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Research Article

Fractals of Generalized F-Hutchinson Operator in b-Metric Spaces

Talat Nazir,[1,2] Sergei Silvestrov,[1] and Xiaomin Qi[1]

[1] *Division of Applied Mathematics, School of Education, Culture and Communication, Mälardalen University, 72123 Västerås, Sweden*
[2] *Department of Mathematics, COMSATS Institute of Information Technology, Abbottabad 22060, Pakistan*

Correspondence should be addressed to Talat Nazir; talat@ciit.net.pk

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The aim of this paper is to construct a fractal with the help of a finite family of generalized F-contraction mappings, a class of mappings more general than contraction mappings, defined in the setup of b-metric space. Consequently, we obtain a variety of results for iterated function system satisfying a different set of contractive conditions. Our results unify, generalize, and extend various results in the existing literature.

1. Introduction and Preliminaries

Iterated function systems are method of constructing fractals and are based on the mathematical foundations laid by Hutchinson [1]. He showed that Hutchinson operator constructed with the help of a finite system of contraction mappings defined on Euclidean space $\mathbb{R}^n$ has closed and bounded subset of $\mathbb{R}^n$ as its fixed point, called attractor of iterated function system (see also [2]). In this context, fixed point theory plays significant and vital role to help in construction of fractals.

Fixed point theory is studied in environment created with appropriate mappings satisfying certain conditions. Recently, many researchers have obtained fixed point results for single and multivalued mappings defined on metrics spaces. Banach contraction principle [3] is of paramount importance in metrical fixed point theory with a wide range of applications, including iterative methods for solving linear, nonlinear, differential, integral, and difference equations. This initiated several researchers to extend and enhance the scope of metric fixed point theory. As a result, Banach contraction principles have been extended either by generalizing the domain of the mapping [4–10] or by extending the contractive condition on the mappings [11–15]. There are certain cases when the range X of a mapping is replaced with a family of sets possessing some topological structure and consequently a single-valued mapping is replaced with a multivalued mapping. Nadler [16] was the first who combined the ideas of multivalued mappings and contractions and hence initiated the study of metric fixed point theory of multivalued operators; see also [17–19]. The fixed point theory of multivalued operators provides important tools and techniques to solve the problems of pure, applied, and computational mathematics which can be restructured as an inclusion equation for an appropriate multivalued operator.

The concept of metric has been generalized further in one to many ways. The concept of b-metric space was introduced by Czerwik in [20]. Since then, several papers have been published on the fixed point theory of various classes of single-valued and multivalued operators in b-metric space [20–30].

In this paper, we construct a fractal set of iterated function system, a certain finite collection of mappings defined on b-metric space which induce compact valued mappings defined on a family of compact subsets of b-metric space. We prove that Hutchinson operator defined with the help of a finite family of generalized F-contraction mappings on a complete b-metric space is itself generalized F-contraction mapping on a family of compact subsets of X. We then obtain a final fractal obtained by successive application of a generalized F-Hutchinson operator in b-metric space.

Definition 1. Let X be a nonempty set and let $b \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric if, for any $x, y, z \in X$, the following conditions hold:

(b_1) $d(x, y) = 0$ if and only if $x = y$,

(b_2) $d(x, y) = d(y, x)$,

(b_3) $d(x, y) \leq b(d(x, z) + d(z, y))$.

The pair (X, d) is called b-metric space with parameter $b \geq 1$.

If $b = 1$, then b-metric space is metric spaces. But the converse does not hold in general [20, 21, 25].

Example 2 (see [31]). Let (X, d) be a metric space, and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. Then ρ is b-metric with $b = 2^{p-1}$.

Obviously conditions (b_1) and (b_2) of above definition are satisfied. If $1 < p < \infty$, then the convexity of the function $f(x) = x^p (x > 0)$ implies

$$\left(\frac{a+b}{2}\right)^p \leq \frac{1}{2}\left(a^p + b^p\right), \tag{1}$$

and hence $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ holds. Thus, for each $x, y, z \in X$ we obtain

$$\begin{aligned}\rho(x, y) &= (d(x, y))^p \leq (d(x, z) + d(z, y))^p \\ &\leq 2^{p-1}\left((d(x, z))^p + (d(z, y))^p\right) \\ &= 2^{p-1}\left(\rho(x, z) + \rho(z, y)\right).\end{aligned} \tag{2}$$

So condition (b_3) of the above definition is satisfied and ρ is b-metric.

If $X = \mathbb{R}$ (set of real numbers) and $d(x, y) = |x - y|$ is the usual metric, then $\rho(x, y) = (x - y)^2$ is b-metric on $\mathbb{R}$ with $b = 2$ but is not a metric on $\mathbb{R}$.

Definition 3 (see [24]). Let (X, d) be b-metric space. Then a subset $C \subseteq X$ is called

(i) closed if and only if, for each sequence $\{x_n\}$ in C which converges to an element x, we have $x \in C$ (i.e., $C = \overline{C}$),

(ii) compact if and only if for every sequence of elements of C there exists a subsequence that converges to an element of C,

(iii) bounded if and only if $\delta(C) := \sup\{d(x, y) : x, y \in C\} < \infty$.

Let $\mathscr{H}(X)$ denote the set of all nonempty compact subsets of X. For $A, B \in \mathscr{H}(X)$, let

$$H(A, B) = \max\left\{\sup_{b \in B} d(b, A), \sup_{a \in A} d(a, B)\right\}, \tag{3}$$

where $d(x, B) = \inf\{d(x, y) : y \in B\}$ is the distance of a point x from the set B. The mapping H is said to be the Pompeiu-Hausdorff metric induced by d. If (X, d) is a complete b-metric space, then $(\mathscr{H}(X), H)$ is also a complete b-metric space.

For the sake of completeness, we state that the following lemma holds in b-metric space [32].

Lemma 4. *Let (X, d) be b-metric space. For all $A, B, C, D \in \mathscr{H}(X)$, the following hold:*

(i) *If $B \subseteq C$, then $\sup_{a \in A} d(a, C) \leq \sup_{a \in A} d(a, B)$.*

(ii) $\sup_{x \in A \cup B} d(x, C) = \max\{\sup_{a \in A} d(a, C), \sup_{b \in B} d(b, C)\}$.

(iii) *One has $H(A \cup B, C \cup D) \leq \max\{H(A, C), H(B, D)\}$.*

The following lemmas from [20, 27, 28] will be needed in the sequel to prove the main result of the paper.

Lemma 5. *Let (X, d) be b-metric space and $CB(X)$ denotes the set of all nonempty closed and bounded subsets of X. For $x, y \in X$ and $A, B \in CB(X)$, the following statements hold:*

(1) *$(CB(X), H)$ is b-metric space.*

(2) *$d(x, B) \leq H(A, B)$ for all $x \in A$.*

(3) *One has $d(x, A) \leq b(d(x, y) + d(y, A))$.*

(4) *For $h > 1$ and $\acute{a} \in A$, there is $\acute{b} \in B$ such that $d(\acute{a}, \acute{b}) \leq hH(A, B)$.*

(5) *For every $h > 0$ and $\acute{a} \in A$, there is $\acute{b} \in B$ such that $d(\acute{a}, \acute{b}) \leq H(A, B) + h$.*

(6) *For every $\lambda > 0$ and $\tilde{a} \in A$, there is $\tilde{b} \in B$ such that $d(\tilde{a}, \tilde{b}) \leq \lambda$.*

(7) *For every $\lambda > 0$ and $\tilde{a} \in A$, there is $\tilde{b} \in B$ such that $d(\tilde{a}, \tilde{b}) \leq \lambda$ implies $H(A, B) \leq \lambda$.*

(8) *$d(x, A) = 0$ if and only if $x \in \overline{A} = A$.*

(9) *For $\{x_n\} \subseteq X$,*

$$\begin{aligned}d(x_0, x_n) \leq bd(x_0, x_1) + \cdots + b^{n-1} d(x_{n-2}, x_{n-1}) \\ + b^{n-1} d(x_{n-1}, x_n).\end{aligned} \tag{4}$$

Definition 6. Let (X, d) be b-metric space. A sequence $\{x_n\}$ in X is called

(i) Cauchy if and only if, for $\varepsilon > 0$, there exists $n(\varepsilon) \in \mathbb{N}$ such that for each $n, m \geq n(\varepsilon)$ one has $d(x_n, x_m) < \varepsilon$,

(ii) convergent if and only if there exists $x \in X$ such that for all $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n(\varepsilon)$ one has $d(x_n, x) < \varepsilon$. In this case one writes $\lim_{n \to \infty} x_n = x$.

It is known that a sequence $\{x_n\}$ in b-metric space X is Cauchy if and only if $\lim_{n \to \infty} d(x_n, x_{n+p}) = 0$ for all $p \in \mathbb{N}$. A sequence $\{x_n\}$ is convergent to $x \in X$ if and only if $\lim_{n \to \infty} d(x_n, x) = 0$, and b-metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent in X.

An et al. [21] studied the topological properties of b-metric spaces and stated the following assertions:

(c_1) In b-metric space (X, d), d is not necessarily continuous in each variable.

(c_2) In b-metric space (X, d), if d is continuous in one variable then d is continuous in other variables.

(c_3) An open ball in b-metric space (X, d) is not necessarily an open set. An open ball is open if d is continuous in one variable.

Wardowski [33] introduced another generalized contraction called F-contraction and proved a fixed point result as interesting generalization of the Banach contraction principle in complete metric space (see also [34]).

Let $\mathcal{F}$ be the collection of all continuous mappings $F : \mathbb{R}^+ \to \mathbb{R}$ that satisfy the following conditions:

(F_1) F is strictly increasing, that is, for all $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha < \beta$ implies that $F(\alpha) < F(\beta)$.

(F_2) For every sequence $\{\alpha_n\}$ of positive real numbers, $\lim_{n\to\infty}\alpha_n = 0$ and $\lim_{n\to\infty}F(\alpha_n) = -\infty$ are equivalent.

(F_3) There exists $h \in (0, 1)$ such that $\lim_{\alpha\to 0^+}\alpha^h F(\alpha) = 0$.

Definition 7 (see [33]). Let (X, d) be a metric space. A self-mapping f on X is called F-contraction if, for any $x, y \in X$, there exist $F \in \mathcal{F}$ and $\tau > 0$ such that

$$\tau + F(d(fx, fy)) \leq F(d(x, y)), \tag{5}$$

whenever $d(fx, fy) > 0$.

From (F_1) and (5), we conclude that

$$d(fx, fy) < d(x, y), \quad \forall x, y \in X, \ fx \neq fy; \tag{6}$$

that is, every F-contraction mapping is contractive, and in particular, every F-contraction mapping is continuous.

Wardowski [33] proved that, in complete metric space (X, d), every F-contractive self-map has a unique fixed point in X and for every x_0 in X a sequence of iterates $\{x_0, fx_0, f^2x_0, \ldots\}$ converges to the fixed point of f.

Let Υ be the set of all mapping $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ that is satisfying $\liminf_{t\to 0}\tau(t) > 0$ for all $t \geq 0$.

Definition 8. Let (X, d) be b-metric space. A self-mapping f on X is called a generalized F-contraction if, for any $x, y \in X$, there exist $F \in \mathcal{F}$ and $\tau \in \Upsilon$ such that

$$\tau(d(x, y)) + F(d(fx, fy)) \leq F(d(x, y)), \tag{7}$$

whenever $d(fx, fy) > 0$.

Theorem 9. *Let (X, d) be b-metric space and let $f : X \to X$ be generalized F-contraction. Then one has the following:*

(1) *f maps elements in $\mathcal{H}(X)$ to elements in $\mathcal{H}(X)$.*

(2) *If, for any $A \in \mathcal{H}(X)$,*

$$f(A) = \{f(x) : x \in A\}, \tag{8}$$

then $f : \mathcal{H}(X) \to \mathcal{H}(X)$ is a generalized F-contraction mapping on $(\mathcal{H}(X), H)$.

Proof. As generalized F-contractive mapping is continuous and the image of a compact subset under $f : X \to X$ is compact, we obtain

$$A \in \mathcal{H}(X) \text{ implies } f(A) \in \mathcal{H}(X). \tag{9}$$

To prove (2), let $A, B \in \mathcal{H}(X)$ with $H(f(A), f(B)) \neq \emptyset$. Since $f : X \to X$ is a generalized F-contraction, we obtain

$$0 < d(fx, fy) < d(x, y) \quad \forall x, y \in X, \ x \neq y. \tag{10}$$

Thus we have

$$d(fx, f(B)) = \inf_{y\in B} d(fx, fy) < \inf_{y\in B} d(x, y) = d(x, B). \tag{11}$$

Also

$$d(fy, f(A)) = \inf_{x\in A} d(fy, fx) < \inf_{x\in A} d(y, x) = d(y, A). \tag{12}$$

Now

$$H(f(A), f(B)) = \max\left\{\sup_{x\in A} d(fx, f(B)), \sup_{y\in B} d(fy, f(A))\right\} < \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\} = H(A, B). \tag{13}$$

Strictly increasing F implies

$$F(H(f(A), f(B))) < F(H(A, B)). \tag{14}$$

Consequently, there exists a function $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ with $\liminf_{t\to 0}\tau(t) > 0$ for all $t \geq 0$ such that

$$\tau(H(A, B)) + F(H(f(A), f(B))) \leq F(H(A, B)). \tag{15}$$

Hence $f : \mathcal{H}(X) \to \mathcal{H}(X)$ is a generalized F-contraction. □

Theorem 10. *Let (X, d) be b-metric space and let $\{f_n : n = 1, 2, \ldots, N\}$ be a finite family of generalized F-contraction self-mappings on X. Define $T : \mathcal{H}(X) \to \mathcal{H}(X)$ by*

$$T(A) = f_1(A) \cup f_2(A) \cup \cdots \cup f_N(A) = \bigcup_{n=1}^{N} f_n(A), \quad \text{for each } A \in \mathcal{H}(X). \tag{16}$$

Then T is a generalized F-contraction on $\mathcal{H}(X)$.

Proof. We demonstrate the claim for $N = 2$. Let $f_1, f_2 : X \to X$ be two F-contractions. Take $A, B \in \mathcal{H}(X)$ with $H(T(A), T(B)) \neq 0$. From Lemma 4 (iii), it follows that

$$\begin{aligned}\tau(H(A,B)) + F(H(T(A),T(B))) &= \tau(H(A,B)) \\ &+ F(H(f_1(A)\cup f_2(A), f_1(B)\cup f_2(B))) \\ &\leq \tau(H(A,B)) \\ &+ F(\max\{H(f_1(A), f_1(B)), H(f_2(A), f_2(B))\}) \\ &\leq F(H(A,B)).\end{aligned} \tag{17}$$

□

Definition 11. Let (X, d) be a metric space. A mapping $T : \mathcal{H}(X) \to \mathcal{H}(X)$ is said to be a Ciric type generalized F-contraction if, for $F \in \mathcal{F}$ and $\tau \in \Upsilon$ such that, for any A, $B \in \mathcal{H}(X)$ with $H(T(A), T(B)) \neq 0$, the following holds:

$$\begin{aligned}&\tau(M_T(A,B)) + F(H(T(A),T(B))) \\ &\quad\leq F(M_T(A,B)),\end{aligned} \tag{18}$$

where

$$\begin{aligned}M_T(A,B) = \max\Big\{&H(A,B), H(A,T(A)), \\ &H(B,T(B)), \frac{H(A,T(B)) + H(B,T(A))}{2b}, \\ &H(T^2(A),T(A)), H(T^2(A),B), \\ &H(T^2(A),T(B))\Big\}.\end{aligned} \tag{19}$$

Theorem 12. *Let (X, d) be b-metric space and let $\{f_n : n = 1, 2, \ldots, N\}$ be a finite sequence of generalized F-contraction mappings on X. If $T : \mathcal{H}(X) \to \mathcal{H}(X)$ is defined by*

$$\begin{aligned}T(A) = f_1(A) \cup f_2(A) \cup \cdots \cup f_N(A) = \bigcup_{n=1}^{N} f_n(A), \\ \text{for each } A \in \mathcal{H}(X),\end{aligned} \tag{20}$$

then T is a Ciric type generalized F-contraction mapping on $\mathcal{H}(X)$.

Proof. Using Theorem 10 with property (F_1), the result follows. □

An operator T in above theorem is called Ciric type generalized F-Hutchinson operator.

Definition 13. Let X be a complete b-metric space. If $f_n : X \to X$, $n = 1, 2, \ldots, N$, are generalized F-contraction mappings, then $(X; f_1, f_2, \ldots, f_N)$ is called generalized F-contractive iterated function system (IFS).

Thus generalized F-contractive iterated function system consists of a complete b-metric space and finite family of generalized F-contraction mappings on X.

Definition 14. A nonempty compact set $A \subset X$ is said to be an attractor of the generalized F-contractive IFS if

(a) $T(A) = A$,

(b) there is an open set $U \subseteq X$ such that $A \subseteq U$ and $\lim_{n\to\infty} T^n(B) = A$ for any compact set $B \subseteq U$, where the limit is taken with respect to the Hausdorff metric.

2. Main Results

We start with the following result.

Theorem 15. *Let (X, d) be a complete b-metric space and let $\{X; f_n,\ n = 1, 2, \ldots, k\}$ be a generalized F-contractive iterated function system. Then the following hold:*

(a) *A mapping $T : \mathcal{H}(X) \to \mathcal{H}(X)$ defined by*

$$T(A) = \bigcup_{n=1}^{k} f_n(A), \quad \forall A \in \mathcal{H}(X), \tag{21}$$

is Ciric type generalized F-Hutchinson operator.

(b) *Operator T has a unique fixed point $U \in \mathcal{H}(X)$; that is,*

$$U = T(U) = \bigcup_{n=1}^{k} f_n(U). \tag{22}$$

(c) *For any initial set $A_0 \in \mathcal{H}(X)$, the sequence of compact sets $\{A_0, T(A_0), T^2(A_0), \ldots\}$ converges to a fixed point of T.*

Proof. Part (a) follows from Theorem 12. For parts (b) and (c), we proceed as follows. Let A_0 be an arbitrary element in $\mathcal{H}(X)$. If $A_0 = T(A_0)$, then the proof is finished. So we assume that $A_0 \neq T(A_0)$. Define

$$A_1 = T(A_0), A_2 = T(A_1), \ldots, A_{m+1} = T(A_m) \tag{23}$$

for $m \in \mathbb{N}$.

We may assume that $A_m \neq A_{m+1}$ for all $m \in \mathbb{N}$. If not, then $A_k = A_{k+1}$ for some k implies $A_k = T(A_k)$ and this completes the proof. Take $A_m \neq A_{m+1}$ for all $m \in \mathbb{N}$. From (18), we have

$$\begin{aligned}\tau(M_T(A_m, A_{m+1})) + F(H(A_{m+1}, A_{m+2})) \\ = \tau(M_T(A_m, A_{m+1})) \\ + F(H(T(A_m), T(A_{m+1}))) \\ \leq F(M_T(A_m, A_{m+1})),\end{aligned} \tag{24}$$

where

$$\begin{aligned} M_T(A_m, A_{m+1}) = \max\Big\{ & H(A_m, A_{m+1}), \\ & H(A_m, T(A_m)), H(A_{m+1}, T(A_{m+1})), \\ & \frac{H(A_m, T(A_{m+1})) + H(A_{m+1}, T(A_m))}{2b}, \\ & H(T^2(A_m), T(A_m)), H(T^2(A_m), A_{m+1}), \\ & H(T^2(A_m), T(A_{m+1}))\Big\} = \max\Big\{H(A_m, A_{m+1}), \\ & H(A_m, A_{m+1}), H(A_{m+1}, A_{m+2}), \\ & \frac{H(A_m, A_{m+2}) + H(A_{m+1}, A_{m+1})}{2b}, \\ & H(A_{m+2}, A_{m+1}), H(A_{m+2}, A_{m+1}), \\ & H(A_{m+2}, A_{m+2})\Big\} = \max\{H(A_m, A_{m+1}), \\ & H(A_{m+1}, A_{m+2})\}. \end{aligned} \tag{25}$$

In case $M_T(A_m, A_{m+1}) = H(A_{m+1}, A_{m+2})$, we have

$$F(H(A_{m+1}, A_{m+2})) \le F(H(A_{m+1}, A_{m+2})) - \tau(H(A_{m+1}, A_{m+2})), \tag{26}$$

a contradiction as $\tau(H(A_{m+1}, A_{m+2})) > 0$. Therefore $M_T(A_m, A_{m+1}) = H(A_m, A_{m+1})$ and we have

$$\begin{aligned} F(H(A_{m+1}, A_{m+2})) &\le F(H(A_m, A_{m+1})) - \tau(H(A_m, A_{m+1})) \\ &< F(H(A_m, A_{m+1})). \end{aligned} \tag{27}$$

Thus $\{H(A_{m+1}, A_{m+2})\}$ is decreasing and hence convergent. We now show that $\lim_{m\to\infty} H(A_{m+1}, A_{m+2}) = 0$. By property of τ, there exists $c > 0$ with $n_0 \in \mathbb{N}$ such that $\tau(H(A_m, A_{m+1})) > c$ for all $m \ge n_0$. Note that

$$\begin{aligned} F(H(A_{m+1}, A_{m+2})) &\le F(H(A_m, A_{m+1})) \\ &- \tau(H(A_m, A_{m+1})) \le F(H(A_{m-1}, A_m)) \\ &- \tau(H(A_{m-1}, A_m)) - \tau(H(A_m, A_{m+1})) \le \cdots \\ &\le H(A_0, A_1) - [\tau(H(A_0, A_1)) \\ &+ \tau(H(A_1, A_2)) + \cdots + \tau(H(A_m, A_{m+1}))] \\ &\le F(H(A_0, A_1)) - n_0 \end{aligned} \tag{28}$$

gives $\lim_{m\to\infty} F(H(A_{m+1}, A_{m+2})) = -\infty$ which together with (F_2) implies that $\lim_{m\to\infty} H(A_{m+1}, A_{m+2}) = 0$. By (F_3), there exists $h \in (0,1)$ such that

$$\lim_{n\to\infty} [H(A_{m+1}, A_{m+2})]^h F(H(A_{m+1}, A_{m+2})) = 0. \tag{29}$$

Thus we have

$$\begin{aligned} & [H(A_m, A_{m+1})]^h F(H(A_m, A_{m+1})) \\ &\quad - [H(A_m, A_{m+1})]^h F(H(A_0, A_1)) \\ &\quad \le [H(A_m, A_{m+1})]^h (F(H(A_0, A_1) - n_0)) \\ &\quad - [H(A_m, A_{m+1})]^h F(H(A_0, A_1)) \\ &\quad \le -n_0 [H(A_m, A_{m+1})]^h \le 0. \end{aligned} \tag{30}$$

On taking limit as $n \to \infty$ we obtain that $\lim_{m\to\infty} m[H(A_{m+1}, A_{m+2})]^h = 0$. Hence $\lim_{m\to\infty} m^{1/h} H(A_{m+1}, A_{m+2}) = 0$. There exists $n_1 \in \mathbb{N}$ such that $m^{1/h} H(A_{m+1}, A_{m+2}) \le 1$ for all $m \ge n_1$ and hence $H(A_{m+1}, A_{m+2}) \le 1/m^{1/h}$ for all $m \ge n_1$. For $m, n \in \mathbb{N}$ with $m > n \ge n_1$, we have

$$\begin{aligned} H(A_n, A_m) &\le H(A_n, A_{n+1}) + H(A_{n+1}, A_{n+2}) + \cdots \\ &+ H(A_{m-1}, A_m) \le \sum_{i=n}^{\infty} \frac{1}{i^{1/h}}. \end{aligned} \tag{31}$$

By the convergence of the series $\sum_{i=1}^{\infty} (1/i^{1/h})$, we get $H(A_n, A_m) \to 0$ as $n, m \to \infty$. Therefore $\{A_n\}$ is a Cauchy sequence in X. Since $(\mathscr{H}(X), d)$ is complete, we have $A_n \to U$ as $n \to \infty$ for some $U \in \mathscr{H}(X)$.

In order to show that U is the fixed point of T, we on the contrary assume that Pompeiu-Hausdorff weight assigned to U and $T(U)$ is not zero. Now

$$\begin{aligned} & \tau(M_T(A_n, U)) + F(H(A_{n+1}, T(U))) \\ &\quad = \tau + F(H(T(A_n), T(U))) \le F(M_T(A_n, U)), \end{aligned} \tag{32}$$

where

$$\begin{aligned} M_T(A_n, U) = \max\Big\{ & H(A_n, U), H(A_n, T(A_n)), \\ & H(U, T(U)), \frac{H(A_n, T(U)) + H(U, T(A_n))}{2b}, \\ & H(T^2(A_n), T(A_n)), H(T^2(A_n), U), \\ & H(T^2(A_n), T(U))\Big\} = \max\Big\{H(A_n, U), \\ & H(A_n, A_{n+1}), H(U, T(U)), \\ & \frac{H(A_n, T(U)) + H(U, A_{n+1})}{2b}, H(A_{n+2}, A_{n+1}), \\ & H(A_{n+2}, U), H(A_{n+2}, T(U))\Big\}. \end{aligned} \tag{33}$$

Now we consider the following cases:

(1) If $M_T(A_n, U) = H(A_n, U)$, then, on taking lower limit as $n \to \infty$ in (32), we have

$$\begin{aligned} & \liminf_{n\to\infty} \tau(H(A_n, U)) + F(H(T(U), U)) \\ &\quad \le F(H(U, U)), \end{aligned} \tag{34}$$

a contradiction as $\liminf_{t\to 0} \tau(t) > 0$ for all $t \ge 0$.

(2) When $M_T(A_n, U) = H(A_n, A_{n+1})$, then, by taking lower limit as $n \to \infty$, we obtain

$$\liminf_{n\to\infty} \tau\left(H\left(A_n, A_{n+1}\right)\right) + F\left(H\left(T\left(U\right), U\right)\right) \leq F\left(H\left(U, U\right)\right), \quad (35)$$

which gives a contradiction.

(3) In case $M_T(A_n, U) = H(U, T(U))$, we get

$$\tau\left(H\left(U, T\left(U\right)\right)\right) + F\left(H\left(T\left(U\right), U\right)\right) \leq F\left(H\left(U, T\left(U\right)\right)\right), \quad (36)$$

a contradiction as $\tau(H(U, T(U))) > 0$.

(4) If $M_T(A_n, U) = (H(A_n, T(U)) + H(U, A_{n+1}))/2b$, then, on taking lower limit as $n \to \infty$, we have

$$\begin{aligned} &\liminf_{n\to\infty} \tau\left(\frac{H\left(A_n, T\left(U\right)\right) + H\left(U, A_{n+1}\right)}{2b}\right) \\ &\quad + F\left(H\left(T\left(U\right), U\right)\right) \\ &\quad \leq F\left(\frac{H\left(U, T\left(U\right)\right) + H\left(U, U\right)}{2b}\right) \\ &\quad = F\left(\frac{H\left(U, T\left(U\right)\right)}{2b}\right), \end{aligned} \quad (37)$$

a contradiction as F is strictly increasing map.

(5) When $M_T(A_n, U) = H(A_{n+2}, A_{n+1})$, then

$$\liminf_{n\to\infty} \tau\left(H\left(A_{n+2}, A_{n+1}\right)\right) + F\left(H\left(T\left(U\right), U\right)\right) \leq F\left(H\left(U, U\right)\right), \quad (38)$$

which gives a contradiction.

(6) In case $M_T(A_n, U) = H(A_{n+2}, U)$, then, on taking lower limit as $n \to \infty$ in (32), we get

$$\liminf_{n\to\infty} \tau\left(H\left(A_{n+2}, U\right)\right) + F\left(H\left(T\left(U\right), U\right)\right) \leq F\left(H\left(U, U\right)\right), \quad (39)$$

a contradiction.

(7) Finally if $M_T(A_n, U) = H(A_{n+2}, T(U))$, then, on taking lower limit as $n \to \infty$, we have

$$\liminf_{n\to\infty} \tau\left(H\left(A_{n+2}, T\left(U\right)\right)\right) + F\left(H\left(T\left(U\right), U\right)\right) \leq F\left(H\left(U, T\left(U\right)\right)\right), \quad (40)$$

a contradiction.

Thus, U is the fixed point of T.

To show the uniqueness of fixed point of T, assume that U and V are two fixed points of T with $H(U, V)$ being not zero. Since T is F-contraction map, we obtain that

$$\begin{aligned} &\tau\left(M_T\left(U, V\right)\right) + F\left(H\left(U, V\right)\right) \\ &\quad = \tau\left(M_T\left(U, V\right)\right) + F\left(H\left(T\left(U\right), T\left(V\right)\right)\right) \\ &\quad \leq F\left(M_T\left(U, V\right)\right), \end{aligned} \quad (41)$$

where

$$\begin{aligned} M_T\left(U, V\right) = \max\Big\{&H\left(U, V\right), H\left(U, T\left(U\right)\right), \\ &H\left(V, T\left(V\right)\right), \frac{H\left(U, T\left(V\right)\right) + H\left(V, T\left(U\right)\right)}{2b}, \\ &H\left(T^2\left(U\right), U\right), H\left(T^2\left(U\right), V\right), H\left(T^2\left(U\right), T\left(V\right)\right)\Big\} \\ = \max\Big\{&H\left(U, V\right), H\left(U, U\right), H\left(V, V\right), \\ &\frac{H\left(U, V\right) + H\left(V, U\right)}{2b}, H\left(U, U\right), H\left(U, V\right), H\left(U, V\right)\Big\} \\ = H&\left(U, V\right); \end{aligned} \quad (42)$$

that is,

$$\tau\left(H\left(U, V\right)\right) + F\left(H\left(U, V\right)\right) \leq F\left(H\left(U, V\right)\right), \quad (43)$$

a contradiction as $\tau(H(U, V)) > 0$. Thus T has a unique fixed point $U \in \mathscr{H}(X)$. □

Remark 16. In Theorem 15, if we take $\mathcal{S}(X)$ the collection of all singleton subsets of X, then clearly $\mathcal{S}(X) \subseteq \mathscr{H}(X)$. Moreover, consider $f_n = f$ for each n, where $f = f_i$ for any $i \in \{1, 2, 3, \ldots, k\}$; then the mapping T becomes

$$T\left(x\right) = f\left(x\right). \quad (44)$$

With this setting we obtain the following fixed point result.

Corollary 17. *Let (X, d) be a complete b-metric space and let $\{X : f_n, n = 1, 2, \ldots, k\}$ be a generalized iterated function system. Let $f : X \to X$ be a mapping defined as in Remark 16. If there exist some $F \in \digamma$ and $\tau \in \Upsilon$ such that, for any $x, y \in \mathscr{H}(X)$ with $d(f(x), f(y)) \neq 0$, the following holds:*

$$\tau\left(M_f\left(x, y\right)\right) + F\left(d\left(fx, fy\right)\right) \leq F\left(M_f\left(x, y\right)\right), \quad (45)$$

where

$$\begin{aligned} M_T\left(x, y\right) = \max\Big\{&d\left(x, y\right), d\left(x, fx\right), d\left(y, fy\right), \\ &\frac{d\left(x, fy\right) + d\left(y, fx\right)}{2b}, d\left(f^2x, y\right), d\left(f^2x, fx\right), \\ &d\left(f^2x, fy\right)\Big\}, \end{aligned} \quad (46)$$

then f has a unique fixed point in X. Moreover, for any initial set $x_0 \in X$, the sequence of compact sets $\{x_0, fx_0, f^2x_0, \ldots\}$ converges to a fixed point of f.

Corollary 18. *Let (X, d) be a complete b-metric space and let $(X; f_n,\ n = 1, 2, \ldots, k)$ be iterated function system where each f_i for $i = 1, 2, \ldots, k$ is a contraction self-mapping on X. Then $T : \mathscr{H}(X) \to \mathscr{H}(X)$ defined in Theorem 15 has a unique fixed point in $\mathscr{H}(X)$. Furthermore, for any set $A_0 \in \mathscr{H}(X)$, the sequence of compact sets $\{A_0, T(A_0), T^2(A_0), \ldots\}$ converges to a fixed point of T.*

Proof. It follows from Theorem 10 that if each f_i for $i = 1, 2, \ldots, k$ is a contraction mapping on X, then the mapping $T : \mathscr{H}(X) \to \mathscr{H}(X)$ defined by

$$T(A) = \bigcup_{n=1}^{k} f_n(A), \quad \forall A \in \mathscr{H}(X), \tag{47}$$

is contraction on $\mathscr{H}(X)$. Using Theorem 15, the result follows. □

Corollary 19. *Let (X, d) be a complete b-metric space and let $(X; f_n,\ n = 1, 2, \ldots, k)$ be an iterated function system. Suppose that each f_i for $i = 1, 2, \ldots, k$ is a mapping on X satisfying*

$$d(f_i x, f_i y)\, e^{d(f_i x, f_i y) - d(x,y)} \le e^{-\tau(d(x,y))} d(x, y), \tag{48}$$

for all $x, y \in X$, $f_i x \ne f_i y$, where $\tau \in \Upsilon$. Then the mapping $T : \mathscr{H}(X) \to \mathscr{H}(X)$ defined in Theorem 15 has a unique fixed point in $\mathscr{H}(X)$. Furthermore, for any set $A_0 \in \mathscr{H}(X)$, the sequence of compact sets $\{A_0, T(A_0), T^2(A_0), \ldots\}$ converges to a fixed point of T.

Proof. Take $F(\lambda) = \ln(\lambda) + \lambda$, $\lambda > 0$ in Theorem 10; then each mapping f_i for $i = 1, 2, \ldots, k$ on X satisfies

$$d(f_i x, f_i y)\, e^{d(f_i x, f_i y) - d(x,y)} \le e^{-\tau(d(x,y))} d(x, y), \tag{49}$$

for all $x, y \in X$, $f_i x \ne f_i y$, where $\tau \in \Upsilon$. Again from Theorem 10, the mapping $T : \mathscr{H}(X) \to \mathscr{H}(X)$ defined by

$$T(A) = \bigcup_{n=1}^{k} f_n(A), \quad \forall A \in \mathscr{H}(X), \tag{50}$$

satisfies

$$H(T(A), T(B))\, e^{H(T(A),T(B)) - H(A,B)} \le e^{-\tau} H(A, B), \tag{51}$$

for all $A, B \in \mathscr{H}(X)$ and $H(T(A), T(B)) \ne 0$. Using Theorem 15, the result follows. □

Corollary 20. *Let (X, d) be a complete b-metric space and let $(X; f_n,\ n = 1, 2, \ldots, k)$ be iterated function system. Suppose that each f_i for $i = 1, 2, \ldots, k$ is a mapping on X satisfying*

$$\begin{aligned} &d(f_i x, f_i y)(d(f_i x, f_i y) + 1) \\ &\quad \le e^{-\tau(d(x,y))} d(x, y)(d(x, y) + 1), \end{aligned} \tag{52}$$

for all $x, y \in X$, $f_i x \ne f_i y$, where $\tau \in \Upsilon$. Then the mapping $T : \mathscr{H}(X) \to \mathscr{H}(X)$ defined in Theorem 15 has a unique fixed point in $\mathscr{H}(X)$. Furthermore, for any set $A_0 \in \mathscr{H}(X)$, the sequence of compact sets $\{A_0, T(A_0), T^2(A_0), \ldots\}$ converges to a fixed point of T.

Proof. By taking $F(\lambda) = \ln(\lambda^2 + \lambda) + \lambda$, $\lambda > 0$, in Theorem 10, we obtain that each mapping f_i for $i = 1, 2, \ldots, k$ on X satisfies

$$\begin{aligned} &d(f_i x, f_i y)(d(f_i x, f_i y) + 1) \\ &\quad \le e^{-\tau(d(x,y))} d(x, y)(d(x, y) + 1), \end{aligned} \tag{53}$$

for all $x, y \in X$, $f_i x \ne f_i y$, where $\tau \in \Upsilon$. Again it follows from Theorem 10 that the mapping $T : \mathscr{H}(X) \to \mathscr{H}(X)$ defined by

$$T(A) = \bigcup_{n=1}^{k} f_n(A), \quad \forall A \in \mathscr{H}(X), \tag{54}$$

satisfies

$$\begin{aligned} &H(T(A), T(B))(H(T(A), T(B)) + 1) \\ &\quad \le e^{-\tau(H(A,B))} H(A, B)(H(A, B) + 1), \end{aligned} \tag{55}$$

for all $A, B \in \mathscr{H}(X)$, $H(T(A), T(B)) \ne 0$. Using Theorem 15, the result follows. □

Corollary 21. *Let (X, d) be a complete b-metric space and let $(X; f_n,\ n = 1, 2, \ldots, k)$ be iterated function system. Suppose that each f_i for $i = 1, 2, \ldots, k$ is a mapping on X satisfying*

$$d(f_i x, f_i y) \le \frac{1}{\left(1 + \tau(d(x, y))\sqrt{d(x, y)}\right)^2} d(x, y), \tag{56}$$

for all $x, y \in X$, $f_i x \ne f_i y$, where $\tau \in \Upsilon$. Then the mapping $T : \mathscr{H}(X) \to \mathscr{H}(X)$ defined in Theorem 15 has a unique fixed point $\mathscr{H}(X)$. Furthermore, for any set $A_0 \in \mathscr{H}(X)$, the sequence of compact sets $\{A_0, T(A_0), T^2(A_0), \ldots\}$ converges to a fixed point of T.

Proof. Take $F(\lambda) = -1/\sqrt{\lambda}$, $\lambda > 0$, in Theorem 10, and then each mapping f_i for $i = 1, 2, \ldots, k$ on X satisfies

$$\begin{aligned} &d(f_i x, f_i y) \le \frac{1}{\left(1 + \tau(d(x, y))\sqrt{d(x, y)}\right)^2} d(x, y), \\ &\qquad \forall x, y \in X,\ f_i x \ne f_i y, \end{aligned} \tag{57}$$

where $\tau \in \Upsilon$. Again it follows from Theorem 10 that the mapping $T : \mathscr{H}(X) \to \mathscr{H}(X)$ defined by

$$T(A) = \bigcup_{n=1}^{k} f_n(A), \quad \forall A \in \mathscr{H}(X), \tag{58}$$

satisfies

$$\begin{aligned} &H(T(A), T(B)) \\ &\quad \le \frac{1}{\left(1 + \tau(H(A, B))\sqrt{H(A, B)}\right)^2} H(A, B), \end{aligned} \tag{59}$$

for all $A, B \in \mathscr{H}(X)$, $H(T(A), T(B)) \ne 0$. Using Theorem 15, the result follows. □

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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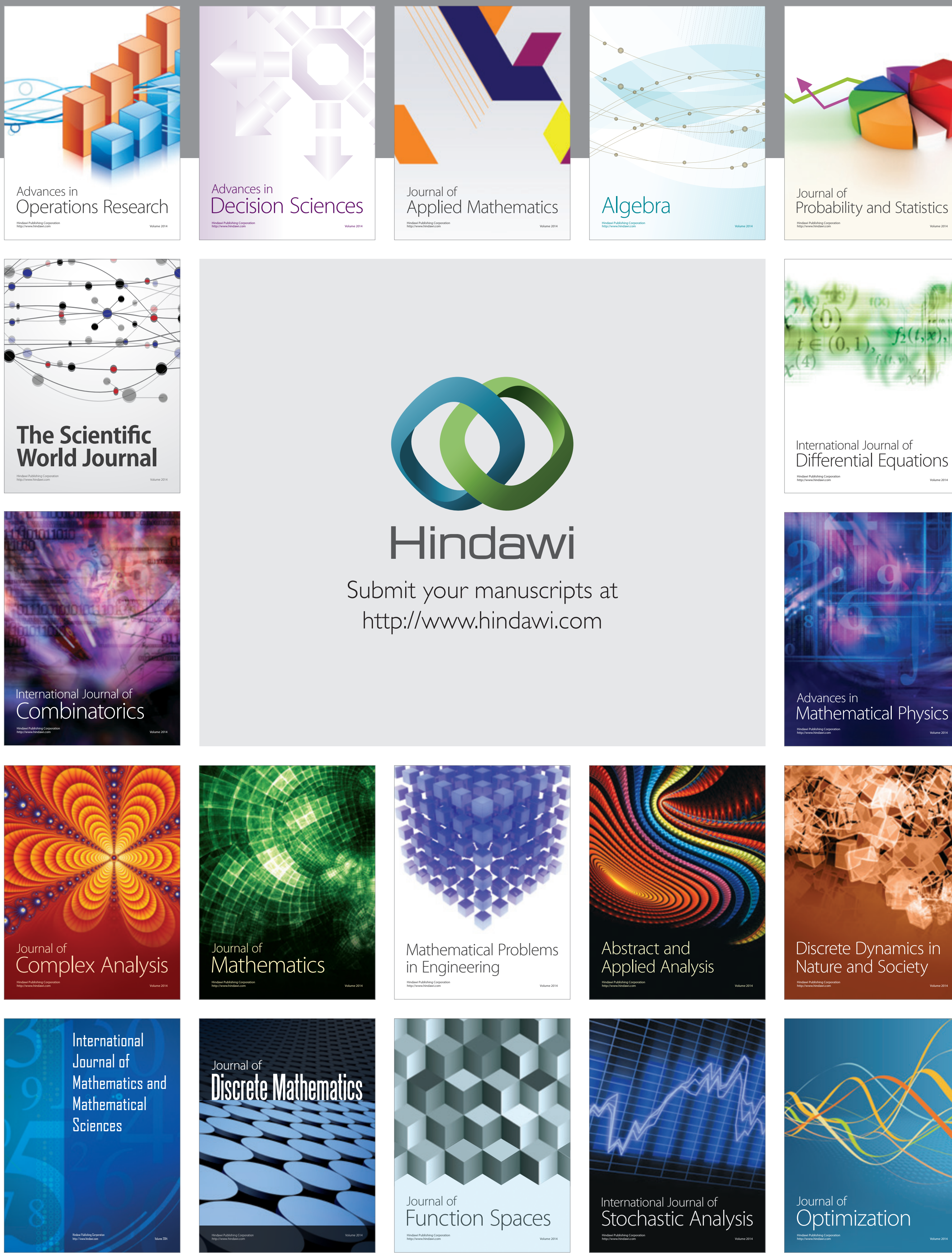
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