Research Article

Optimal Multiplicative Generalized Linear Search Plan for a Discrete Random Walker

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This paper formulates a search model that gives the optimal search plan for the problem of finding a discrete random walk target in minimum time. The target moves through one of n -disjoint real lines in $\mathbb{R}^n$: we have n -searchers starting the searching process for the target from any point rather than the origin. We find the conditions that make the expected value of the first meeting time between one of the searchers and the target finite. Furthermore, we show the existence of the optimal search plan that minimizes the expected value of the first meeting time and find it. The effectiveness of this model is illustrated using numerical example.

1. Introduction

The search problem for a randomly moving target has a remarkable importance in our life due to its great applicability. This problem is very interesting because it may arise in many real world situations such as searching for randomly moving persons or targets on roads. That mathematical analysis collected all the researches derived from searching applications of the II World War. They solved with beauty two complementary objectives of the search: find the target with (1) the smallest cost and (2) in minimum time. Readers are referred to Koopman [1] for the early works and to Benkoski et al. [2] and Frost and Stone [3] for a more recent survey.

In the linear search problem, the target moves on the real line according to a known random process, and its initial position is given by the value of a random variable X_0 which has known probability distribution function. A searcher starts looking for the target at a point H_0 ($|H_0| < \infty$). The searcher moves continuously along the line in both directions of the starting point H_0 until the target is met. The searcher would change its direction at suitable points many times before meeting its goal. Thus, we consider that the path length which we represented is the cost of the search. The aim

of the searcher is to minimize E_{τ_ϕ} , that is, the expected value of the first meeting time τ_ϕ between the searcher and the target. The problem is to find a search plan $\phi(t)$, such that $E_{\tau_\phi} < \infty$; in this case, we call that $\phi(t)$ is a finite search plan and if $E_{\tau_{\phi^*}} < E_{\tau_\phi}$ for all $\phi(t) \in \Phi(t)$, where $\Phi(t)$ terms to the class of all search plans then we call that $\phi^*(t)$ is an optimal search plan.

MC Cabe [4] found a finite search plan for a one-dimensional random walk target when the searcher starts the search from the origin, and the initial position of the target has a standard normal distribution. Mohamed [5] discussed the existence of a finite search plan for a one-dimensional random walk target in general case which means that the search may start from any point on the real line, and the initial position of the target has any distribution. El-Rayes and Mohamed [6] have shown the existence of a search plan which minimizes the expected value of the first meeting time between the searcher and the randomly moving target with imposed conditions. Fristedt and Heath [7] derived the conditions for optimal search path which minimizes the cost of effort of finding a randomly moving target on the real line. Ohsumi [8] presented an optimal search plan for a target

Let I be a set of integer numbers and I^+ a nonnegative part of I . We also assume that $\{X_{ij}\}_{j \geq 1}$ are sequences of independent identically distributed random variables in L_i , $i = 1, 2, \dots, n$, respectively. In addition, we let a value of “-1” indicate a step to the left and a value “1” a step to the right, so that, for any $j \geq 1$, we have the n -dimensional probability vectors, $[P(X_{1j} = 1) = p_1, P(X_{2j} = 1) = p_2, \dots, P(X_{nj} = 1) = p_n]$ and $[P(X_{1j} = -1) = q_1, P(X_{2j} = -1) = q_2, \dots, P(X_{nj} = -1) = q_n]$, where $[p_1, p_2, \dots, p_n] + [q_1, q_2, \dots, q_n] = [1, 1, \dots, 1]$.

Supposing that, for $t > 0$, $t \in I^+$, $W(t) = \sum_{j=1}^t X_{ij}$, $W(0) = 0$ and X_0 is a random variable that represent the target initial position, valued in $2I$ or $(2I + 1)$ and independent of $W(t)$, $t \geq 0$; if $\xi_i > 0$ and X_{ij} such that $0 \leq K_{i1} = (\xi_i + x_i)/2 \leq \xi_i$, where K_{i1} is an integer, then

$$P(W(\xi_i) = K_{i1}) = \begin{cases} \binom{\xi_i}{K_{i1}} p_i^{K_{i1}} q_i^{\xi_i - K_{i1}}, \\ 0, \end{cases} \quad \text{if } K_{i1} \text{ does not exist.} \quad (1)$$

The target is assumed to move randomly on one of n -disjoint real lines according to the process $\{W(t), t \in I^+\}$, where I^+ is the set of positive integer numbers and $W(t)$ is a one-dimensional random walk motion. The initial position of the target is unknown but the searchers know its probability distribution (i.e., the probability distribution of the target is given at time 0) and the process $\{W(t), t \in I^+\}$ which controls the target's motion.

Figure 1 gives an illustration of the search plan that the n -searchers S_i , $i = 1, 2, \dots, n$, follow it. Moreover, it is to be noted that this search plan $\phi(t) = (\phi_1(t), \phi_2(t), \dots, \phi_n(t))$ is a combination of continuous functions $\phi_i(t)$ with speed v_i and given by $\phi_i(t) : I^+ \rightarrow I$, $i = 1, 2, \dots, n$, such that

$$|\phi_i(t_1) - \phi_i(t_2)| < v_i |t_1 - t_2| \quad \forall t_1, t_2 \in I^+, \quad (2)$$

$$\phi_i(0) = 0, \quad i = 1, 2, \dots, n.$$

The first meeting time τ_ϕ is a random variable valued in I^+ , and it is defined by

$$\tau_\phi = \begin{cases} \inf \{t : \text{either one of } \phi_i(t) = X_0 + W(t), \\ i = 1, 2, \dots, n\}, \\ \infty, \end{cases} \quad \text{if the set is empty,} \quad (3)$$

where $X_0 \in I$, that is, a random variable independent of $W(t)$, and it represents the initial position of the target. We suppose that $X_0 = X_i$ if the target moves on L_i , $i = 1, 2, \dots, n$. Also assume that the set of all search plans of the searchers S_i with speeds v_i , $i = 1, 2, \dots, n$, respectively, satisfying condition (2) be $\Phi_{v_i}(t)$, $i = 1, 2, \dots, n$. The problem is to find a search plan $\phi(t) = (\phi_1(t), \phi_2(t), \dots, \phi_n(t)) \in \Phi(t)$ for the n -searchers such that $E_{\tau_\phi} < \infty$, where E_{τ_ϕ} is the expected value of the first meeting time between one of the searchers and the target, and $\Phi(t) = \{(\phi_1(t), \phi_2(t), \dots, \phi_n(t)); \phi_i(t) \in \Phi_{v_i}(t), i = 1, 2, \dots, n\}$, is called a class of all sets of the search plans.

Assuming that ϑ_i, λ_i , $i = 1, 2, \dots, n$, are positive numbers such that $\vartheta_i > 1$, $C_i = (\vartheta_i - 1)/(\vartheta_i + 1) > |p_i - q_i|$, $\lambda_i = b\bar{\omega}_i$, $b = 1, 2, \dots$, and $\bar{\omega}_i(1 - C_i)/2$ is positive number also. In this problem, we assume that $v_i = 1$, $i = 1, 2, \dots, n$. In addition, we define the sequences $\{G_{ij}\}$, $\{d_{ij}\}$, and $\{H_{ij}\}$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots$ by $G_{ij} = \lambda_i(\vartheta_i^j - 1)$, $d_{ij} = (-1)^{j+1}C_i[G_{ij} + 1 + (-1)^{j+1}]$, and $H_{ij} = d_{ij} + \delta_{i0}$. We also use the notations $\psi_i(G_{i(2j+1)}) = W(G_{i(2j+1)}) - K_{i1}(G_{i(2j+1)})$, $\hat{\psi}_i(G_{i(2j)}) = W(G_{i(2j)}) + K_{i2}(G_{i(2j)})$, where $K_{i1}(G_{i(2j+1)}) = C_i G_{i(2j+1)}$, $K_{i2}(G_{i(2j)}) = C_i G_{i(2j)}$ which are positive functions and the following remark.

Remark 1. If $0 < \kappa_\$ < 1$, $\$ = 1, 2, \dots, \kappa$, then $\prod_{\$=1}^\kappa \kappa_\$ < \sum_{\$=1}^\kappa \kappa_\$$.

There are known probability measures γ_i , $i = 1, 2, \dots, n$, such that $\gamma_1 + \gamma_2 + \dots + \gamma_n = 1$ on $L_1 \cup L_2 \cup \dots \cup L_n$. And, they describe the location of the target, where γ_i is induced by the position of the target on L_i .

The objective is to obtain the conditions that make the search plan $\phi(t) = (\phi_1(t), \phi_2(t), \dots, \phi_n(t))$ be finite (i.e., $E_{\tau_\phi} < \infty$). We also need to show the existence of the optimal search plan $\phi^*(t) = (\phi_1^*(t), \phi_2^*(t), \dots, \phi_n^*(t))$ and find it, that is, to give the minimum expected value of the first meeting time.

3. Properties of the Search Model

In this section we start to discuss some properties that the search model for a one-dimensional random walk target should satisfy.

The searcher S_i would conduct its search as in the above manner that is detailed in Section 2. Consequently, for any line L_i , $i = 1, 2, \dots, n$, $t \in I^+$, we have the following.

Case 1. If we consider that $\mathbb{Q}$ is a set of positive even numbers such that $\mathbb{Q} = \{2, 4, 6, \dots, \hat{n}\}$, for $\mathcal{F} \in \mathbb{Q}$, $j \in I^+$, then we have

$$\begin{aligned} \dots < H_{i(j+2)} < 0 < H_{ij} < H_{i(j-2)} < \dots < H_{i2} < \delta_{i0} \\ < H_{i1} < H_{i3} < \dots; \end{aligned} \quad (4)$$

for any $t \in I^+$, if $G_{i(2j-1)} \leq t \leq G_{i(2j)}$, $1 \leq j \leq \mathcal{F}/2$, then

$$\phi_i(t) = H_{i(2j+1)} - \delta_{i0} - [t - G_{i(2j-1)}]; \quad (5)$$

if $G_{ij} \leq t \leq G_{i(j+1)}$, $j \geq \mathcal{F} + 1$, then

$$\phi_i(t) = H_{ij} + (-1)^j [t - G_{ij}]; \quad (6)$$

if $G_{i(2j)} \leq t \leq G_{i(2j+1)}$, $1 \leq j \leq \mathcal{F}/2$, then

$$\phi_i(t) = \delta_{i0} - H_{i(2j)} + [t - G_{i(2j)}]. \quad (7)$$

Case 2. Also, if O is a set of positive odd numbers such that $O = \{1, 3, 5, \dots, \hat{m}\}$, for $\mathcal{F} \in O$, $j \in I^+$, then we have

$$\begin{aligned} \dots < H_{i2} < \delta_{i0} < H_{i1} < H_{i3} < \dots < H_{i(j-2)} < H_{ij} \\ < 0 < H_{i(j+2)} < \dots; \end{aligned} \quad (8)$$

for any $t \in I^+$, if $G_{i(2j-1)} \leq t \leq G_{i(2j)}$, $1 \leq j \leq (\mathcal{F} + 1)/2$, then

$$\phi_i(t) = \delta_{i0} - H_{i(2j-1)} - [t - G_{i(2j-1)}]; \quad (9)$$

if $G_{ij} \leq t \leq G_{i(j+1)}$, $j \geq \mathcal{F} + 1$, then

$$\phi_i(t) = H_{ij} + (-1)^j [t - G_{ij}]; \quad (10)$$

if $G_{i(2j)} \leq t \leq G_{i(2j+1)}$, $1 \leq j \leq (\mathcal{F} - 1)/2$, then

$$\phi_i(t) = H_{i(2j)} - \delta_{i0} + [t - G_{i(2j)}]. \quad (11)$$

It is clear that the search path of S_i depends on λ_i , ϑ_i and $j \in I^+$. Since $X_0 \in I$ then the first meeting time is done at $x_i \in I$ where x_i is an integer number. For this reason we will prove the conditions (properties) that make S_i meet the target at x_i as in the following theorems.

Theorem 2. If C_i is a rational number different from 1 to -1 and, for all $\xi_i \geq 1$, $V(\xi_i) = (W(\xi_i \vartheta_i) - C_i \xi_i \vartheta_i)/2$ then

- (i) there exists a sequence $\{Y_{ij}\}_{j \geq 1}$ of independent identically distributed random variables such that $V(\xi_i) = \sum_{j=1}^{\mathcal{F}} Y_{ij}$, and the distribution of Y_{ij} is concentrated on the integers $E(Y_{ij}) = (\vartheta_i/2)[E(X_{ij}) - C_i]$ and the probability vector $[p(Y_{1j} = \mathcal{F}) > 0, p(Y_{2j} = \mathcal{F}) > 0, \dots, p(Y_{nj} = \mathcal{F}) > 0]$, if and only if $-\vartheta_i(1 + C_i)/2 \leq \mathcal{F} \leq \vartheta_i(1 - C_i)/2$.
- (ii) The probability vector, $[p(V(\xi_1) = x_1) > 0, p(V(\xi_2) = x_2) > 0, \dots, p(V(\xi_n) = x_n) > 0]$, holds if and only if x_i is an integer such that $-\xi_i \vartheta_i(1 + C_i)/2 \leq x_i \leq \xi_i \vartheta_i(1 - C_i)/2$.
- (iii) If $C_i \neq p_i - q_i$ then there exist constants r_{i1} and r_{i2} depending on C_i , p_i such that, for any $x_i \in \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, if $\xi_i > r_{i1}x_i + r_{i2}$ and if $x_i \geq 0$ we have $[p(0 \leq V(\xi_1 + 1) \leq x_1), p(0 \leq V(\xi_2 + 1) \leq x_2), \dots, p(0 \leq V(\xi_n + 1) \leq x_n)] \leq [p(0 \leq V(\xi_1) \leq x_1), p(0 \leq V(\xi_2) \leq x_2), \dots, p(0 \leq V(\xi_n) \leq x_n)]$, and if $x_i < 0$ we have, $[p(x_1 \leq V(\xi_1 + 1) < 0), p(x_2 \leq V(\xi_2 + 1) < 0), \dots, p(x_n \leq V(\xi_n + 1) < 0)] \leq [p(x_1 \leq V(\xi_1) < 0), p(x_2 \leq V(\xi_2) < 0), \dots, p(x_n \leq V(\xi_n) < 0)]$.

Proof. (i) Define $Y_{ij} = \sum_{\mathcal{F}=1}^{\vartheta_i} ((X_{i(\mathcal{F}+(j-1)\vartheta_i)} - C_i)/2)$, $j \geq 1$, and $[p(Y_{1j} = x_1), p(Y_{2j} = x_2), \dots, p(Y_{nj} = x_n)] = [p(W(\vartheta_1)) = 2x_1 + \vartheta_1 C_1, p(W(\vartheta_2)) = 2x_2 + \vartheta_2 C_2, \dots, p(W(\vartheta_n)) = 2x_n + \vartheta_n C_n]$.

Consequently, if $[p(Y_{1j} = x_1) > 0, p(Y_{2j} = x_2) > 0, \dots, p(Y_{nj} = x_n) > 0]$, then from (1) we have $x_i + \vartheta_i(1 + C_i)/2$ that is an integer, and since x_i is an integer also then we have $0 \leq x_i + \vartheta_i(1 + C_i)/2 \leq \vartheta_i$. Therefore, $-\vartheta_i(1 + C_i)/2 \leq x_i \leq \vartheta_i(1 - C_i)/2$. If $x_i = \mathcal{F}$ then we have $[p(Y_{1j} = \mathcal{F}) > 0, p(Y_{2j} = \mathcal{F}) > 0, \dots, p(Y_{nj} = \mathcal{F}) > 0]$, if and only if $-\vartheta_i(1 + C_i)/2 \leq \mathcal{F} \leq \vartheta_i(1 - C_i)/2$. In addition, by using (1), $E(\sum_{\mathcal{F}=1}^{\vartheta_i} ((X_{i(\mathcal{F}+(j-1)\vartheta_i)} - C_i)/2)) = (\vartheta_i/2)[E(X_{ij}) - C_i]$ is proved.

(ii) Since $V(\xi_i) = (W(\xi_i \vartheta_i) - C_i \xi_i \vartheta_i)/2$ then we have $[p(V(\xi_1) = x_1), p(V(\xi_2) = x_2), \dots, p(V(\xi_n) = x_n)] = [p((W(\xi_1 \vartheta_1) - C_1 \xi_1 \vartheta_1)/2 = x_1), p((W(\xi_2 \vartheta_2) - C_2 \xi_2 \vartheta_2)/2 = x_2), \dots, p((W(\xi_n \vartheta_n) - C_n \xi_n \vartheta_n)/2 = x_n)] = [p(W(\xi_1 \vartheta_1)) = 2x_1 + C_1 \xi_1 \vartheta_1, p(W(\xi_2 \vartheta_2)) = 2x_2 + C_2 \xi_2 \vartheta_2, \dots, p(W(\xi_n \vartheta_n)) = 2x_n + C_n \xi_n \vartheta_n]$, and using (1) the prove is completed.

(iii) By using (ii), if $x_i \geq 0$ then we have $[p(0 \leq V(\xi_1) \leq x_1), p(0 \leq V(\xi_2) \leq x_2), \dots, p(0 \leq V(\xi_n) \leq x_n)] = [\sum_{j=0}^{[x_1]} p(V(\xi_1) = j), \sum_{j=0}^{[x_2]} p(V(\xi_2) = j), \dots, \sum_{j=0}^{[x_n]} p(V(\xi_n) = j)]$, and if $x_i < 0$ we get $[p(x_1 \leq V(\xi_1 + 1) < 0), p(x_2 \leq V(\xi_2 + 1) < 0), \dots, p(x_n \leq V(\xi_n + 1) < 0)] = [\sum_{j=[x_1]}^0 p(V(\xi_1) = j), \sum_{j=[x_2]}^0 p(V(\xi_2) = j), \dots, \sum_{j=[x_n]}^0 p(V(\xi_n) = j)]$, where $[x_i]$ means the greatest integer less than or equal to x_i , $i = 1, 2, \dots, n$. It is sufficient to show that

$$\begin{aligned} & [p(V(\xi_1 + 1) = \mathcal{F}), p(V(\xi_2 + 1) = \mathcal{F}), \dots, \\ & p(V(\xi_n + 1) = \mathcal{F})] \\ & \leq [p(V(\xi_1) = \mathcal{F}), p(V(\xi_2) = \mathcal{F}), \dots, \\ & p(V(\xi_n) = \mathcal{F})] \quad \text{if } \xi_i > r_{i1}x_i + r_{i2}. \end{aligned} \quad (12)$$

We have the following cases.

Case (a). If $C_i < -1$, then, from (ii), $[p(V(\xi_1) = \mathcal{F}) > 0, p(V(\xi_2) = \mathcal{F}) > 0, \dots, p(V(\xi_n) = \mathcal{F}) > 0]$, if and only if $2\mathcal{F}/\vartheta_i(1 - C_i) \leq \xi_i \leq -2\mathcal{F}/\vartheta_i(1 + C_i)$. We take $r_{i1} = -2/\vartheta_i(1 + C_i)$, $r_{i2} = 0$ and then $\xi_i > r_{i1}|\mathcal{F}| \Rightarrow \xi_i > -2\mathcal{F}/\vartheta_i(1 + C_i)$ that leads to $[p(V(\xi_1) = \mathcal{F}) = 0, p(V(\xi_2) = \mathcal{F}) = 0, \dots, p(V(\xi_n) = \mathcal{F}) = 0]$. Consequently, (12) holds.

Case (b). If $C_i > 1$, hence $[p(V(\xi_1) = \mathcal{F}) > 0, p(V(\xi_2) = \mathcal{F}) > 0, \dots, p(V(\xi_n) = \mathcal{F}) > 0]$, if and only if $-2\mathcal{F}/\vartheta_i(1 + C_i) \leq \xi_i \leq 2\mathcal{F}/\vartheta_i(1 - C_i)$. From (ii) putting $r_{i1} = -2/\vartheta_i(1 - C_i)$, $r_{i2} = 0$ then $\xi_i > r_{i1}|\mathcal{F}| \Rightarrow \xi_i > 2\mathcal{F}/\vartheta_i(1 - C_i)$ that leads to $[p(V(\xi_1) = \mathcal{F}) = 0, p(V(\xi_2) = \mathcal{F}) = 0, \dots, p(V(\xi_n) = \mathcal{F}) = 0]$.

Then, (12) holds.

Case (c). If $-1 < C_i < 1$, then let $\alpha_i = (1 - C_i)/2$, $\beta_i = 1 - \alpha_i$ and $\Delta_i = (\alpha_i/q_i)^{\alpha_i}(\beta_i/p_i)^{\beta_i}$, where $[p_1, p_2, \dots, p_n] + [q_1, q_2, \dots, q_n] = [1, 1, \dots, 1]$. In addition, put $r_{i2} = 1/(\Delta_i - 1)$, $r_{i1} = \Delta_i \max[(1/\alpha_i, 1/\beta_i)\vartheta_i(\Delta_i - 1)]$. Since $E(X_{ij}) \neq C_i$ and $0 < \alpha_i < 1$ then $\alpha_i \neq q_i$ and then $\Delta_i > 1$. Consequently, r_{i1} and r_{i2} are positive and well defined. Assuming that $\xi_i > r_{i1}|\mathcal{F}|$ then $-\beta_i \xi_i \vartheta_i < \mathcal{F} < \alpha_i \xi_i \vartheta_i$; therefore, from (ii), we have $[p(V(\xi_1) = \mathcal{F}) > 0, p(V(\xi_2) = \mathcal{F}) > 0, \dots, p(V(\xi_n) = \mathcal{F}) > 0]$.

It remains to prove that if $[p(V(\xi_1) = \mathcal{F}) > 0, p(V(\xi_2) = \mathcal{F}) > 0, \dots, p(V(\xi_n) = \mathcal{F}) > 0]$ and $\xi_i > r_{i1}|\mathcal{F}| + r_{i2}$ then (2) holds.

Considering that $Z_{ij} = (X_{ij} - C_i)/2$ then $[p(Z_{1j} = \alpha_1) = p_1, p(Z_{2j} = \alpha_2) = p_2, \dots, p(Z_{nj} = \alpha_n) = p_n]$, $[p(Z_{1j} = -\beta_1) = q_1, p(Z_{2j} = -\beta_2) = q_2, \dots, p(Z_{nj} = -\beta_n) = q_n]$,

and $[p(V(\xi_1) = \mathcal{F}), p(V(\xi_2) = \mathcal{F}), \dots, p(V(\xi_n) = \mathcal{F})] = [p(\sum_{j=1}^{\xi_1 \vartheta_1} Z_{1j} = \mathcal{F}), p(\sum_{j=1}^{\xi_2 \vartheta_2} Z_{2j} = \mathcal{F}), \dots, p(\sum_{j=1}^{\xi_n \vartheta_n} Z_{nj} = \mathcal{F})]$. By using (1) we have

$$\begin{aligned} & \left[\frac{p(V(\xi_1 + 1) = \mathcal{F})}{p(V(\xi_1) = \mathcal{F})}, \frac{p(V(\xi_2 + 1) = \mathcal{F})}{p(V(\xi_2) = \mathcal{F})}, \dots, \right. \\ & \left. \frac{p(V(\xi_n + 1) = \mathcal{F})}{p(V(\xi_n) = \mathcal{F})} \right] \\ &= \left[\prod_{j=1}^{\vartheta_1 \beta_1} \frac{\xi_1 \vartheta_1 + j}{\Delta_1(\xi_1 \vartheta_1 + j + (j\alpha_1 + \mathcal{F})/\beta_1)} \right. \\ & \quad \times \prod_{j=1}^{\vartheta_1 \alpha_1} \frac{\xi_1 \vartheta_1 + j + \vartheta_1 \beta_1}{\Delta_1(\xi_1 \vartheta_1 + j + (i\beta_1 - \mathcal{F})/\alpha_1)}, \\ & \quad \prod_{j=1}^{\vartheta_2 \beta_2} \frac{\xi_2 \vartheta_2 + j}{\Delta_2(\xi_2 \vartheta_2 + j + (j\alpha_2 + \mathcal{F})/\beta_2)} \\ & \quad \times \prod_{j=1}^{\vartheta_2 \alpha_2} \frac{\xi_2 \vartheta_2 + j + \vartheta_2 \beta_2}{\Delta_2(\xi_2 \vartheta_2 + j + (i\beta_2 - \mathcal{F})/\alpha_2)}, \dots, \\ & \quad \prod_{j=1}^{\vartheta_n \beta_n} \frac{\xi_n \vartheta_n + j}{\Delta_n(\xi_n \vartheta_n + j + (j\alpha_n + \mathcal{F})/\beta_n)} \\ & \quad \times \left. \prod_{j=1}^{\vartheta_n \alpha_n} \frac{\xi_n \vartheta_n + j + \vartheta_n \beta_n}{\Delta_n(\xi_n \vartheta_n + j + (i\beta_n - \mathcal{F})/\alpha_n)} \right]. \end{aligned} \quad (13)$$

Since $\xi_i > r_{i1}|\mathcal{F}| + r_{i2}$ then every term is strictly less than 1. Consequently, $[p(V(\xi_1 + 1) = \mathcal{F}), p(V(\xi_2 + 1) = \mathcal{F}), \dots, p(V(\xi_n + 1) = \mathcal{F})] < [p(V(\xi_1) = \mathcal{F}), p(V(\xi_2) = \mathcal{F}), \dots, p(V(\xi_n) = \mathcal{F})]$. $\square$

Theorem 3. If $E(X_{ij}) < C_i$, $C_i \in \mathbb{R}$ (the set of real numbers), then there exists ε_i , $0 < \varepsilon_i < 1$, such that $[p(W(\xi_1)), p(W(\xi_2)), \dots, p(W(\xi_n))] \leq [\varepsilon_1^{\xi_1}, \varepsilon_2^{\xi_2}, \dots, \varepsilon_n^{\xi_n}]$, for all ξ_i , $i = 1, 2, \dots, n$.

Proof. For $\zeta_i > 0$, $[p(W(\xi_1) \geq C_1 \xi_1), p(W(\xi_2) \geq C_2 \xi_2), \dots, p(W(\xi_n) \geq C_n \xi_n)] = [p(\exp\{\zeta_1(W(\xi_1) - C_1 \xi_1)\} \geq 1), p(\exp\{\zeta_2(W(\xi_2) - C_2 \xi_2)\} \geq 1), \dots, p(\exp\{\zeta_n(W(\xi_n) - C_n \xi_n)\} \geq 1)] \leq [E(\exp\{\zeta_1(W(\xi_1) - C_1 \xi_1)\}), E(\exp\{\zeta_2(W(\xi_2) - C_2 \xi_2)\}), \dots, E(\exp\{\zeta_n(W(\xi_n) - C_n \xi_n)\})] = [\{f_1(\zeta_1)\}^{\xi_1}, \{f_2(\zeta_2)\}^{\xi_2}, \dots, \{f_n(\zeta_n)\}^{\xi_n}]$, where $\{f_1(\zeta_1)\}, \{f_2(\zeta_2)\}, \dots, \{f_n(\zeta_n)\} = [E(\exp\{\zeta_1(X_{1j} - C_1)\}), E(\exp\{\zeta_2(X_{2j} - C_2)\}), \dots, E(\exp\{\zeta_n(X_{nj} - C_n)\})] = [p_1 \exp\{\zeta_1(1 - C_1)\} + q_1 \exp\{\zeta_1(-1 - C_1)\}, p_2 \exp\{\zeta_2(1 - C_2)\} + q_2 \exp\{\zeta_2(-1 - C_2)\}, \dots, p_n \exp\{\zeta_n(1 - C_n)\} + q_n \exp\{\zeta_n(-1 - C_n)\}]$; if $E(X_{ij}) < C_i$, $i = 1, 2, \dots, n$, then $\{f'_1(0)\}, \{f'_2(0)\}, \dots, \{f'_n(0)\} < [0, 0, \dots, 0]$, and since $\{f_1(0)\}, \{f_2(0)\}, \dots, \{f_n(0)\} < [1, 1, \dots, 1]$, then $[\min_{\zeta_1 > 0} \{f_1(\zeta_1)\}, \min_{\zeta_2 > 0} \{f_2(\zeta_2)\}, \dots, \min_{\zeta_n > 0} \{f_n(\zeta_n)\}] = [\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n] < [1, 1, \dots, 1]$.

By similar arguments if $E(X_{ij}) > C_i$, $i = 1, 2, \dots, n$, then there exist $\varepsilon_i < 1$ such that $[p(W(\xi_1)), p(W(\xi_2)), \dots, p(W(\xi_n))] \leq [\varepsilon_1^{\xi_1}, \varepsilon_2^{\xi_2}, \dots, \varepsilon_n^{\xi_n}]$ for all ξ_i , $i = 1, 2, \dots, n$. $\square$

4. Existence of a Finite Search Plan

In this section, we find the conditions that make the search plan be finite. In addition, we will discuss that under what these conditions are indeed finite. This is a crucial issue related to the existence of a finite search plan. So, we will provide useful theorems that help us to do it.

Theorem 4. Let γ_i be the measure defined on $\mathbb{R}$ by X_{ij} , $i = 1, 2, \dots, n$, and if $\phi(t) = (\phi_1(t), \phi_2(t), \dots, \phi_n(t))$ is a search plan defined previously, the expectation E_{τ_ϕ} is finite if

$$\begin{aligned} & \int_{-\infty}^{\delta_{i0}} \left[\sum_{j=1}^{\mathcal{F}/2} \vartheta_i^{2j} p(\psi_i(G_{i(2j)}) < -x_i) \right. \\ & \quad \left. + \sum_{j=\mathcal{F}/2+1}^{\infty} \vartheta_i^{2j} p(\tilde{\psi}_i(G_{i(2j)}) \leq -x_i) \right] \gamma_i(dx_i), \end{aligned} \quad (14)$$

$\int_{\delta_{i0}}^{\infty} \sum_{j=1}^{\infty} \vartheta_i^{2j+1} p(\psi_i(G_{i(2j+1)}) > -x_i) \gamma_i(dx_i)$, are finite when $\delta_{i0} > 0$. And if $\delta_{i0} < 0$, then E_{τ_ϕ} is finite if

$$\begin{aligned} & \int_{\delta_{i0}}^{\infty} \left[\sum_{j=(\mathcal{F}+1)/2}^{\infty} \vartheta_i^{2j+1} p(\psi_i(G_{i(2j+1)}) > -x_i) \right. \\ & \quad \left. + \sum_{j=1}^{(\mathcal{F}-1)/2} \vartheta_i^{2j+1} p(\tilde{\psi}_i(G_{i(2j+1)}) \leq -x_i) \right] \gamma_i(dx_i), \end{aligned} \quad (15)$$

$\int_{-\infty}^{\delta_{i0}} \sum_{j=1}^{\infty} \vartheta_i^{2j} p(\tilde{\psi}_i(G_{i(2j)}) < -x_i) \gamma_i(dx_i)$, are finite.

Proof. The hypotheses X_i and δ_{i0} are valued in $2I$ or $(2I + 1)$ then $X_i + W(t)$ is greater than $\phi_i(t)$ until the first meeting; also if X_i is smaller than δ_{i0} then $X_i + W(t)$ is smaller than $\phi_i(t)$ until the first meeting. Since $\tau_{\phi_i} > t$, $i = 1, 2, \dots, n$, are mutually exclusive events then $p(\tau_{\phi_1} > t) = p(\tau_{\phi_1} > t \text{ or } \tau_{\phi_2} > t \text{ or } \dots \text{ or } \tau_{\phi_n} > t) = \sum_{i=1}^n p(\tau_{\phi_i} > t)$, and, for any $j \geq 0$ we have

$$\begin{aligned} & p(\tau_{\phi_i} > G_{i(2j+1)}) \\ & \leq \int_{-\infty}^{\delta_{i0}} p(X_i + W(G_{i(2j)}) < H_{i(2j)} \mid X_i = x_i) \gamma_i(dx_i) \\ & \quad + \int_{\delta_{i0}}^{\infty} p(X_i + W(G_{i(2j+1)}) > H_{i(2j+1)} \mid X_i = x_i) \gamma_i(dx_i). \end{aligned} \quad (16)$$

Using the notation $\tilde{\psi}_i(G_{i(2j)}) = W(G_{i(2j)}) + C_i G_{i(2j)}$, we obtain $W(G_{i(2j)}) - H_{i(2j)} < -X_i = x_i$; then $W(G_{i(2j)}) - (-1)^{2j+1} C_i [G_{i(2j)} + 1 + (-1)^{2j+1}] < -x_i$ leads to $W(G_{i(2j)}) - (-1) C_i [G_{i(2j)} + 1 + (-1)] = W(G_{i(2j)}) + C_i [G_{i(2j)}] =$

$\tilde{\psi}_i(G_{i(2j)}) < -x_i$. Similarly, by using the notation $\psi_i(G_{i(2j+1)}) = W(G_{i(2j+1)}) - C_i G_{i(2j+1)}$, we get $\psi_i(G_{i(2j+1)}) > -x_i$. Consequently,

$$\begin{aligned} & p(\tau_{\phi_i} > G_{i(2j+1)}) \\ & \leq \int_{-\infty}^{\delta_{i0}} p(\tilde{\psi}_i(G_{i(2j)}) < -x_i) \gamma_i(dx_i) \\ & \quad + \int_{\delta_{i0}}^{\infty} p(\psi_i(G_{i(2j+1)}) > -x_i) \gamma_i(dx_i), \end{aligned} \quad (17)$$

also,

$$\begin{aligned} & p(\tau_{\phi_i} > G_{i(2j)}) \\ & \leq \int_{-\infty}^{\delta_{i0}} p(\tilde{\psi}_i(G_{i(2j)}) < -x_i) \gamma_i(dx_i) \\ & \quad + \int_{\delta_{i0}}^{\infty} p(\psi_i(G_{i(2j-1)}) > -x_i) \gamma_i(dx_i). \end{aligned} \quad (18)$$

From Remark 1, we obtain

$$\begin{aligned} E_{\tau_\phi} &= \int_0^\infty p(\tau_\phi > t) dt \\ &\leq \sum_{j=0}^\infty \int_{G_{1j}}^{G_{1(j+1)}} p(\tau_{\phi_1} > t) dt + \cdots \\ &\quad + \sum_{j=0}^\infty \int_{G_{nj}}^{G_{n(j+1)}} p(\tau_{\phi_n} > t) dt, \end{aligned} \quad (19)$$

where $G_{i0} = 0$; then

$$\begin{aligned} E_{\tau_\phi} &\leq \sum_{j=0}^\infty \int_{G_{1j}}^{G_{1(j+1)}} p(\tau_{\phi_1} > G_{1j}) dt + \cdots \\ &\quad + \sum_{j=0}^\infty \int_{G_{nj}}^{G_{n(j+1)}} p(\tau_{\phi_n} > t) dt \\ &= \sum_{i=1}^n \sum_{j=0}^\infty (G_{i(j+1)} - G_{ij}) p(\tau_{\phi_i} > G_{ij}) \\ &= \sum_{i=1}^n \sum_{j=0}^\infty [\lambda_i(\vartheta_i^{j+1} - 1) - \lambda_i(\vartheta_i^j - 1)] p(\tau_{\phi_i} > G_{ij}) \\ &= \sum_{i=1}^n \sum_{j=0}^\infty \lambda_i \vartheta_i^j (\vartheta_i - 1) p(\tau_{\phi_i} > G_{ij}) \\ &= \sum_{i=1}^n \left[\lambda_i (\vartheta_i - 1) \sum_{j=0}^\infty \vartheta_i^j p(\tau_{\phi_i} > G_{ij}) \right] \\ &= \sum_{i=1}^n [\lambda_i (\vartheta_i - 1)] \end{aligned}$$

$$\begin{aligned} & \times \left(p(\tau_{\phi_i} > D) + \vartheta_i p(\tau_{\phi_i} > G_{i1}) \right. \\ & \quad \left. + \vartheta_i^2 p(\tau_{\phi_i} > G_{i2}) + \vartheta_i^3 p(\tau_{\phi_i} > G_{i3}) + \cdots \right). \end{aligned} \quad (20)$$

If $\delta_{i0} < 0$, then we get

$$\begin{aligned} E_{\tau_\phi} &\leq \sum_{i=1}^n \left[\lambda_i (\vartheta_i - 1) \right. \\ &\quad \times \left(p(\tau_{\phi_i} > 0) + \vartheta_i p(\tau_{\phi_i} > G_{i1}) \right. \\ &\quad + \vartheta_i^2 \left\{ \int_{\delta_{i0}}^\infty p(\psi_i(G_{i1}) > -x_i) \gamma_i(dx_i) \right. \\ &\quad \left. + \int_{-\infty}^{\delta_{i0}} p(\tilde{\psi}_i(G_{i2}) < -x_i) \gamma_i(dx_i) \right\} \\ &\quad + \vartheta_i^3 \left\{ \int_{\delta_{i0}}^\infty p(\psi_i(G_{i3}) > -x_i) \gamma_i(dx_i) \right. \\ &\quad \left. + \int_{-\infty}^{\delta_{i0}} p(\tilde{\psi}_i(G_{i2}) < -x_i) \gamma_i(dx_i) \right\} \\ &\quad + \vartheta_i^4 \left\{ \int_{\delta_{i0}}^\infty p(\psi_i(G_{i3}) > -x_i) \gamma_i(dx_i) \right. \\ &\quad \left. + \int_{-\infty}^{\delta_{i0}} p(\tilde{\psi}_i(G_{i4}) < -x_i) \gamma_i(dx_i) \right\} \\ &\quad + \cdots \\ &\quad + \vartheta_i^{\mathcal{F}} \left\{ \int_{\delta_{i0}}^\infty p(\psi_i(G_{i(\mathcal{F}-1)}) > -x_i) \gamma_i(dx_i) \right. \\ &\quad \left. + \int_{-\infty}^{\delta_{i0}} p(\tilde{\psi}_i(G_{i\mathcal{F}}) < -x_i) \gamma_i(dx_i) \right\} \\ &\quad + \vartheta_i^{\mathcal{F}+1} \left\{ \int_{\delta_{i0}}^\infty p(\psi_i(G_{i(\mathcal{F}+1)}) > -x_i) \gamma_i(dx_i) \right. \\ &\quad \left. + \int_{-\infty}^{\delta_{i0}} p(\tilde{\psi}_i(G_{i\mathcal{F}}) < -x_i) \gamma_i(dx_i) \right\} \\ &\quad + \vartheta_i^{\mathcal{F}+2} \left\{ \int_{\delta_{i0}}^\infty p(\psi_i(G_{i(\mathcal{F}+1)}) > -x_i) \gamma_i(dx_i) \right. \\ &\quad \left. + \int_{-\infty}^{\delta_{i0}} p(\tilde{\psi}_i(G_{i(\mathcal{F}+2)}) < -x_i) \gamma_i(dx_i) \right\} \\ &\quad + \vartheta_i^{\mathcal{F}+3} \left\{ \int_{\delta_{i0}}^\infty p(\psi_i(G_{i(\mathcal{F}+3)}) > -x_i) \gamma_i(dx_i) \right. \\ &\quad \left. + \int_{-\infty}^{\delta_{i0}} p(\tilde{\psi}_i(G_{i(\mathcal{F}+2)}) < -x_i) \gamma_i(dx_i) \right\} \\ &\quad \left. + \cdots \right) \Big]; \end{aligned} \quad (21)$$

thus,

$$\begin{aligned}
E_{\tau_\phi} &\leq \sum_{i=1}^n \left[\lambda_i (\vartheta_i - 1) \right. \\
&\quad \times \left(p(\tau_{\phi_i} > 0) + \vartheta_i p(\tau_{\phi_i} > G_{i1}) \right. \\
&\quad \left. + \vartheta_i^2 (\vartheta_i + 1) \right. \\
&\quad \times \int_{-\infty}^{\delta_{i0}} p(\tilde{\psi}_i(G_{i2}) < -x_i) \gamma_i(dx_i) + \cdots \\
&\quad \left. + \vartheta_i^{\mathcal{F}} (\vartheta_i + 1) \right. \\
&\quad \times \int_{-\infty}^{\delta_{i0}} p(\tilde{\psi}_i(G_{i\mathcal{F}}) < -x_i) \gamma_i(dx_i) + \cdots \\
&\quad \left. + \vartheta_i^3 (\vartheta_i + 1) \int_{\delta_{i0}}^{\infty} p(\psi_i(G_{i3}) > -x_i) \gamma_i(dx_i) \right. \\
&\quad \left. + \vartheta_i^5 (\vartheta_i + 1) \right. \\
&\quad \times \int_{\delta_{i0}}^{\infty} p(\psi_i(G_{i5}) > -x_i) \gamma_i(dx_i) + \cdots \\
&\quad \left. + \vartheta_i^{\mathcal{F}+3} (\vartheta_i + 1) \right. \\
&\quad \left. \times \int_{\delta_{i0}}^{\infty} p(\psi_i(G_{i(\mathcal{F}+3)}) > -x_i) \gamma_i(dx_i) + \cdots \right) \Big]. \quad (22)
\end{aligned}$$

Leads to

$$\begin{aligned}
E_{\tau_\phi} &\leq \sum_{i=1}^n \left[\lambda_i (\vartheta_i - 1) \right. \\
&\quad \times \left[{}^1\hat{g}_i + (\vartheta_i + 1) \right. \\
&\quad \times \left(\int_{-\infty}^{\delta_{i0}} {}^1\hat{w}_i(x_i) \gamma_i(dx_i) \right. \\
&\quad \left. + \int_{\delta_{i0}}^{\infty} {}^1\hat{q}_i(x_i) \gamma_i(dx_i) \right. \\
&\quad \left. \left. + \int_{\delta_{i0}}^{\infty} {}^1\hat{h}_i(x_i) \gamma_i(dx_i) \right) \right] \\
&= \sum_{i=1}^n \lambda_i (\vartheta_i - 1) ({}^1\hat{g}_i)
\end{aligned}$$

$$\begin{aligned}
&+ \sum_{i=1}^n \left[\lambda_i (\vartheta_i - 1) (\vartheta_i + 1) \right. \\
&\quad \times \left(\int_{-\infty}^{\delta_{i0}} {}^1\hat{w}_i(x_i) \gamma_i(dx_i) \right. \\
&\quad \left. + \int_{\delta_{i0}}^{\infty} {}^1\hat{q}_i(x_i) \gamma_i(dx_i) \right. \\
&\quad \left. \left. + \int_{\delta_{i0}}^{\infty} {}^1\hat{h}_i(x_i) \gamma_i(dx_i) \right) \right], \quad (23)
\end{aligned}$$

where

$$\begin{aligned}
{}^1\hat{g}_i &= p(\tau_{\phi_i} > 0) + \vartheta_i p(\tau_{\phi_i} > G_{i1}) \\
&\quad + \vartheta_i^2 \int_{\delta_{i0}}^{\infty} p(\psi_i(G_{i1}) > -x_i) \gamma_i(dx_i), \\
{}^1\hat{w}_i(x_i) &= \sum_{j=1}^{\infty} \vartheta_i^{2j} p(\tilde{\psi}_i(G_{i(2j)}) < -x_i), \\
{}^1\hat{q}_i(x_i) &= \sum_{j=1}^{(\mathcal{F}-1)/2} \vartheta_i^{2j+1} p(\tilde{\psi}_i(G_{i(2j+1)}) \leq -x_i), \\
{}^1\hat{h}_i(x_i) &= \sum_{j=(\mathcal{F}+1)/2}^{\infty} \vartheta_i^{2j+1} p(\psi_i(G_{i(2j+1)}) > -x_i).
\end{aligned} \quad (24)$$

Then E_{τ_ϕ} is finite if

$$\begin{aligned}
&\int_{\delta_{i0}}^{\infty} \left[\sum_{j=(\mathcal{F}+1)/2}^{\infty} \vartheta_i^{2j+1} p(\psi_i(G_{i(2j+1)}) > -x_i) \right. \\
&\quad \left. + \sum_{j=1}^{(\mathcal{F}-1)/2} \vartheta_i^{2j+1} p(\tilde{\psi}_i(G_{i(2j+1)}) \leq -x_i) \right] \gamma_i(dx_i), \quad (25)
\end{aligned}$$

$\int_{-\infty}^{\delta_{i0}} \sum_{j=1}^{\infty} \vartheta_i^{2j} p(\tilde{\psi}_i(G_{i(2j)}) < -x_i) \gamma_i(dx_i)$, are finite.

By similar way if $\delta_{i0} > 0$, then E_{τ_ϕ} is finite if

$$\begin{aligned}
E_{\tau_\phi} &\leq \sum_{i=1}^n \left[\lambda_i (\vartheta_i - 1) ({}^2\hat{g}_i) + \lambda_i (\vartheta_i - 1) (\vartheta_i + 1) \right. \\
&\quad \times \left(\int_{-\infty}^{\delta_{i0}} {}^2\hat{w}_i(x_i) \gamma_i(dx_i) \right. \\
&\quad \left. + \int_{-\infty}^{\delta_{i0}} {}^2\hat{q}_i(x_i) \gamma_i(dx_i) \right. \\
&\quad \left. \left. + \int_{\delta_{i0}}^{\infty} {}^2\hat{h}_i(x_i) \gamma_i(dx_i) \right) \right], \quad (26)
\end{aligned}$$

where

$$\begin{aligned}
{}^2\hat{g}_i &= p(\tau_{\phi_i} > 0) + \vartheta_i p(\tau_{\phi_i} > G_{i1}) \\
&\quad + \vartheta_i^2 \int_{\delta_{i0}}^{\infty} p(\psi_i(G_{i1}) > -x_i) \gamma_i(dx_i), \\
{}^2\hat{w}_i(x_i) &= \sum_{j=1}^{\mathcal{F}/2} \vartheta_i^{2j} p(\psi_i(G_{i(2j)}) < -x_i), \\
{}^2\hat{q}_i(x_i) &= \sum_{j=\mathcal{F}/2+1}^{\infty} \vartheta_i^{2j} p(\tilde{\psi}_i(G_{i(2j)}) \leq -x_i), \\
{}^2\hat{h}_i(x_i) &= \sum_{j=1}^{\infty} \vartheta_i^{2j+1} p(\psi_i(G_{i(2j+1)}) > -x_i).
\end{aligned} \tag{27}$$

And $E_{\tau_{\phi}}$ is finite if

$$\begin{aligned}
&\int_{\delta_{i0}}^{\infty} \left[\sum_{j=1}^{\mathcal{F}/2} \vartheta_i^{2j} p(\tilde{\psi}_i(G_{i(2j)}) < -x_i) \right. \\
&\quad \left. + \sum_{j=\mathcal{F}/2+1}^{\infty} \vartheta_i^{2j} p(\tilde{\psi}_i(G_{i(2j)}) \leq -x_i) \right] \gamma_i(dx_i),
\end{aligned} \tag{28}$$

$\int_{\delta_{i0}}^{\infty} \sum_{j=1}^{\infty} \vartheta_i^{2j+1} p(\psi_i(G_{i(2j+1)}) > -x_i)$, are finite. $\square$

Lemma 5 (see El-Rayes et al. [9]). If $a_{\bar{m}} \geq 0$, $a_{\bar{m}+1} \leq a_{\bar{m}}$, $\bar{m} = 1, 2, \dots$, and $\{\bar{d}_{\bar{m}}\}$, $\bar{m} = 1, 2, \dots$, are a strictly increasing sequence of integer numbers with $\bar{d}_0 = 0$, then, for any $\bar{k} = 1, 2, \dots$,

$$\begin{aligned}
\sum_{\bar{m}=\bar{k}}^{\infty} [\bar{d}_{\bar{m}+1} - \bar{d}_{\bar{m}}] a_{\bar{d}_{\bar{m}+1}} &\leq \sum_{\bar{m}=\bar{k}}^{\infty} a_{\bar{m}} \\
&\leq \sum_{\bar{m}=\bar{k}}^{\infty} [\bar{d}_{\bar{m}+1} - \bar{d}_{\bar{m}}] a_{\bar{d}_{\bar{m}}},
\end{aligned} \tag{29}$$

where $\sum_{\bar{m}=\bar{k}}^{\infty} [\bar{d}_{\bar{m}+1} - \bar{d}_{\bar{m}}] a_{\bar{d}_{\bar{m}+1}}$, $\sum_{\bar{m}=\bar{k}}^{\infty} a_{\bar{m}}$, and $\sum_{\bar{m}=\bar{k}}^{\infty} [\bar{d}_{\bar{m}+1} - \bar{d}_{\bar{m}}] a_{\bar{d}_{\bar{m}}}$ are vectors in formulas and they are taken to be row vectors. These vectors are defined as follows:

$$\begin{aligned}
&\sum_{\bar{m}=\bar{k}}^{\infty} [\bar{d}_{\bar{m}+1} - \bar{d}_{\bar{m}}] a_{\bar{d}_{\bar{m}+1}} \\
&= \left[\sum_{\bar{m}=\bar{k}}^{\infty} [\bar{d}_{1(\bar{m}+1)} - \bar{d}_{1\bar{m}}] a_{1\bar{d}_{1(\bar{m}+1)}}, \right. \\
&\quad \sum_{\bar{m}=\bar{k}}^{\infty} [\bar{d}_{2(\bar{m}+1)} - \bar{d}_{2\bar{m}}] a_{2\bar{d}_{2(\bar{m}+1)}}, \dots, \\
&\quad \left. \sum_{\bar{m}=\bar{k}}^{\infty} [\bar{d}_{n(\bar{m}+1)} - \bar{d}_{n\bar{m}}] a_{n\bar{d}_{n(\bar{m}+1)}} \right],
\end{aligned}$$

$$\begin{aligned}
\sum_{\bar{m}=\bar{k}}^{\infty} a_{\bar{m}} &= \left[\sum_{\bar{m}=\bar{d}_{1\bar{k}}}^{\infty} a_{1\bar{m}}, \sum_{\bar{m}=\bar{d}_{2\bar{k}}}^{\infty} a_{2\bar{m}}, \dots, \sum_{\bar{m}=\bar{d}_{n\bar{k}}}^{\infty} a_{n\bar{m}} \right], \\
&\sum_{\bar{m}=\bar{k}}^{\infty} [\bar{d}_{\bar{m}+1} - \bar{d}_{\bar{m}}] a_{\bar{d}_{\bar{m}}} \\
&= \left[\sum_{\bar{m}=\bar{k}}^{\infty} [\bar{d}_{1(\bar{m}+1)} - \bar{d}_{1\bar{m}}] a_{1\bar{d}_{1\bar{m}}}, \right. \\
&\quad \sum_{\bar{m}=\bar{k}}^{\infty} [\bar{d}_{2(\bar{m}+1)} - \bar{d}_{2\bar{m}}] a_{2\bar{d}_{2\bar{m}}}, \dots, \\
&\quad \left. \sum_{\bar{m}=\bar{k}}^{\infty} [\bar{d}_{n(\bar{m}+1)} - \bar{d}_{n\bar{m}}] a_{n\bar{d}_{n\bar{m}}} \right].
\end{aligned} \tag{30}$$

Now, we will discuss under what conditions of the chosen search plan should be satisfied to make the previous integrals in Theorem 4 are indeed finite. This is a crucial issue related to the existence of a finite search plan.

Theorem 6. The chosen search plan should satisfy ${}^2\hat{h}(x) \leq L(|x|)$, ${}^2\hat{q}(x) \leq \tilde{L}(|x|)$, if $\delta_{i0} > 0$ and ${}^1\hat{w}(x) \leq M(|x|)$, ${}^1\hat{h}(x) \leq \tilde{M}(|x|)$ if $\delta_{i0} < 0$, where $L(|x|)$, $\tilde{L}(|x|)$, $M(|x|)$, and $\tilde{M}(|x|)$ are vectors of linear functions given by ${}^1\hat{w}(x) = [{}^1\hat{w}_1(x_1), {}^1\hat{w}_2(x_2), \dots, {}^1\hat{w}_n(x_n)]$, ${}^1\hat{h}(x) = [{}^1\hat{h}_1(x_1), {}^1\hat{h}_2(x_2), \dots, {}^1\hat{h}_n(x_n)]$, ${}^2\hat{h}(x) = [{}^2\hat{h}_1(x_1), {}^2\hat{h}_2(x_2), \dots, {}^2\hat{h}_n(x_n)]$, ${}^2\hat{q}(x) = [{}^2\hat{q}_1(x_1), {}^2\hat{q}_2(x_2), \dots, {}^2\hat{q}_n(x_n)]$, $L(|x|) = [L_1(|x_1|), L_2(|x_2|), \dots, L_n(|x_n|)]$, $\tilde{L}(|x|) = [\tilde{L}_1(|x_1|), \tilde{L}_2(|x_2|), \dots, \tilde{L}_n(|x_n|)]$, $M(|x|) = [M_1(|x_1|), M_2(|x_2|), \dots, M_n(|x_n|)]$ and $\tilde{M}(|x|) = [\tilde{M}_1(|x_1|), \tilde{M}_2(|x_2|), \dots, \tilde{M}_n(|x_n|)]$.

Proof. We will prove this theorem for ${}^2\hat{h}(x)$ when $\delta_{i0} > 0$, where ${}^2\hat{h}(x) = [{}^2\hat{h}_1(x_1), {}^2\hat{h}_2(x_2), \dots, {}^2\hat{h}_n(x_n)] = [\sum_{j=1}^{\infty} \vartheta_1^{2j+1} p(\psi_1(G_{1(2j+1)}) > -x_1), \sum_{j=1}^{\infty} \vartheta_2^{2j+1} p(\psi_2(G_{2(2j+1)}) > -x_2), \dots, \sum_{j=1}^{\infty} \vartheta_n^{2j+1} p(\psi_n(G_{n(2j+1)}) > -x_n)]$, and we obtain the following cases.

(I) If $x_i > \delta_{i0}$, then we have $[{}^2\hat{h}_1(x_1), {}^2\hat{h}_2(x_2), \dots, {}^2\hat{h}_n(x_n)] = [{}^2\hat{h}_1(\delta_{10}), {}^2\hat{h}_2(\delta_{20}), \dots, {}^2\hat{h}_n(\delta_{n0})] + [\sum_{j=1}^{\infty} \vartheta_1^{2j+1} p(-x_1 < \psi_1(G_{1(2j+1)}) \leq -\delta_{10}), \sum_{j=1}^{\infty} \vartheta_2^{2j+1} p(-x_2 < \psi_2(G_{2(2j+1)}) \leq -\delta_{20}), \dots, \sum_{j=1}^{\infty} \vartheta_n^{2j+1} p(-x_n < \psi_n(G_{n(2j+1)}) \leq -\delta_{n0})]$.

(II) If $0 \leq x_i \leq \delta_{i0}$, then we have $[{}^2\hat{h}_1(x_1), {}^2\hat{h}_2(x_2), \dots, {}^2\hat{h}_n(x_n)] = [{}^2\hat{h}_1(0), {}^2\hat{h}_2(0), \dots, {}^2\hat{h}_n(0)] + [\sum_{j=1}^{\infty} \vartheta_1^{2j+1} p(-x_1 < \psi_1(G_{1(2j+1)}) \leq 0), \sum_{j=1}^{\infty} \vartheta_2^{2j+1} p(-x_2 < \psi_2(G_{2(2j+1)}) \leq 0), \dots, \sum_{j=1}^{\infty} \vartheta_n^{2j+1} p(-x_n < \psi_n(G_{n(2j+1)}) \leq 0)]$.

(III) If $x_i \leq \delta_{i0}$, then we get $[\hat{h}_1(x_1), \hat{h}_2(x_2), \dots, \hat{h}_n(x_n)] = [\hat{h}_1(0), \hat{h}_2(0), \dots, \hat{h}_n(0)] - [\sum_{j=1}^{\infty} \vartheta_1^{2j+1} p(0 < \psi_1(G_{1(2j+1)}) \leq -x_1), \sum_{j=1}^{\infty} \vartheta_2^{2j+1} p(0 < \psi_2(G_{2(2j+1)}) \leq -x_2), \dots, \sum_{j=1}^{\infty} \vartheta_n^{2j+1} p(0 < \psi_n(G_{n(2j+1)}) \leq -x_n)]$.

From (II) we have $[\hat{h}_1(x_1), \hat{h}_2(x_2), \dots, \hat{h}_n(x_n)] \leq [\hat{h}_1(0), \hat{h}_2(0), \dots, \hat{h}_n(0)]$, and for $x_i \geq \delta_{i0}$ $[\hat{h}_1(x_1), \hat{h}_2(x_2), \dots, \hat{h}_n(x_n)] = [\hat{h}_1(\delta_{10}), \hat{h}_2(\delta_{20}), \dots, \hat{h}_n(\delta_{n0})] + [\sum_{j=1}^{\infty} \vartheta_1^{2j+1} p(-x_1 < \psi_1(G_{1(2j+1)}) \leq -\delta_{10}), \sum_{j=1}^{\infty} \vartheta_2^{2j+1} p(-x_2 < \psi_2(G_{2(2j+1)}) \leq -\delta_{20}), \dots, \sum_{j=1}^{\infty} \vartheta_n^{2j+1} p(-x_n < \psi_n(G_{n(2j+1)}) \leq -\delta_{n0})]$, but $[\hat{h}_1(x_1), \hat{h}_2(x_2), \dots, \hat{h}_n(x_n)] = [\hat{h}_1(0), \hat{h}_2(0), \dots, \hat{h}_n(0)] + [\sum_{j=1}^{\infty} \vartheta_1^{2j+1} p(-x_1 < \psi_1(G_{1(2j+1)}) \leq 0), \sum_{j=1}^{\infty} \vartheta_2^{2j+1} p(-x_2 < \psi_2(G_{2(2j+1)}) \leq 0), \dots, \sum_{j=1}^{\infty} \vartheta_n^{2j+1} p(-x_n < \psi_n(G_{n(2j+1)}) \leq 0)]$. Consequently, from Theorem 3, we get $[\hat{h}_1(0), \hat{h}_2(0), \dots, \hat{h}_n(0)] = [\sum_{j=1}^{\infty} \vartheta_1^{2j+1} p(\psi_1(G_{1(2j+1)}) > 0), \sum_{j=1}^{\infty} \vartheta_2^{2j+1} p(\psi_2(G_{2(2j+1)}) > 0), \dots, \sum_{j=1}^{\infty} \vartheta_n^{2j+1} p(\psi_n(G_{n(2j+1)}) > 0)] \leq [\sum_{j=1}^{\infty} \vartheta_1^{2j+1} \varepsilon_1^{G_{1(2j+1)}}, \sum_{j=1}^{\infty} \vartheta_2^{2j+1} \varepsilon_2^{G_{2(2j+1)}}, \dots, \sum_{j=1}^{\infty} \vartheta_n^{2j+1} \varepsilon_n^{G_{n(2j+1)}}] \leq [\vartheta_1^3/(\vartheta_1^2 - 1), \vartheta_2^3/(\vartheta_2^2 - 1), \dots, \vartheta_n^3/(\vartheta_n^2 - 1)]$, $0 < \varepsilon_i < 1$.

Suppose the following holds:

- (i) $\bar{d}_{i\xi_i} = G_{i(2\xi_i+1)}/\sigma_i = \mu_i(\vartheta_i^{2\xi_i+1} - 1)$,
- (ii) $\mu_i(\xi_i) = \psi_i(\xi_i \sigma_i)/2 = \sum_{j=1}^{\xi_i} X_{ij}$, where $\{X_{ij}\}$, $j = 1, 2, \dots$, is a sequence of independent identically distributed random variables,
- (iii) $a_i(\xi_i) = p(-x_i/2 < \mu_i(\xi_i) \leq 0) = \sum_{\mathcal{F}=0}^{|x_i|/2} p(-(\mathcal{F} + 1) < \mu_i(\xi_i) \leq -\mathcal{F})$,
- (iv) $\bar{m}$ is an integer such that $\bar{d}_{i\bar{m}} = r_{i1}|x_i| + r_{i2}$,
- (v) $\bar{\alpha}_i = \vartheta_i^2/\mu_i(\vartheta_i - 1)$,
- (vi) $U_i(\mathcal{F}, \mathcal{F} + 1) = \sum_{\xi_i=0}^{\infty} p(-(\mathcal{F} + 1) < \mu_i(\xi_i) \leq -\mathcal{F})$.

Thus, from Theorem 2, we have $a_i(\xi_i)$ is nonincreasing if $\xi_i > \bar{d}_{i\xi_i}$. By applying Lemma 5, we have $[\hat{h}_1(x_1), \hat{h}_2(x_2), \dots, \hat{h}_n(x_n)] - [\hat{h}_1(0), \hat{h}_2(0), \dots, \hat{h}_n(0)] = [\sum_{j=1}^{\infty} \vartheta_1^{2j+1} p(-x_1 < \psi_1(G_{1(2j+1)}) \leq 0), \sum_{j=1}^{\infty} \vartheta_2^{2j+1} p(-x_2 < \psi_2(G_{2(2j+1)}) \leq 0), \dots, \sum_{j=1}^{\infty} \vartheta_n^{2j+1} p(-x_n < \psi_n(G_{n(2j+1)}) \leq 0)] = [\sum_{\xi_1=1}^{\bar{m}} \vartheta_1^{2\xi_1+1} a_1(\bar{d}_{1\xi_1}), \sum_{\xi_2=1}^{\bar{m}} \vartheta_2^{2\xi_2+1} a_2(\bar{d}_{2\xi_2}), \dots, \sum_{\xi_n=1}^{\bar{m}} \vartheta_n^{2\xi_n+1} a_n(\bar{d}_{n\xi_n})] + [\sum_{\xi_1=\bar{m}+1}^{\infty} \vartheta_1^{2\xi_1+1} a_1(\bar{d}_{1\xi_1}), \sum_{\xi_2=\bar{m}+1}^{\infty} \vartheta_2^{2\xi_2+1} a_2(\bar{d}_{2\xi_2}), \dots, \sum_{\xi_n=\bar{m}+1}^{\infty} \vartheta_n^{2\xi_n+1} a_n(\bar{d}_{n\xi_n})] \leq [\sum_{\xi_1=1}^{\bar{m}} \vartheta_1^{2\xi_1+1}, \sum_{\xi_2=1}^{\bar{m}} \vartheta_2^{2\xi_2+1}, \dots, \sum_{\xi_n=1}^{\bar{m}} \vartheta_n^{2\xi_n+1}] + [\bar{\alpha}_1 \sum_{\xi_1=\bar{m}+1}^{\infty} (\bar{d}_{1\xi_1} - \bar{d}_{1(\xi_1-1)}) a_1(\bar{d}_{1\xi_1}), \bar{\alpha}_2 \sum_{\xi_2=\bar{m}+1}^{\infty} (\bar{d}_{2\xi_2} - \bar{d}_{2(\xi_2-1)}) a_2(\bar{d}_{2\xi_2}), \dots, \bar{\alpha}_n \sum_{\xi_n=\bar{m}+1}^{\infty} (\bar{d}_{n\xi_n} - \bar{d}_{n(\xi_n-1)}) a_n(\bar{d}_{n\xi_n})] \leq [\sum_{\xi_1=1}^{\bar{m}} \vartheta_1^{2\xi_1+1}, \sum_{\xi_2=1}^{\bar{m}} \vartheta_2^{2\xi_2+1}, \dots, \sum_{\xi_n=1}^{\bar{m}} \vartheta_n^{2\xi_n+1}] + [\bar{\alpha}_1 \sum_{\xi_1=\bar{d}_{1\bar{m}}}^{\infty} a_1(\xi_1), \bar{\alpha}_2 \sum_{\xi_2=\bar{d}_{2\bar{m}}}^{\infty} a_2(\xi_2), \dots, \bar{\alpha}_n \sum_{\xi_n=\bar{d}_{n\bar{m}}}^{\infty} a_n(\xi_n)] \leq [\sum_{\xi_1=1}^{\bar{m}} \vartheta_1^{2\xi_1+1}, \sum_{\xi_2=1}^{\bar{m}} \vartheta_2^{2\xi_2+1}, \dots, \sum_{\xi_n=1}^{\bar{m}} \vartheta_n^{2\xi_n+1}] + [\bar{\alpha}_1 \sum_{\mathcal{F}=0}^{|x_1|/2} U_1(\mathcal{F}, \mathcal{F} + 1), \bar{\alpha}_2 \sum_{\mathcal{F}=0}^{|x_2|/2} U_2(\mathcal{F}, \mathcal{F} + 1), \dots, \bar{\alpha}_n \sum_{\mathcal{F}=0}^{|x_n|/2} U_n(\mathcal{F}, \mathcal{F} + 1)]$

Since $U_i(\mathcal{F}, \mathcal{F} + 1)$ satisfies the conditions of renewal theorem as in Feller [28], then $U_i(\mathcal{F}, \mathcal{F} + 1)$ is bounded

for all $\mathcal{F}$, $i = 1, 2, \dots, n$, by a constant so ${}^2\hat{h}(x) = [q_1(x_1), q_2(x_2), \dots, q_n(x_n)] \leq [{}^2\hat{h}_1(\delta_{10}), {}^2\hat{h}_2(\delta_{20}), \dots, {}^2\hat{h}_n(\delta_{n0})] + [{}^2\Lambda_{11}, {}^2\Lambda_{21}, \dots, {}^2\Lambda_{n1}] + [{}^2\Lambda_{12}|x_1|, {}^2\Lambda_{22}|x_2|, \dots, {}^2\Lambda_{n2}|x_n|] = [L_1(|x_1|), L_2(|x_2|), \dots, L_n(|x_n|)] = L(|x|)$, then, ${}^2\hat{h}(x) \leq L(|x|)$. Similar to the previous analysis we can prove that ${}^2\hat{q}(x) \leq \bar{L}(|x|)$ and also ${}^1\hat{w}(x) \leq M(|x|)$, ${}^1\hat{h}(x) \leq \bar{M}(|x|)$ if $\delta_{i0} < 0$. Therefore, the proof is completed. $\square$

Theorem 7. If there exists a finite search plan $\phi(t) = (\phi_1(t), \phi_2(t), \dots, \phi_n(t)) \in \Phi(t)$, then $E(|X_0|)$ is finite.

Proof. For $E_{\tau_{\phi_i}} < \infty$, $i = 1, 2, \dots, n$, we have $p(\tau_{\phi_i} \text{ is finite}) = 1$ and so $p(\text{one of } \tau_{\phi_i} \text{ is finite}, i = 1, 2, \dots, n) = 1$. Therefore,

$$p\left(\bigcup_{i=1}^n (\tau_{\phi_i} < \infty)\right) = \sum_{i=1}^n p(\tau_{\phi_i} < \infty) - \sum_{i, \bar{\beta}} p((\tau_{\phi_i} < \infty) \cap (\tau_{\phi_{\bar{\beta}}} < \infty)) + \dots + (-1)^n p\left(\bigcap_{i=1}^n (\tau_{\phi_i} < \infty)\right). \quad (31)$$

Since $\tau_{\phi_i} < \infty$, $i = 1, 2, \dots, n$, are mutually exclusive events, then

$$\sum_{i=1}^n p(\tau_{\phi_i} < \infty) = 1. \quad (32)$$

Consequently, $p(\tau_{\phi_i} \text{ is finite}) = 1$ or $p(\tau_{\phi_{\theta}} \text{ is finite}) = 1$ for all $i \neq \theta$, $i, \theta = 1, 2, \dots, n$. If $p(\tau_{\phi_i} \text{ is finite}) = 1$ then $X_0 = \phi_i(\tau_{\phi_i}) - W(\tau_{\phi_i})$ with probability one and $|X_0| \leq |\phi_i(\tau_{\phi_i})| + |W(\tau_{\phi_i})| \leq \tau_{\phi_i} + |W(\tau_{\phi_i})|$ that leads to $E(|X_0|) \leq E_{\tau_{\phi_i}} + E(|W(\tau_{\phi_i})|)$. But $W(\tau_{\phi_i}) < \tau_{\phi_i}$, then $E(|W(\tau_{\phi_i})|) < E_{\tau_{\phi_i}}$, and $E(|X_0|)$ is finite.

On the other hand, if $p(\tau_{\phi_{\theta}} \text{ is finite}) = 1$ then $X_0 = \phi_{\theta}(\tau_{\phi_{\theta}}) - W(\tau_{\phi_{\theta}})$ with probability one, and by the same manner we can get $E(|X_0|)$ is finite. $\square$

Remark 8. A direct consequence of Theorems 4, 6, and 7 satisfies the existence of a finite search plan if and only if $E(|X_0|)$ is finite.

5. Existence of an Optimal Search Plan

The goal of the searching strategy could minimize the expected value of the first meeting time between one of the searchers and the target. Therefore, the main problem here is to find a search paths $\phi_i(t)$, $i = 1, 2, \dots, m$. If such a search paths exists, we call it optimal search paths.

Definition 9. Let $\{\phi_{ih}(t)\}_{h \geq 1} \in \Phi_1(t)$, $i = 1, 2, \dots, m$, be sequences of search plans; we say that $\phi_{ih}(t)$ converges to $\phi_i(t)$ as h tends to ∞ if and only if, for any $t \in I^+$, $\phi_{ih}(t)$ converges to $\phi_i(t)$ uniformly on every compact space (El-Rayes et al. [9]).

Theorem 10. Let for any $t \in I^+$ and $W(t)$ be a one-dimensional random walk. Then, the mapping

$$(\phi_1(t), \phi_2(t), \dots, \phi_n(t)) \longrightarrow E_{\tau_\phi} \in I^+, \quad (33)$$

is lower semicontinuous on $\Phi_1(t)$.

Proof. Let $B(\phi_i, t)$ be the indicator function of the set $\{\tau_{\phi_i} > t, i = 1, 2, \dots, n\}$; by the Fatou-Lesbesque theorem, we get

$$\begin{aligned} E_{\tau_\phi} &= E \left[\sum_{t=0}^{\infty} B(\phi_i, t) \right] \\ &= E \left[\sum_{t=0}^{\infty} \lim_{h \rightarrow \infty} \inf B(\phi_{ih}(t), t) \right] \\ &\leq \sum_{t=0}^{\infty} \lim_{h \rightarrow \infty} \inf E(1_{\{\tau_{\phi_h} > t\}}), \end{aligned} \quad (34)$$

for any sequences $\phi_{ih} \rightarrow \phi_i$ in $\Phi_1(t)$, where $\Phi_1(t)$ is sequentially compact; see El-Rayes et al. [9]. Thus the mapping $(\phi_1(t), \phi_2(t), \dots, \phi_n(t)) \rightarrow E_{\tau_\phi}$ is lower semicontinuous mapping on $\Phi_1(t)$ and this completed the proof. $\square$

6. Optimal Search Plan

In this section, we will find the optimal search plan that minimize the expected value of the first meeting time.

Since τ_ϕ depends on infimum either one of the functions $\phi_i(t)$, $i = 1, 2, \dots, n$, then we need to minimize E_{τ_ϕ} . That happens when we obtain the optimal search plan $\phi^*(t) = \{\phi_1^*(t), \phi_2^*(t), \dots, \phi_n^*(t)\}$, where $\phi_i(t)$ is a function on a distance H_{ij} , $i = 1, 2, \dots, n$, $1 \leq j \leq \mathcal{F}/2$. Therefore, it can be seen that the vector $(\phi_1(t), \phi_2(t), \dots, \phi_n(t))$ are n -distinct objective functions. Since the speed of the n -searchers (search team) is $v_i = 1$, $i = 1, 2, \dots, n$, then the objective functions $\phi_i(t)$, $i = 1, 2, \dots, n$, are distinct set of constrains given from Cases 1 and 2. In this problem, we consider the constraint $G_{i(2j-1)} \leq t \leq G_{i(2j)}$, $i = 1, 2, \dots, n$, $1 \leq j \leq \mathcal{F}/2$. Then, the problem takes the form

$$\begin{aligned} \min \quad & (\phi_1(t), \phi_2(t), \dots, \phi_n(t)) \\ \text{subject to} \quad & t \in T = \left\{ t \in I^+ \mid G_{i(2j-1)} \leq t \leq G_{i(2j)}, \right. \\ & \left. i = 1, 2, \dots, n, 1 \leq j \leq \frac{\mathcal{F}}{2} \right\}, \end{aligned} \quad (35)$$

where t is an n -dimensional vector of the decision variables, $\phi_1(t), \phi_2(t), \dots, \phi_n(t)$ are n -distinct objective functions of the decision vector of inequality constraints, and T is the feasible set of constrained decisions.

But $v_i = 1$, $i = 1, 2, \dots, n$, that leads to $t = H_{ij}$, $i = 1, 2, \dots, n$, $1 \leq j \leq \mathcal{F}/2$; then the above problem

is represented as the following multiobjective nonlinear programming problem (MONLP):

MONLP:

$$\begin{aligned} \min \quad & (\Xi_1(H_{1j}), \Xi_2(H_{2j}), \dots, \Xi_n(H_{nj})), \\ \text{subject to} \quad & G_{i(2j-1)} \leq H_{ij} \leq G_{i(2j)}, \\ & H_{ij} \in I^+, \quad i = 1, 2, \dots, n, 1 \leq j \leq \frac{\mathcal{F}}{2}, \\ \text{where} \quad & \Xi_i(H_{ij}) = H_{i(2j+1)} - \delta_{i0} - [t - G_{i(2j-1)}]. \end{aligned} \quad (36)$$

However, on formulating the MONLP which closely describes and represents the real decision situation, various factors of the real system should be reflected in the description of the objective functions and the constraints. Naturally, these objective functions and the constraints involve many variables and parameters whose possible values may be assigned by the experts. In the conventional approach, such parameters are changed according to the variables in an experimental and/or subjective manner through the experts understanding the nature of the variables.

Substituting $G_{ij} = b\omega_i(\vartheta_i^j - 1)$ in the above MONLP we have

MONLP(I):

$$\begin{aligned} \min \quad & (\Xi_1(\vartheta_1; \omega_1), \Xi_2(\vartheta_2; \omega_2), \dots, \Xi_n(\vartheta_n; \omega_n)), \\ \text{subject to} \quad & b\omega_i(\vartheta_i^{2j-1} - 1) \leq H_{ij}, \quad H_{ij} \leq b\omega_i(\vartheta_i^{2j} - 1), \\ & H_{ij} \in I^+, \quad i = 1, 2, \dots, n, 1 \leq j \leq \frac{\mathcal{F}}{2}, \\ \text{where} \quad & \Xi_i(H_{ij}) = H_{i(2j+1)} - \delta_{i0} \\ & - [t - b\omega_i(\vartheta_i^{2j-1} - 1)], \end{aligned} \quad (37)$$

where the variable ϑ_i takes +ve numbers greater than 1 and the parameter ω_i is the least positive integer such that $\omega_i(1 - (\vartheta_i - 1)/(\vartheta_i + 1))/2$ is positive number; $H_{ij} \geq 0$ for all $i = 1, 2, \dots, n$, $1 \leq j \leq \mathcal{F}/2$. Furthermore, H_{ij} is a function of the variable ϑ_i and the parameter ω_i , where

$$\begin{aligned} H_{ij} &= (-1)^{j+1} C_i [G_{ij} + 1 + (-1)^{j+1}] + \delta_{i0} \\ &= (-1)^{j+1} \left(\frac{\vartheta_i - 1}{\vartheta_i + 1} \right) [b\omega_i(\vartheta_i^j - 1) + 1 + (-1)^{j+1}] \\ &\quad + \delta_{i0}. \end{aligned} \quad (38)$$

Then, the previous MONLP(I) takes the form

MONLP(II):

$$\min \quad (\Xi_1(\vartheta_1; \omega_1), \Xi_2(\vartheta_2; \omega_2), \dots, \Xi_n(\vartheta_n; \omega_n)),$$

$$\begin{aligned} \text{subject to} \quad & b\omega_i(\vartheta_i^{2j-1} - 1) \\ & \leq (h_1 - 1) \left(\frac{\vartheta_i - 1}{\vartheta_i + 1} \right) \\ & \quad \times [b\omega_i(\vartheta_i^j - 1) + h_1] + \delta_{i0}, \\ & b\omega_i(\vartheta_i^{2j} - 1) \geq (h_1 - 1) \left(\frac{\vartheta_i - 1}{\vartheta_i + 1} \right) \\ & \quad \times [b\omega_i(\vartheta_i^j - 1) + h_1] \\ & \quad + \delta_{i0}, \end{aligned}$$

$$\begin{aligned} \text{where} \quad \Xi_i(\vartheta_i; \omega_i) &= (h_3) \left(\frac{\vartheta_i - 1}{\vartheta_i + 1} \right) \\ & \quad \times [b\omega_i(\vartheta_i^{2j-1} - 1) + h_4] \\ & \quad - 2\delta_{i0} + (h_2) \left(\frac{\vartheta_i - 1}{\vartheta_i + 1} \right) \\ & \quad \times [b\omega_i(\vartheta_i^j - 1) + h_5] \\ & \quad + b\omega_i(\vartheta_i^{2j-1} - 1), \\ h_1 &= 1 + (-1)^{j+1}, \quad h_2 = (-1)^{j+2}, \quad h_3 = (-1)^{2j+1}, \\ h_4 &= 1 + (-1)^{2j}, \quad h_5 = 1 + (-1)^j, \quad \vartheta_i > 1, \\ \omega_i &> 0 \quad \forall i = 1, 2, \dots, n, \quad b = 1, 2, \dots, \quad 1 \leq j \leq \frac{\mathcal{F}}{2}. \end{aligned} \quad (39)$$

Choosing a certain weights $\mathcal{U}_i$, $i = 1, 2, \dots, n$, such that $\sum_{i=1}^n \mathcal{U}_i = 1$ and the certain parameters $\omega_i = 1$, $i = 1, 2, \dots, n$, then the previous MONLP(II) takes the form

$$\begin{aligned} \text{MONLP(III):} \quad \min \quad & \mathcal{U}_i \left\{ (h_3) \left(\frac{\vartheta_i - 1}{\vartheta_i + 1} \right) \right. \\ & \times [b(\vartheta_i^{2j-1} - 1) + h_4] \\ & - 2\delta_{i0} + (h_2) \left(\frac{\vartheta_i - 1}{\vartheta_i + 1} \right) \\ & \times [b(\vartheta_i^j - 1) + h_5] \\ & \left. + b(\vartheta_i^{2j-1} - 1) \right\}, \end{aligned}$$

$$\begin{aligned} \text{subject to} \quad & b(\vartheta_i^{2j-1} - 1) \\ & \leq (h_1 - 1) \left(\frac{\vartheta_i - 1}{\vartheta_i + 1} \right) \\ & \quad \times [b(\vartheta_i^j - 1) + h_1] \\ & \quad + \delta_{i0}, \\ & b(\vartheta_i^{2j} - 1) \\ & \geq (h_1 - 1) \left(\frac{\vartheta_i - 1}{\vartheta_i + 1} \right) \\ & \quad \times [b(\vartheta_i^j - 1) + h_1] \\ & \quad + \delta_{i0}, \\ & 1 - \vartheta_i < 0, \quad \forall b = 1, 2, \dots, \quad 1 \leq j \leq \frac{\mathcal{F}}{2}. \end{aligned} \quad (40)$$

From the Kuhn-Tucker conditions we have

$$\begin{aligned} & \frac{1}{2} \left\{ \frac{2}{(\vartheta_i + 1)^2} [(h_3 - 1) [b(\vartheta_i^{2j-1} - 1) + h_4] \right. \\ & \quad \left. + (h_2 - 1) [b(\vartheta_i^j - 1) + h_5]] \right. \\ & \quad \left. + \left(\frac{\vartheta_i - 1}{\vartheta_i + 1} \right) b [(h_3 - 1) (2j - 1) \vartheta_i^{2j-2} + (h_2 - 1) j \vartheta_i^{j-1}] \right. \\ & \quad \left. + b(2j - 1) \vartheta_i^{2j-2} \right\} \\ & + u_1 \left\{ b(2j - 1) \vartheta_i^{2j-2} - \frac{(h_1 - 1)}{\vartheta_i + 1} \right. \\ & \quad \left. \times \left[\frac{2}{\vartheta_i + 1} [b(\vartheta_i^j - 1) + h_1] + (\vartheta_i - 1) j b \vartheta_i^{j-1} \right] \right\} \\ & + u_2 \left\{ -b(2j - 1) \vartheta_i^{2j-2} + \frac{(h_1 - 1)}{\vartheta_i + 1} \right. \\ & \quad \left. \times \left[\frac{2}{\vartheta_i + 1} [b(\vartheta_i^j - 1) + h_1] + (\vartheta_i - 1) j b \vartheta_i^{j-1} \right] \right\} \\ & - u_3 = 0, \end{aligned} \quad (41)$$

$$\begin{aligned} & u_1 \{ b(\vartheta_i^{2j-1} - 1) (\vartheta_i + 1) - (h_1 - 1) (\vartheta_i - 1) \\ & \quad \times [b(\vartheta_i^j - 1) + h_1] \} = 0, \end{aligned} \quad (42)$$

$$\begin{aligned} & u_2 \{ (h_1 - 1) (\vartheta_i - 1) [b(\vartheta_i^j - 1) + h_1] \\ & \quad - b(\vartheta_i^{2j-1} - 1) (\vartheta_i + 1) \} = 0, \end{aligned} \quad (43)$$

$$u_3 \{ 1 - \vartheta_i \} = 0. \quad (44)$$

FIGURE 2: (a) and (b) give the optimal expected value of the first meeting time between S_i and the target that moves with discrete random walk in one dimension.

Solving (41)–(44), we found the optimal values of ϑ_i that give the optimal distances H_{ij} for all $b = 1, 2, \dots$, $1 \leq j \leq \mathcal{F}/2$, and they are given by

$$\vartheta_i = \sqrt{\frac{\hat{\Upsilon} + \hat{\Lambda}}{b(1-2j)\vartheta_i^{2j-2}}} - 1, \quad \vartheta_i > 1, \quad (45)$$

where

$$\begin{aligned} \hat{\Upsilon} &= 2 \left[(h_3 - 1) \left[b(\vartheta_i^{2j-1} - 1) + h_4 \right] \right. \\ &\quad \left. + (h_2 - 1) \left[b(\vartheta_i^j - 1) + h_5 \right] \right], \\ \hat{\Lambda} &= b(h_3 - 1)(2j - 1)\vartheta_i^{2j} - b(h_3 - 1)(2j - 1)\vartheta_i^{2j-2} \\ &\quad + b(h_2 - 1)j\vartheta_i^{j+1} - b(h_2 - 1)j\vartheta_i^{j-1}. \end{aligned} \quad (46)$$

7. An Illustrative Example

In this section, we clarify the effectiveness of this model by considering the following example.

Example 1. Suppose that a criminal drunk (target) leaves its cache and walks up and down through one of n -disjoint streets, totally disoriented. We model the streets as real lines with cache at the point X_0 . In addition, assume that the criminal drunk takes unit steps, so we may record his position with an integer. Thus, for example, if he takes 5 steps to the left, he will be at a position $X_0 - 5$. If we consider the previous conditions in Theorems 2 and 3 hold, then the optimal expected value of the first meeting time depends on the optimal values of the distances H_{ij} that the searcher S_i should do it and the optimal values of G_{ij} , $j = 2, 4, 6$. Using the constraint $G_{i(2j-1)} \leq t \leq G_{i(2j)}$, $i = 1, 2, \dots, n$, $1 \leq j \leq \mathcal{F}/2$, from Case 2, such that $j = 2, 4, 6$ and for $b = 1, 2$, in MONLP(III), and also using (38), we obtain the optimal values of ϑ_i that give the optimal values of H_{ij} , G_{ij} in Table 1.

It is clear that the optimal distances H_{ij} that S_i should do it depend on b where $\lambda_i = b\omega_i$ and $G_{ij} = \lambda_i(\vartheta_i^j - 1)$.

TABLE 1: The optimal values of ϑ_i , H_{ij} and G_{ij} , $j = 2, 4, 6$, $b = 1, 2$.

b	j	ϑ_i	G_{ij}	H_{ij}
1	2	1.8836	0.8836	$0.27075 + \delta_{i0}$
	4	2.5265	1.5265	$0.6607 - \delta_{i0}$
	6	2.6982	1.6982	$0.7798 + \delta_{i0}$
2	2	2.5226	3.0452	$1.3162 + \delta_{i0}$
	4	2.7897	3.5794	$2.2963 - \delta_{i0}$
	6	2.8652	3.7304	$1.8 + \delta_{i0}$

From the previous numerical calculations and by considering $\delta_{i0} = 10$, we get Figure 2 that shows the projection of the target's motion (one-dimensional random walk) on the S_i path with $b = 1, 2$ where the current position on the line represented in the vertical axis versus the time steps at horizontal axis.

8. Discussion and Conclusions

A multiplicative generalized linear search plan for a one-dimensional random walk target on one of n -disjoint real lines has been presented, where the target initial position is given by a random variable X_0 . Therefore, the target will be met if $E_{\tau_\phi} < \infty$, where $\phi(t) = (\phi_1(t), \phi_2(t), \dots, \phi_n(t))$ and E_{τ_ϕ} is the expected value of the first meeting time between one of the searchers and the target. We discuss some properties that the search model should satisfy in Theorems 2 and 3. We introduce the proof of conditions that make a search plan finite in Theorem 4, based on the continuity of the search plan and the conditions in Theorems 2 and 3 to show that $E_{\tau_\phi} < \infty$. We provide more analysis by using Lemma 5, Theorems 2 and 3 in Theorem 6 to show that the search plan $\phi(t)$ is finite if the conditions ${}^2\hat{h}(x) \leq L(|x|)$, ${}^2\hat{q}(x) \leq \tilde{L}(|x|)$, if $\delta_{i0} > 0$ and ${}^1\hat{w}(x) \leq M(|x|)$, ${}^1\hat{h}(x) \leq \tilde{M}(|x|)$ if $\delta_{i0} < 0$, where $L(|x|)$, $\tilde{L}(|x|)$, $M(|x|)$, and $\tilde{M}(|x|)$ are vectors of linear functions. We use Theorem 7 to show that if there exist a finite search plan then the expected value of the target initial position $E(|X_0|)$ is

finite. It will also be interesting to see a direct consequence of Theorems 4, 6, and 7 satisfying the existence of a finite search plan if and only if $E(|X_0|)$ is finite. The existence of an optimal search plan has been proved in Theorem 10.

To find the optimal distances H_{ij} , $i = 1, \dots, n$, $j = 1, 2, \dots$, that give the optimal search plan $\phi^*(t) = (\phi_1^*(t), \phi_2^*(t), \dots, \phi_n^*(t))$ we solve the multiobjective nonlinear programming problem (MONLP) which contains the variables ϑ_i and the parameters λ_i , $i = 1, \dots, n$. We use the Kuhn-Tucker conditions to obtain the optimal values of ϑ_i that can give the optimal distances, after substituting (38), and then minimize the expected value of the first meeting time. The effectiveness of this model is illustrated using numerical example.

In future research, it seems that the proposed model will be extendible to the multiple searchers case by considering the combinations of movement of multiple targets.

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