕ , and θ directions, respectively. From both considerations of (1) and Hooke's law, it is written that

$$\begin{pmatrix} \sigma_{\zeta\zeta} \\ \sigma_{\phi\phi} \\ \sigma_{\theta\theta} \\ \sigma_{\zeta\phi} \\ \sigma_{\zeta\theta} \\ \sigma_{\phi\theta} \end{pmatrix} - \mu \nabla^2 \begin{pmatrix} \sigma_{\zeta\zeta} \\ \sigma_{\phi\phi} \\ \sigma_{\theta\theta} \\ \sigma_{\zeta\phi} \\ \sigma_{\zeta\theta} \\ \sigma_{\phi\theta} \end{pmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \begin{pmatrix} \varepsilon_{\zeta\zeta} \\ \varepsilon_{\phi\phi} \\ \varepsilon_{\theta\theta} \\ \gamma_{\zeta\phi} \\ \gamma_{\zeta\theta} \\ \gamma_{\phi\theta} \end{pmatrix}. \quad (5)$$

By considering (5), (3), and strain displacement relations (4), the nondimensional stress components may be expressed in terms of the three nondimensional displacement components and nonlocal parameter μ as

$$\begin{aligned}
\tau_{\zeta\zeta} &= \frac{1}{\hat{\mu}} \frac{\partial u}{\partial \zeta} + \frac{1}{\hat{\mu}} \frac{\nu}{1-2\nu} \vartheta, \\
\tau_{\phi\phi} &= \frac{1}{\hat{\mu}} \left(\frac{u}{\zeta} + \frac{1}{\zeta} \frac{\partial v}{\partial \phi} \right) + \frac{1}{\hat{\mu}} \frac{\nu}{1-2\nu} \vartheta, \\
\tau_{\theta\theta} &= \frac{1}{\hat{\mu}} \frac{c}{\rho} \chi + \frac{1}{\hat{\mu}} \frac{\nu}{1-2\nu} \vartheta, \\
\tau_{\zeta\phi} &= \frac{1}{2\hat{\mu}} \left(\frac{1}{\zeta} \frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial \zeta} - \frac{v}{\zeta} \right), \\
\tau_{\zeta\theta} &= \frac{1}{2\hat{\mu}} \left(\frac{\partial w}{\partial \zeta} + \frac{c}{\rho} \frac{\partial u}{\partial \theta} - \frac{c}{\rho} w \cos \phi \right), \\
\tau_{\phi\theta} &= \frac{1}{2\hat{\mu}} \left(\frac{1}{\zeta} \frac{\partial w}{\partial \phi} + \frac{c}{\rho} \frac{\partial v}{\partial \theta} + \frac{c}{\rho} w \sin \phi \right),
\end{aligned} \quad (6)$$

where

$$\begin{aligned}
\vartheta &= \psi + \frac{c}{\rho} \chi, \\
\psi &= \frac{\partial u}{\partial \zeta} + \frac{u}{\zeta} + \frac{1}{\zeta} \frac{\partial v}{\partial \phi}, \\
\chi &= u \cos \phi - v \sin \phi + \frac{\partial w}{\partial \theta}, \\
\hat{\mu} &= 1 + \frac{n^2}{\zeta^2} \mu.
\end{aligned} \quad (7)$$

Three equilibrium equations in stress form in the toroidal coordinate are represented as [34]

$$\begin{aligned}
&\frac{\partial \tau_{\zeta\zeta}}{\partial \zeta} + \frac{1}{\zeta} \frac{\partial \tau_{\zeta\phi}}{\partial \phi} + \frac{1}{\zeta} (\tau_{\zeta\zeta} - \tau_{\phi\phi}) \\
&\quad + \frac{c}{\rho} \left(\frac{\partial \tau_{\zeta\theta}}{\partial \theta} + (\tau_{\zeta\zeta} - \tau_{\phi\phi}) \cos \phi - \tau_{\zeta\phi} \sin \phi \right) = 0, \\
&\frac{\partial \tau_{\zeta\phi}}{\partial \zeta} + \frac{1}{\zeta} \frac{\partial \tau_{\phi\phi}}{\partial \phi} + \frac{2}{\zeta} \tau_{\zeta\phi} \\
&\quad + \frac{c}{\rho} \left(\frac{\partial \tau_{\phi\theta}}{\partial \theta} + \tau_{\zeta\phi} \cos \phi - (\tau_{\phi\phi} - \tau_{\theta\theta}) \sin \phi \right) = 0, \\
&\frac{\partial \tau_{\zeta\theta}}{\partial \zeta} + \frac{1}{\zeta} \frac{\partial \tau_{\phi\theta}}{\partial \phi} + \frac{1}{\zeta} \tau_{\zeta\theta} \\
&\quad + \frac{c}{\rho} \left(\frac{\partial \tau_{\theta\theta}}{\partial \theta} + 2\tau_{\zeta\theta} \cos \phi - 2\tau_{\phi\theta} \sin \phi \right) = 0,
\end{aligned} \quad (8)$$

where $\rho = R + r \cos \phi$ is the radial cylindrical coordinate of the general point. Substituting the stress-displacement relations (6) into the equilibrium equations (8), the government Navier equations in toroidal coordinate for the curved nanotube are obtained as

$$\begin{aligned}
U &= -\left(\frac{c}{\rho}\right)^1 \bar{U} - \left(\frac{c}{\rho}\right)^2 \hat{U}, \\
V &= -\left(\frac{c}{\rho}\right)^1 \bar{V} - \left(\frac{c}{\rho}\right)^2 \hat{V}, \\
W &= -\left(\frac{c}{\rho}\right)^1 \bar{W} - \left(\frac{c}{\rho}\right)^2 \hat{W},
\end{aligned} \quad (9)$$

where

$$\begin{aligned}
U &= \frac{1}{\hat{\mu}} \nabla^2 u - \frac{1}{\hat{\mu}} \frac{u}{\zeta^2} - \frac{2}{\zeta^2} \frac{1}{\hat{\mu}} \frac{\partial v}{\partial \phi} + \frac{1}{1-2\nu} \frac{1}{\hat{\mu}} \frac{\partial \psi}{\partial \zeta}, \\
\bar{U} &= \cos \phi \frac{1}{\hat{\mu}} \frac{\partial u}{\partial \zeta} - \sin \phi \frac{1}{\hat{\mu}} \left(\frac{1}{\zeta} \frac{\partial u}{\partial \phi} - \frac{v}{\zeta} \right) \\
&\quad + \frac{1}{1-2\nu} \frac{1}{\hat{\mu}} \frac{\partial \chi}{\partial \zeta},
\end{aligned}$$

$$\begin{aligned}
\widehat{U} &= \frac{1}{\widehat{\mu}} \frac{\partial^2 u}{\partial \theta^2} - \cos \phi \frac{1}{\widehat{\mu}} \frac{\partial w}{\partial \theta} - \frac{2(1-\nu)}{1-2\nu} \cos \phi \frac{1}{\widehat{\mu}} \chi, \\
V &= \frac{1}{\widehat{\mu}} \nabla^2 v - \frac{1}{\widehat{\mu}} \frac{\nu}{\varsigma^2} + \frac{2}{\varsigma^2} \frac{1}{\widehat{\mu}} \frac{\partial u}{\partial \phi} + \frac{1}{1-2\nu} \frac{1}{\varsigma} \frac{1}{\widehat{\mu}} \frac{\partial \psi}{\partial \phi}, \\
\overline{V} &= \cos \phi \frac{1}{\widehat{\mu}} \frac{\partial v}{\partial \varsigma} - \sin \phi \frac{1}{\widehat{\mu}} \left(\frac{1}{\varsigma} \frac{\partial v}{\partial \phi} + \frac{u}{\varsigma} \right) \\
&\quad + \frac{1}{1-2\nu} \frac{1}{\varsigma} \frac{1}{\widehat{\mu}} \frac{\partial \chi}{\partial \phi}, \\
\widehat{V} &= \frac{1}{\widehat{\mu}} \frac{\partial^2 v}{\partial \theta^2} + \sin \phi \frac{1}{\widehat{\mu}} \frac{\partial w}{\partial \theta} + \frac{2(1-\nu)}{1-2\nu} \sin \phi \frac{1}{\widehat{\mu}} \chi, \\
W &= \frac{1}{\widehat{\mu}} \nabla^2 w, \\
\overline{W} &= \cos \phi \frac{1}{\widehat{\mu}} \frac{\partial w}{\partial \varsigma} - \sin \phi \frac{1}{\varsigma} \frac{1}{\widehat{\mu}} \frac{\partial w}{\partial \phi} + \frac{1}{1-2\nu} \frac{1}{\widehat{\mu}} \frac{\partial \psi}{\partial \theta}, \\
\widehat{W} &= -\frac{1}{\widehat{\mu}} \frac{\partial^2 w}{\partial \theta^2} - \frac{1}{\widehat{\mu}} w + \frac{3-4\nu}{1-2\nu} \frac{1}{\widehat{\mu}} \frac{\partial \chi}{\partial \theta}, \\
\nabla^2(\dots) &= \frac{\partial^2(\dots)}{\partial \varsigma^2} + \frac{1}{\varsigma} \frac{\partial(\dots)}{\partial \varsigma} + \frac{1}{\varsigma^2} \frac{\partial^2(\dots)}{\partial \phi^2}.
\end{aligned} \tag{10}$$

The three Navier equations serve as the fundamental equations for the curved nanotube. These equations are composed of three parts. The first part is independent of c/ρ . The second and the third parts are the linear and nonlinear parts of c/ρ . As it is impossible to find an exact solution for the Navier equations, the method of successive approximation is used to obtain an approximate solution.

4. Method of Successive Approximation

The method of successive approximation is employed to solve the Navier equations in the toroidal coordinate system. The solution for each component of the displacement is assumed to be a series in terms of a small parameter ε . The parameter ε is chosen as

$$\varepsilon = \frac{c}{R}. \tag{11}$$

Therefore the nondimensional displacement components take the form as

$$\begin{aligned}
u &= u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \dots, \\
v &= v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 + \dots, \\
w &= w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \varepsilon^3 w_3 + \dots,
\end{aligned} \tag{12}$$

where the indexes of the right hand side terms indicate the order of the displacement components in the series. By substituting the series form displacement functions (12) into

(3), the corresponding nondimensional stress components are derived as

$$\begin{aligned}
\tau_{\varsigma\varsigma} &= \tau_{\varsigma\varsigma 0} + \varepsilon \tau_{\varsigma\varsigma 1} + \varepsilon^2 \tau_{\varsigma\varsigma 2} + \varepsilon^3 \tau_{\varsigma\varsigma 3} + \dots, \\
\tau_{\phi\phi} &= \tau_{\phi\phi 0} + \varepsilon \tau_{\phi\phi 1} + \varepsilon^2 \tau_{\phi\phi 2} + \varepsilon^3 \tau_{\phi\phi 3} + \dots, \\
\tau_{\theta\theta} &= \tau_{\theta\theta 0} + \varepsilon \tau_{\theta\theta 1} + \varepsilon^2 \tau_{\theta\theta 2} + \varepsilon^3 \tau_{\theta\theta 3} + \dots, \\
\tau_{\varsigma\phi} &= \tau_{\varsigma\phi 0} + \varepsilon \tau_{\varsigma\phi 1} + \varepsilon^2 \tau_{\varsigma\phi 2} + \varepsilon^3 \tau_{\varsigma\phi 3} + \dots, \\
\tau_{\varsigma\theta} &= \tau_{\varsigma\theta 0} + \varepsilon \tau_{\varsigma\theta 1} + \varepsilon^2 \tau_{\varsigma\theta 2} + \varepsilon^3 \tau_{\varsigma\theta 3} + \dots, \\
\tau_{\phi\theta} &= \tau_{\phi\theta 0} + \varepsilon \tau_{\phi\theta 1} + \varepsilon^2 \tau_{\phi\theta 2} + \varepsilon^3 \tau_{\phi\theta 3} + \dots.
\end{aligned} \tag{13}$$

The functions of Navier equations are expanded as

$$\begin{aligned}
U &= U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \varepsilon^3 U_3 + \dots, \\
\overline{U} &= \overline{U}_0 + \varepsilon \overline{U}_1 + \varepsilon^2 \overline{U}_2 + \varepsilon^3 \overline{U}_3 + \dots, \\
\widehat{U} &= \widehat{U}_0 + \varepsilon \widehat{U}_1 + \varepsilon^2 \widehat{U}_2 + \varepsilon^3 \widehat{U}_3 + \dots, \\
V &= V_0 + \varepsilon V_1 + \varepsilon^2 V_2 + \varepsilon^3 V_3 + \dots, \\
\overline{V} &= \overline{V}_0 + \varepsilon \overline{V}_1 + \varepsilon^2 \overline{V}_2 + \varepsilon^3 \overline{V}_3 + \dots, \\
\widehat{V} &= \widehat{V}_0 + \varepsilon \widehat{V}_1 + \varepsilon^2 \widehat{V}_2 + \varepsilon^3 \widehat{V}_3 + \dots, \\
W &= W_0 + \varepsilon W_1 + \varepsilon^2 W_2 + \varepsilon^3 W_3 + \dots, \\
\overline{W} &= \overline{W}_0 + \varepsilon \overline{W}_1 + \varepsilon^2 \overline{W}_2 + \varepsilon^3 \overline{W}_3 + \dots, \\
\widehat{W} &= \widehat{W}_0 + \varepsilon \widehat{W}_1 + \varepsilon^2 \widehat{W}_2 + \varepsilon^3 \widehat{W}_3 + \dots.
\end{aligned} \tag{14}$$

By substituting the expanded Navier functions (14) into Navier equations (9), matching terms in ε , and setting each order of ε to zero, the governing equations for the different orders are obtained. The details to obtain the solution are explained in Appendix A.

5. General Solution for Curved Nanotubes Subjected to Pure Bending

A direct solution is formulated to avoid the complications that arise from the use of stress function. The complementary solution for the displacement is given in the following form:

$$\begin{aligned}
u_0 &= a \varsigma^m \cos(n\phi) \cos(\theta), \\
v_0 &= b \varsigma^m \sin(n\phi) \cos(\theta), \\
w_0 &= c \varsigma^{\overline{m}} \cos(n\phi) \sin(\theta).
\end{aligned} \tag{15}$$

FIGURE 2: End couples acting on the curved nanotube.

The quantities $a, b, c, m, \bar{m}$, and n are constants determined from zeroth-order Navier equations (13) and stress free boundary conditions on the free surfaces. The results are

$$\begin{aligned}
 u_0 = & \left[a_0 \zeta^{-1} + b_0 \zeta + \{a_1 + b_1 (1 - 4\nu) \zeta^2 + c_1 \zeta^{-2} \right. \\
 & + d_1 (3 - 4\nu) \ln \zeta \} \cos \phi + \sum_{n=2}^{\infty} \{a_n \zeta^{n-1} \\
 & + b_n (2 - n - 4\nu) \zeta^{n+1} + c_n \zeta^{-(n+1)} \\
 & \left. + d_n (2 - n + 4\nu) \zeta^{-(n-1)} \} \cos n\phi \right] \cos(\theta), \\
 v_0 = & \left[\{-a_1 + b_1 (5 - 4\nu) \zeta^2 + c_1 \zeta^{-2} - d_1 (3 - 4\nu) \ln \zeta \right. \\
 & - d_1 \} \sin \phi + \sum_{n=2}^{\infty} \{-a_n \zeta^{n-1} + b_n (4 + n - 4\nu) \zeta^{n+1} \\
 & + c_n \zeta^{-(n+1)} + d_n (4 - n - 4\nu) \zeta^{-(n-1)} \} \sin n\phi \left. \right] \cos(\theta), \\
 w_0 = & \left[e_0 + f_0 \ln \zeta + \{e_1 \zeta^{-1} + f_1 \zeta \} \cos \phi + \sum_{n=2}^{\infty} \{e_n \zeta^{-n} \right. \\
 & \left. + f_n \zeta^n \} \cos n\phi \right] \sin(\theta),
 \end{aligned} \quad (16)$$

where the quantities $a_0, b_0, a_1, b_1, c_1, d_1, e_0, f_0, e_1, f_1, a_n, b_n, c_n, d_n, e_n$, and f_n are constants to be found from the boundary conditions. The detailed solution is presented for the curved nanotube subjected to bending moments, M_0 , as shown in Figure 2.

Solutions are proposed successively for the various orders, starting from the zeroth order. The zeroth-order

solution is taken from the complementary solution. The particular part is arranged using trial functions. The displacement functions for that order are obtained by summing the complementary and particular parts. The constants in the complementary solution of each order are determined separately from the boundary conditions. The total solution is the sum of the particular and complementary solution of all the orders times the appropriate power of ϵ . Any constants remaining in the total solution are determined from end or edge boundary conditions.

5.1. The Zeroth-Order Solution. Considering the boundary conditions on the inner and outer free surfaces, the displacement components are obtained as

$$\begin{aligned}
 u_0 &= a_1 \cos \phi \cos(\theta), \\
 v_0 &= -a_1 \sin \phi \cos(\theta), \\
 w_0 &= 0.
 \end{aligned} \quad (17)$$

Constant a_1 of this order is to be determined later from the boundary conditions on the ends of the curved nanotube structure.

5.2. The First-Order Solution. The first-order displacement is obtained as

$$\begin{aligned}
 u_1 &= 0, \\
 v_1 &= 0, \\
 w_1 &= 0.
 \end{aligned} \quad (18)$$

5.3. The Second-Order Solution. The second-order displacement is obtained as

$$\begin{aligned}
 u_2 &= A_1 \zeta^2 \cos \phi \cos(\theta), \\
 v_2 &= B_1 \zeta^2 \sin \phi \cos(\theta), \\
 w_2 &= 0,
 \end{aligned} \quad (19)$$

where

$$\begin{aligned}
 A_1 &= \frac{1}{\bar{\mu}} \frac{4v^2 - 13v + 4}{4v - 2} a_1, \\
 B_1 &= \frac{1}{\bar{\mu}} \frac{4v^2 - 9v + 8}{4v - 2} a_1.
 \end{aligned} \quad (20)$$

5.4. The Third-Order Solution. The third-order solution then takes the form

$$\begin{aligned}
 u_3 &= \{a_2 \zeta + b_2 (1 - 4\nu) \zeta^3 + c_2 \zeta^{-3} + d_2 (3 - 4\nu) \zeta^{-1} \\
 &+ A_{22} \zeta^3 \} \cos 2\phi \cos(\theta), \\
 v_3 &= \{-a_2 \zeta + b_2 (6 - 4\nu) \zeta^3 + c_2 \zeta^{-3} + d_2 (2 - 4\nu) \zeta^{-1} \\
 &+ C_{22} \zeta^3 \} \sin 2\phi \cos(\theta), \\
 w_3 &= \{e_2 \zeta^{-2} + f_2 \zeta^2 + D_{22} \zeta^3 \} \cos \phi \sin(\theta),
 \end{aligned} \quad (21)$$

where a_2 , b_2 , c_2 , d_2 , e_2 , and f_2 are free constants. These are determined from the boundary conditions and governing equations. These constants are functions of the material property, ν , the ratio $\lambda = a/b$, and the nonlocal parameter μ . The coefficients appearing in (21) are defined in Appendix B.

5.5. The Fourth-Order Solution. The fourth-order solution takes the form

$$\begin{aligned}
 u_4 = & \left[a_3 \zeta^2 - b_3 (1 + 4\nu) \zeta^4 + c_3 \zeta^{-4} - d_3 (5 - 4\nu) \zeta^{-2} \right] \\
 & \cdot \cos 3\phi \cos(\theta) + \frac{1 - 2\nu}{20\nu - 11} \left[L_1 \zeta^4 + L_2 \zeta^{-2} + L_3 \right] \\
 & \cdot \cos 3\phi \cos(\theta) + \frac{1 - 2\nu}{4\nu - 3} \left[L_4 \zeta^2 + L_5 \zeta^4 + L_6 \right] \cos \phi \\
 & \cdot \cos(\theta), \\
 v_4 = & \left[-a_3 \zeta^2 + b_3 (7 - 4\nu) \zeta^4 + c_3 \zeta^{-4} + d_3 (1 - 4\nu) \zeta^{-2} \right] \\
 & \cdot \sin 3\phi \cos(\theta) + \frac{1 - 2\nu}{20\nu - 19} \left[L_7 \zeta^4 + L_8 \zeta^{-2} + L_9 \right] \\
 & \cdot \sin 3\phi \cos(\theta) \\
 & + \frac{1 - 2\nu}{4\nu - 3} \left[L_{10} \zeta^2 + L_{11} \zeta^4 + L_{12} + L_{13} \zeta^{-1} + L_{14} \zeta^{13} \right] \\
 & \cdot \sin \phi \cos(\theta), \\
 w_4 = & \left[e_3 \zeta^{-3} + f_3 \zeta^3 \right] \cos 2\phi \sin(\theta) \\
 & + \frac{-1}{4} \left[L_{15} \zeta^{-1} + L_{16} \zeta^3 + L_{18} + L_{17} \zeta^4 \right] \cos 2\phi \\
 & \cdot \sin(\theta).
 \end{aligned} \tag{22}$$

It should be noted that a_3 , b_3 , c_3 , d_3 , e_3 , and f_3 are free constants obtained in Appendix C by applying the boundary conditions and governing equations. Finally, substituting the solution for the various orders into (12) yields the total solution for the problem.

6. Results and Discussion

The theoretical investigations are performed for the curved single-walled carbon nanotube (SWCNT) subjected to bending moment (see Figure 2). The mechanical properties of SWCNT are given in Table 1 [32]. Furthermore, the displacement components are normalized as $\bar{u}_i = u_i/u_0$, where $u_0 = L * \sigma_0/E$ and $\sigma_0 = (M_0 \cdot r)/(\pi/64) * (OD^4 - ID^4)$ where OD and ID are outer and inner radius of the curved SWCNT. The innermost radius and thickness of SWCNT are assumed to be 8.5 nm and 0.34 nm, respectively [33]. In addition, L is the length of the curved nanotube. L/h is defined as an aspect ratio.

Tube length, thickness, bend radius, and nonlocal parameter have effects on the nondimensional displacement components. The numerical results for curved SWCNT under bending moment (M_0) at the point in which $\phi = 180^\circ$ are presented in Tables 2, 3, 4, and 5.

TABLE 1: Mechanical properties of SWCNT [32].

Nanotubes	Mechanical properties	
	E (TPa)	ν (Poisson ratio)
SWCNT	1.05	0.19

TABLE 2: Nondimensional displacement components of curved SWCNT ($R/D = 4$ and $L = r * \pi/2$).

L/h	μ	$\bar{u}_\theta$	$\bar{u}_\phi (\times 10^{-2})$	$\bar{u}_r (\times 10^{-4})$
100	0	1.158	1.390	1.454
	1.0	2.045	2.356	1.998
	2.0	2.081	2.505	2.455
	3.0	2.305	2.890	2.531
	4.0	2.712	3.104	3.104
	5.0	3.019	3.505	3.402
200	0	1.401	1.682	1.759
	1.0	2.474	2.851	2.418
	2.0	2.518	3.031	2.971
	3.0	2.789	3.497	3.063
	4.0	3.282	3.756	3.756
	5.0	3.653	4.241	4.116
500	0	1.760	2.113	2.210
	1.0	3.108	3.581	3.037
	2.0	3.163	3.808	3.732
	3.0	3.504	4.393	3.847
	4.0	4.122	4.718	4.718
	5.0	4.589	5.328	5.171

TABLE 3: Nondimensional displacement components of curved SWCNT ($R/D = 4$ and $L = r * \pi$).

L/h	μ	$\bar{u}_\theta$	$\bar{u}_\phi (\times 10^{-2})$	$\bar{u}_r (\times 10^{-4})$
100	0	1.285	1.543	1.614
	1.0	2.270	2.615	2.218
	2.0	2.310	2.781	2.725
	3.0	2.559	3.208	2.809
	4.0	3.010	3.445	3.445
	5.0	3.351	3.891	3.776
200	0	1.581	1.898	1.985
	1.0	2.792	3.216	2.728
	2.0	2.841	3.421	3.352
	3.0	3.148	3.946	3.455
	4.0	3.702	4.237	4.237
	5.0	4.122	4.786	4.644
500	0	1.979	2.376	2.486
	1.0	3.496	4.027	3.416
	2.0	3.557	4.283	4.197
	3.0	3.941	4.940	4.326
	4.0	4.635	5.305	5.305
	5.0	5.161	5.992	5.815

The effects of thickness and nonlocal parameter (μ) on the nondimensional displacement components are investigated while assuming $R/OD = 4$ and $L = r * \pi/2$ as shown

TABLE 4: Nondimensional displacement components of curved SWCNT ($\mu = 3.0$ and $L = r * \pi/2$).

L/h	R/D	$\bar{u}_\theta$	$\bar{u}_\phi (\times 10^{-2})$	$\bar{u}_r (\times 10^{-4})$
100	4	2.305	2.890	2.531
	6	2.265	2.701	2.508
	8	2.201	2.692	2.420
	10	2.165	2.669	2.398
200	4	2.881	3.613	3.164
	6	2.831	3.376	3.135
	8	2.751	3.365	3.025
	10	2.706	3.336	2.998
500	4	3.596	4.508	3.948
	6	3.533	4.214	3.912
	8	3.434	4.200	3.775
	10	3.377	4.164	3.741

TABLE 5: Nondimensional displacement components of curved SWCNT ($\mu = 3.0$ and $L = r * \pi$).

L/h	R/D	$\bar{u}_\theta$	$\bar{u}_\phi (\times 10^{-2})$	$\bar{u}_r (\times 10^{-4})$
100	4	2.582	3.237	2.835
	6	2.537	3.025	2.809
	8	2.465	3.015	2.710
	10	2.425	2.989	2.686
200	4	3.202	4.014	3.515
	6	3.146	3.751	3.483
	8	3.057	3.739	3.360
	10	3.007	3.706	3.331
500	4	4.002	5.017	4.394
	6	3.932	4.689	4.354
	8	3.821	4.673	4.201
	10	3.759	4.633	4.163

in Table 2. It is found from Table 2 that the displacement components increase with increasing the aspect ratio (L/h) while curved SWCNT length is kept constant. The displacement components increase with the increase of the nonlocal parameter (μ) while the length and thickness of curved SWCNT are kept constant as shown in Table 2.

Table 3 shows the effects of thickness and nonlocal parameter (μ) on the nondimensional displacement components for $R/OD = 4$ and $L = r * \pi$. It is observed from Table 3 that the displacement components increase with increasing the aspect ratio (L/h) while curved SWCNT length is kept constant. Observing Table 3, the displacement components increase as the nonlocal parameter (μ) increases where length and thickness of curved SWCNT are fixed.

Comparing Table 2 with Table 3, it is clear that the nondimensional displacement components increase with increasing the curved SWCNT length.

The effects of thickness and bend radius ratio (R/D) on the nondimensional displacement components are studied for $\mu = 3.0$ and $L = r * \pi/2$ as shown in Table 4. Observing Table 4, the displacement components decrease with the increase of bend radius ratio (R/D).

Table 5 shows the effects of thickness and bend radius ratio (R/D) on the nondimensional displacement components for $\mu = 3.0$ and $L = r * \pi/2$. It is observed from Table 5 that the displacement components increase with increasing the aspect ratio (L/h) while curved SWCNT length is kept constant. Table 5 shows that the displacement components decrease as the bend radius ratio (R/D) increases where length and thickness of curved SWCNT are fixed.

7. Conclusions

We have presented a new displacement-based solution by using the nonlocal Eringen theory and Toroidal Elasticity to study curved nanotube structures under mechanical loadings. Equations of motion were developed based on Eringen's differential constitutive equations of nonlocal elasticity for curved nanotube structures. The successive approximation method was employed to simplify the governing equations for curved nanotube structures under pure bending moment. The numerical results demonstrate the effectiveness of length, thickness, bend radius, and nonlocal parameter on all displacement components of curved nanotubes. It was observed that for curved SWCNT the values of displacement component in the longitudinal direction ($\bar{u}_\theta$) were greater than those for the displacement components in the circumferential ($\bar{u}_\phi$) and radial ($\bar{u}_r$) directions.

Appendices

A. Governing Equations for the Different Orders

The equations are expressed for the zeroth order as

$$\begin{aligned} U_0 &= 0; \\ V_0 &= 0; \\ W_0 &= 0. \end{aligned} \quad (\text{A.1})$$

For the first order

$$\begin{aligned} U_1 &= -\bar{U}_0; \\ V_1 &= -\bar{V}_0; \\ W_1 &= -\bar{W}_0. \end{aligned} \quad (\text{A.2})$$

For the second order

$$\begin{aligned} U_2 &= -\bar{U}_1 + \bar{U}_0 \zeta \cos \phi - \hat{U}_0, \\ V_2 &= -\bar{V}_1 + \bar{V}_0 \zeta \cos \phi - \hat{V}_0, \\ W_2 &= -\bar{W}_1 + \bar{W}_0 \zeta \cos \phi - \hat{W}_0. \end{aligned} \quad (\text{A.3})$$

For the third order

$$\begin{aligned}
 U_3 &= -\bar{U}_2 + \bar{U}_1 \zeta \cos \phi - \bar{U}_0 \zeta^2 \cos^2 \phi - \widehat{U}_1 \\
 &\quad + 2\widehat{U}_0 \zeta \cos \phi, \\
 V_3 &= -\bar{V}_2 + \bar{V}_1 \zeta \cos \phi - \bar{V}_0 \zeta^2 \cos^2 \phi - \widehat{V}_1 \\
 &\quad + 2\widehat{V}_0 \zeta \cos \phi, \\
 W_3 &= -\bar{W}_2 + \bar{W}_1 \zeta \cos \phi - \bar{W}_0 \zeta^2 \cos^2 \phi - \widehat{W}_1 \\
 &\quad + 2\widehat{W}_0 \zeta \cos \phi.
 \end{aligned} \tag{A.4}$$

And for the fourth order

$$\begin{aligned}
 U_4 &= -\bar{U}_3 + \bar{U}_2 \zeta \cos \phi - \bar{U}_1 \zeta^2 \cos^2 \phi + \bar{U}_0 \zeta^3 \cos^3 \phi \\
 &\quad - \widehat{U}_2 + 2\widehat{U}_1 \zeta \cos \phi - 3\widehat{U}_0 \zeta^2 \cos^2 \phi, \\
 V_4 &= -\bar{V}_3 + \bar{V}_2 \zeta \cos \phi - \bar{V}_1 \zeta^2 \cos^2 \phi + \bar{V}_0 \zeta^3 \cos^3 \phi \\
 &\quad - \widehat{V}_2 + 2\widehat{V}_1 \zeta \cos \phi - 3\widehat{V}_0 \zeta^2 \cos^2 \phi, \\
 W_4 &= -\bar{W}_3 + \bar{W}_2 \zeta \cos \phi - \bar{W}_1 \zeta^2 \cos^2 \phi + \bar{W}_0 \zeta^3 \cos^3 \phi \\
 &\quad - \widehat{W}_2 + 2\widehat{W}_1 \zeta \cos \phi - 3\widehat{W}_0 \zeta^2 \cos^2 \phi.
 \end{aligned} \tag{A.5}$$

The Navier functions of order k ($k = 0, 1, 2, \dots$) in (A.1)–(A.5) are defined as

$$\begin{aligned}
 U_k &= \frac{1}{\bar{\mu}} \nabla^2 u_k - \frac{1}{\bar{\mu}} \frac{u_k}{\zeta^2} - \frac{2}{\zeta^2} \frac{1}{\bar{\mu}} \frac{\partial v_k}{\partial \phi} \\
 &\quad + \frac{1}{1-2\nu} \frac{1}{\bar{\mu}} \frac{\partial \psi_k}{\partial \zeta}, \\
 \bar{U}_k &= \cos \phi \frac{1}{\bar{\mu}} \frac{\partial u_k}{\partial \zeta} - \sin \phi \frac{1}{\bar{\mu}} \left(\frac{1}{\zeta} \frac{\partial u_k}{\partial \phi} - \frac{v_k}{\zeta} \right) \\
 &\quad + \frac{1}{1-2\nu} \frac{1}{\bar{\mu}} \frac{\partial \chi_k}{\partial \zeta}, \\
 \widehat{U}_k &= \frac{1}{\bar{\mu}} \frac{\partial^2 u_k}{\partial \theta^2} - \cos \phi \frac{1}{\bar{\mu}} \frac{\partial w_k}{\partial \theta} \\
 &\quad - \frac{2(1-\nu)}{1-2\nu} \cos \phi \frac{1}{\bar{\mu}} \chi_k, \\
 V_k &= \frac{1}{\bar{\mu}} \nabla^2 v_k - \frac{1}{\bar{\mu}} \frac{v_k}{\zeta^2} + \frac{2}{\zeta^2} \frac{1}{\bar{\mu}} \frac{\partial u_k}{\partial \phi} \\
 &\quad + \frac{1}{1-2\nu} \frac{1}{\bar{\mu}} \frac{1}{\zeta} \frac{\partial \psi_k}{\partial \phi}, \\
 \bar{V}_k &= \cos \phi \frac{1}{\bar{\mu}} \frac{\partial v_k}{\partial \zeta} - \sin \phi \frac{1}{\bar{\mu}} \left(\frac{1}{\zeta} \frac{\partial v_k}{\partial \phi} + \frac{u_k}{\zeta} \right) \\
 &\quad + \frac{1}{1-2\nu} \frac{1}{\bar{\mu}} \frac{1}{\zeta} \frac{\partial \chi_k}{\partial \phi},
 \end{aligned}$$

$$\begin{aligned}
 \widehat{V}_k &= \frac{1}{\bar{\mu}} \frac{\partial^2 v_k}{\partial \theta^2} + \sin \phi \frac{1}{\bar{\mu}} \frac{\partial w_k}{\partial \theta} \\
 &\quad + \frac{2(1-\nu)}{1-2\nu} \sin \phi \frac{1}{\bar{\mu}} \chi_k,
 \end{aligned}$$

$$W_k = \frac{1}{\bar{\mu}} \nabla^2 w_k,$$

$$\begin{aligned}
 \bar{W}_k &= \cos \phi \frac{1}{\bar{\mu}} \frac{\partial w_k}{\partial \zeta} - \sin \phi \frac{1}{\bar{\mu}} \frac{1}{\zeta} \frac{\partial w_k}{\partial \phi} \\
 &\quad + \frac{1}{1-2\nu} \frac{1}{\bar{\mu}} \frac{\partial \psi_k}{\partial \theta},
 \end{aligned}$$

$$\widehat{W}_k = -\frac{1}{\bar{\mu}} \frac{\partial^2 w_k}{\partial \theta^2} - \frac{1}{\bar{\mu}} w_k + \frac{3-4\nu}{1-2\nu} \frac{1}{\bar{\mu}} \frac{\partial \chi_k}{\partial \theta},$$

$$\psi_k = \frac{1}{\bar{\mu}} \frac{\partial u_k}{\partial \zeta} + \frac{1}{\bar{\mu}} \frac{u_k}{\zeta} + \frac{1}{\zeta} \frac{1}{\bar{\mu}} \frac{\partial v_k}{\partial \phi},$$

$$\chi_k = \frac{1}{\bar{\mu}} u_k \cos \phi - \frac{1}{\bar{\mu}} v_k \sin \phi + \frac{1}{\bar{\mu}} \frac{\partial w_k}{\partial \theta},$$

$$\nabla^2 (\dots) = \frac{\partial^2 (\dots)}{\partial \zeta^2} + \frac{1}{\zeta} \frac{\partial (\dots)}{\partial \zeta} + \frac{1}{\zeta^2} \frac{\partial^2 (\dots)}{\partial \phi^2}. \tag{A.6}$$

The displacement functions on the right hand members with negative indexes are assumed to be zero. It is noted from (A.1)–(A.5) that the zeroth-order governing equations are homogenous equations. The right hand side terms for orders greater than zero contain portions from the lower orders, and the complication of the right hand side terms increases with order. There is only a complementary solution for the zeroth-order governing equations. For orders greater than the zeroth, the solutions are composed of a complementary and a particular part. The complementary part is determined from the homogeneous part of that order's governing equations. The particular part is derived to satisfy the right hand side terms of the governing equations. The particular part is developed by using trial displacement functions with free constants, to match the right hand side terms of the governing equations of that order.

B. The Coefficients for the First up to the Third-Order Solution

The coefficients appearing in (21) are obtained as follows:

$$\begin{aligned}
 a_2 &= \frac{1}{\bar{\mu}} \left[3A_{22} + \frac{\nu}{1-2\nu} \left(4A_{22} + 2C_{22} + \frac{1}{2}a_1 \right) \right] \\
 &\quad \cdot \left[\frac{-2(\lambda^{10} - \lambda^2 + 1)}{\lambda^2(\lambda^4 - 1)(\lambda^4 - \lambda^2 + 4)} \right],
 \end{aligned}$$

$$\begin{aligned}
b_2 &= \frac{1}{6\hat{\mu}} \left[3A_{22} + \frac{v}{1-2v} \left(4A_{22} + 2C_{22} + \frac{1}{2}a_1 \right) \right] \\
&\cdot \left[\frac{-2\lambda^{10} - \lambda^8 + 2\lambda^6 + 2\lambda^4 - 3}{\lambda^2(\lambda^4 - 1)(\lambda^4 - \lambda^2 + 4)} \right] \\
&- \frac{1}{6\hat{\mu}} [C_{22} - A_{22}], \\
c_2 &= \frac{1}{3\hat{\mu}} \left[3A_{22} + \frac{v}{1-2v} \left(4A_{22} + 2C_{22} + \frac{1}{2}a_1 \right) \right] \\
&\cdot \left[1 + \frac{2(1 - \lambda^2 - \lambda^4)}{(\lambda^4 - 1)(\lambda^4 - \lambda^2 + 4)} \right], \\
d_2 &= \frac{1}{2\hat{\mu}} \left[3A_{22} + \frac{v}{1-2v} \left(4A_{22} + 2C_{22} + \frac{1}{2}a_1 \right) \right] \\
&\cdot \left[\frac{\lambda^6 - 1}{\lambda^2(\lambda^4 - \lambda^2 + 4)} \right], \\
e_2 &= \frac{1}{2\hat{\mu}} \left[\left(\frac{4v^2 - 13v + 4}{2 - 4v} \right) a_1 + 3D_{22} \right] \frac{1 - \lambda}{1 - \lambda^{-4}}, \\
f_2 &= \frac{1}{2\hat{\mu}} \left[\left(\frac{4v^2 - 13v + 4}{2 - 4v} \right) a_1 + 3D_{22} \right] \frac{\lambda(\lambda^{-5} - 1)}{1 - \lambda^{-4}},
\end{aligned} \tag{B.1}$$

where

$$\begin{aligned}
A_{11} &= \frac{-1}{1-2v} \left[A_1(3-2v) + \frac{1}{\hat{\mu}} a_1(6-8v) \right], \\
B_{11} &= \frac{2v-3}{1-2v} B_1, \\
C_{11} &= \frac{1}{1-2v} \left[\frac{1}{2} A_1(3-2v) + \frac{1}{2} A_1(1+2v) \right. \\
&\quad \left. + \frac{1}{2\hat{\mu}} a_1(6-8v) \right], \\
D_{11} &= \frac{1}{1-2v} \left[(3A_1 + B_1) - \frac{1}{\hat{\mu}} a_1(6-8v) \right], \\
A_{22} &= \frac{1}{2} \frac{1-2v}{6-10v} [B_{11} - A_{11}] - \frac{1}{2} \frac{1-2v}{2-2v} [A_{11} + B_{11} \\
&\quad + 4A_1], \\
C_{22} &= -C_{11} \frac{1-2v}{9-10v}, \\
D_{22} &= -D_{11}.
\end{aligned} \tag{B.2}$$

C. The Coefficients for the Fourth-Order Solution

The coefficients appearing in (22) are obtained as follows:

$$a_3 = 4c_3 - 9d_3 - \frac{1}{4} (H_2(1) + 2H_1(1)),$$

$$\begin{aligned}
b_3 &= c_3 - 2d_3 - \frac{1}{8} (H_2(1) + H_1(1)), \\
c_3 &= \frac{X_3 X_5 - X_6 X_2}{X_1 X_5 - X_4 X_2}, \\
d_3 &= \frac{X_6 X_1 - X_3 X_4}{X_1 X_5 - X_4 X_2}, \\
e_3 &= \frac{(2/3) [H_3(\lambda) - \lambda^2 H_3(1)]}{\lambda^2 - \lambda^{-4}}, \\
f_3 &= \frac{(2/3) [H_3(\lambda) - \lambda^{-4} H_3(1)]}{\lambda^2 - \lambda^{-4}},
\end{aligned} \tag{C.1}$$

where

$$\begin{aligned}
X_1 &= 8\lambda - 4\lambda^{-3} - 4\lambda^{-5}, \\
X_2 &= -18\lambda + 8\lambda^3, \\
X_3 &= -H_1(\lambda) + \frac{1}{2} [H_2(1) + 2H_1(1)] \lambda - \frac{1}{2} [H_2(1) \\
&\quad + H_1(1)] \lambda^3, \\
X_4 &= -8\lambda + 12\lambda^3 - 4\lambda^{-5}, \\
X_6 &= 18\lambda - 24\lambda^3 + 6\lambda^{-3}, \\
X_6 &= -H_2(\lambda) - \frac{1}{2} [H_2(1) + 2H_1(1)] \lambda + \frac{3}{2} [H_2(1) \\
&\quad + H_1(1)] \lambda^3, \\
H_1(\varsigma) &= \left[\frac{4L_{21} - 3v(L_{21} - L_{24})}{1-2v} \varsigma^3 \right. \\
&\quad \left. + \frac{3v(L_{22} + L_{25}) - 2L_{22}}{1-2v} \varsigma^{-3} \right. \\
&\quad \left. + \frac{v(L_{23} + 3L_{26})}{1-2v} \varsigma^{-1} \right] + \left[b_2(3-4v) \varsigma^3 + c_2 \varsigma^{-3} \right. \\
&\quad \left. - d_2 \varsigma^{-1} + \frac{1}{2} A_{22} \varsigma^3 + \frac{1}{2} C_{22} \varsigma^3 + \frac{1}{4} (B_1 - A_1) \varsigma^3 \right. \\
&\quad \left. - \frac{1}{\hat{\mu}} a_1 \varsigma^3 \right], \\
H_2(\varsigma) &= \frac{-1}{2} [3(L_{21} - L_{24}) \varsigma^3 + 3(L_{22} + L_{25}) \varsigma^{-3} \\
&\quad + (3L_{23} + L_{26}) \varsigma^{-1}], \\
H_3(\varsigma) &= -\frac{1}{8} [-L_{18} \varsigma^{-2} + 3L_{16} \varsigma^2 + 4L_{17} \varsigma^3] \\
&\quad - \frac{1}{4} [e_2 \varsigma^{-2} + f_2 \varsigma^2 + D_{22} \varsigma^3] + \frac{1}{4} A_1 \varsigma^3 - \frac{1}{2} [a_2 \varsigma
\end{aligned} \tag{C.2}$$

$$\begin{aligned}
& + b_2 (-4v) \zeta^3 - c_2 \zeta^{-3} + d_2 (4v - 4) \zeta^{-1} + A_{22} \zeta^3 \\
& + \frac{1}{4\hat{\mu}} a_1 \zeta^3.
\end{aligned}
\tag{C.3}$$

In addition,

$$\begin{aligned}
L_5 &= \frac{1}{\hat{\mu}} \frac{-1}{1-2v} \left[3b_2 (8v^2 - 6v + 1) + \frac{1}{2} A_{22} (1-2v) \right. \\
&\quad \left. - C_{22} (1+v) + \frac{3}{4} B_1 + A_1 \left(7v - \frac{23}{4} \right) \right. \\
&\quad \left. - \frac{9}{4} a_1 (3-4v) \right], \\
L_6 &= -d_2, \\
L_7 &= \frac{1}{\hat{\mu}} \frac{1}{1-2v} \left[\frac{-3}{4} a_1 (3-4v) + B_1 (4v-1) \right. \\
&\quad \left. + A_1 (4v-5) - 6b_2 (v-1) + C_{22} (v+1) \right. \\
&\quad \left. + A_{22} (v-1) \right], \\
L_8 &= \frac{1}{\hat{\mu}} \frac{1}{1-2v} [6c_2 (1-v)], \\
L_9 &= \frac{1}{\hat{\mu}} \frac{-1}{1-2v} [d_2 (-8v^2 + 6v + 2)], \\
L_{10} &= \frac{1}{\hat{\mu}} \frac{-1}{1-2v} [2a_2 (1-v)], \\
L_{11} &= \frac{1}{\hat{\mu}} \frac{1}{1-2v} \left[\frac{-3}{4} a_1 (3-4v) + \frac{1}{4} B_1 (16v-5) \right. \\
&\quad \left. + \frac{1}{4} A_1 (4v-5) - 2b_2 (10v^2 - 20v + 9) \right. \\
&\quad \left. - C_{22} (3-5v) - A_{22} v + D_{22} \right], \\
L_{12} &= \frac{1}{\hat{\mu}} \frac{-1}{1-2v} [2d_2 (1-v)], \\
L_{13} &= \frac{1}{1-2v} e_2, \\
L_{14} &= \frac{1}{1-2v} f_2, \\
L_{15} &= \frac{3}{2} e_2, \\
L_{16} &= \frac{-1}{2} f_2, \\
L_{17} &= \frac{1}{\hat{\mu}} \frac{1}{1-2v} \left[-D_{22} (1-2v) + \frac{3}{2} a_1 (3-4v) \right. \\
&\quad \left. - \frac{1}{2} A_1 (3+4v) - \frac{1}{2} B_1 (5-4v) + 12b_2 (1-2v) \right.
\end{aligned}$$

$$+ 4A_{22} + 2C_{22} \Big],$$

$$L_{18} = \frac{1}{\hat{\mu}} \frac{1}{1-2v} [4d_2 (1-2v)],$$

$$L_{19} = \frac{-3}{2} f_2,$$

$$\begin{aligned}
L_{20} &= \frac{1}{\hat{\mu}} \frac{1}{1-2v} \left[-2D_2 (1-2v) + \frac{3}{4} a_1 (3-4v) \right. \\
&\quad \left. - \frac{1}{2} A_1 (3+4v) + \frac{1}{2} B_1 (1-4v) \right].
\end{aligned}$$

(C.4)

The coefficients appearing in (C.3) are given as

$$\begin{aligned}
L_{21} &= \frac{-1}{20v-11} L_1, \\
L_{22} &= \frac{1}{20v-11} L_2, \\
L_{23} &= \frac{-1}{20v-11} L_3, \\
L_{24} &= \frac{1}{20v-19} L_7, \\
L_{25} &= \frac{1}{20v-19} L_8, \\
L_{26} &= \frac{-1}{20v-19} L_9.
\end{aligned}
\tag{C.5}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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