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Research Article

Central Limit Theorem for Coloured Hard Dimers

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We study the central limit theorem for a class of coloured graphs. This means that we investigate the limit behavior of certain random variables whose values are combinatorial parameters associated to these graphs. The techniques used at arriving this result comprise combinatorics, generating functions, and conditional expectations.

1. Introduction

In this paper we want to verify the central limit theorem (CLT) in the context of a combinatorial problem for coloured hard dimer configurations, which comprise a certain class of labeled graphs on $\mathbb{R}$. We will consider two combinatorial parameters that characterize our hard dimers and will therefore investigate a bivariate mass function. The problem is interesting as far as it is difficult to reformulate the task in such a manner that a general CLT becomes applicable. The proof here is based on the explicit knowledge of the bivariate mass function and is a generalization of the original De Moivre-Laplace theorem. By the way, coloured hard dimers are applied in the framework of causally triangulated (2 + 1)-dimensional quantum gravity. We believe that our result could be employed for further study on the asymptotics of the one-step propagator, as it has been defined in [1] by means of an inversion formula.

Let us describe the objects we are interested in. Given a sequence ξ_N of length N of consecutive blue and red vertices on the one-dimensional lattice $\mathbb{Z}$, one defines a dimer to be an edge connecting two nearest sites of the same colour. The dimer's colour is given by the colour of its boundary vertices. A coloured hard dimer configuration (CHDC) is defined to be a sequence ξ_N together with a sequence of dimers on it, which must not intersect. We include also the empty CHDC, that is, the configuration when no dimers are present. In Figure 1, an example of a CHDC is shown.

Figure 1: A CHDC ($N = 13$) with two red dimers and one blue dimer.

In graph-theoretic language, CHDCs form a subclass of labeled graphs whose vertices and edges carry one of two possible labels (here termed colours). For a given CHDC D, let $n_b(D), n_r(D)$ denote the numbers of blue, red dimers and $n_{br}(D)$ the number of inner vertices, that is, vertices inside dimers that are not boundary points. Below we shall consider the numbers (n_b, n_r, n_{br}) as random variables (r.v.s) investigating their joint probability generating function. Due to the symmetry w.r.t. n_b and n_r, this will lead us to the joint mass function of the r.v.s $(n_b + n_r, n_b + n_r + \gamma_b + \gamma_r)$, where $\gamma_b(D)$ and $\gamma_r(D)$ denote the number of blue and red vertices which are not occupied by dimers ("single points").

The paper is organized as follows. In Section 2, we give an explicit formula for the combinatorial generating function and define the probability mass function associated with CHDCs, with an exact expression for its normalizing constant, that is, $C_N = (3/2)^{N-1}$. Moreover, we find the right probability distribution for the r.v. $n_b + n_r + n_{br}$. In Section 3, we calculate the first two moments of the r.v. $n_b + n_r$. In Section 4, we prove a CLT for the pair of r.v.s $(n_b + n_r, n_b + n_r + \gamma_b + \gamma_r)$. The limit distribution is a bivariate Gaussian distribution with correlation coefficient equal to $-1/\sqrt{3}$.

2. Coloured Hard Dimers and Probability Distributions

With the definitions made in the introduction, the following constraint

$$2n_b(D) + 2n_r(D) + n_{br}(D) + \gamma_b(D) + \gamma_r(D) = N \tag{2.1}$$

holds. In the example above (Figure 1),

$$n_b(D) = 1, \qquad n_r(D) = 2, \qquad n_{br}(D) = 3, \qquad \gamma_b(D) = 3, \qquad \gamma_r(D) = 1. \tag{2.2}$$

First, we want to find the combinatorial generating function of the variables n_b, n_r, n_{br}. This is given as

$$\widetilde{F}_N(u, v, w) = \sum_{\xi_N} F_{\xi_N}(u, v, w), \tag{2.3}$$

where

$$F_{\xi_N}(u, v, w) = \sum_{\text{CHDC's on } \xi_N} u^{n_b(D)} v^{n_r(D)} w^{n_{br}(D)}, \quad u, v, w \in \mathbb{R}. \tag{2.4}$$

It is useful to define further variables t and s, where t corresponds to the number of sites occupied by dimers and s to the number of coloured hard dimers

$$t = N - \gamma_b(D) - \gamma_r(D), \qquad s = n_b(D) + n_r(D). \tag{2.5}$$

In the next proposition, we prove an exact formula for $\widetilde{F}_N$, using combinatorial arguments.

Lemma 2.1. *The combinatorial generating function $\widetilde{F}_N$ has the following explicit expression, for any N:*

$$\widetilde{F}_N(u,v,w) = 2^N\left(1+\sum_{t=1}^{N}\sum_{s=1}^{[t/2]}\binom{N-t+s}{s}\binom{t-s-1}{s-1}\left(\frac{u+v}{4}\right)^s\left(\frac{w}{2}\right)^{t-2s}\right), \tag{2.6}$$

where $[\cdot]$ denotes the integer part.

Proof of Lemma 2.1. Consider a CHDC D on any ξ_N with fixed but arbitrary N. We set $n_b(D) = g$, $n_r(D) = h$, $n_{br}(D) = m$, $\gamma_b(D) = \gamma_b$, and $\gamma_r(D) = \gamma_r$. By (2.1) and (2.5) the following equalities

$$2g+2h+m+\gamma_b+\gamma_r = N, \qquad t = 2g+2h+m \tag{2.7}$$

hold.

First, we fix the number of blue and red dimers and that of single points γ_b and γ_r. Note that by (2.7) m is also fixed. The dimers of the same colour are considered indistinguishable. Then, we calculate all possible permutations of g blue dimers, h red dimers, γ_b blue single points, and γ_r red single points, that is,

$$\frac{(g+h+\gamma_b+\gamma_r)!}{g!h!\gamma_b!\gamma_r!}. \tag{2.8}$$

Now, for any nontrivial given permutation of coloured dimers and single points, that is, $n_b + n_r \neq 0$ (the empty CHDCs contribute a factor 2^N), we have to see in how many ways we can distribute the given m inner vertices over the given dimers. The number of all these combinations is

$$\binom{m+g+h-1}{g+h-1}. \tag{2.9}$$

Therefore, all contributions are summed to

$$2^N + \sum_{\substack{g,h,m:\\ 2g+2h+m=1}}^{N}\sum_{\gamma_b+\gamma_r=N-2g-2h-m}\frac{(g+h+\gamma_b+\gamma_r)!}{g!h!\gamma_b!\gamma_r!}\binom{m+g+h-1}{g+h-1}u^g v^h w^m. \tag{2.10}$$

Since $g+h+\gamma_b+\gamma_r = N-g-h-m$ and $\gamma_r = N-2g-2h-m-\gamma_b$, we have

$$\begin{aligned}
&\widetilde{F}_N(u,v,w) \\
&= 2^N + \sum_{\substack{g,h,m:\\ 2g+2h+m=1}}^{N} \frac{(N-g-h-m)!}{g!h!(N-2g-2h-m)!}\binom{m+g+h-1}{g+h-1} u^g v^h w^m \\
&\quad \times \sum_{\gamma_b=0}^{N-2g-2h-m} \binom{N-2g-2h-m}{\gamma_b}.
\end{aligned} \tag{2.11}$$

In obtaining (2.11), we have multiplied and divided the generic term of the previous sum by $(N-2g-2h-m)!$. The binomial formula yields $2^{N-2g-2h-m}$, and (2.11) becomes

$$2^N\left(1+\sum_{\substack{g,h,m:\\ 2g+2h+m=1}}^{N} \frac{(N-g-h-m)!}{g!h!(N-2g-2h-m)!}\binom{m+g+h-1}{g+h-1}\left(\frac{u}{4}\right)^g\left(\frac{v}{4}\right)^h\left(\frac{w}{2}\right)^m\right). \tag{2.12}$$

Now by multiplying and dividing the generic term of the sum by $(g+h)!$, we get

$$2^N\left(1+\sum_{\substack{g,h,m:\\ 2g+2h+m=1}}^{N} \binom{N-g-h-m}{g+h}\binom{m+g+h-1}{g+h-1}\binom{g+h}{g}\left(\frac{u}{4}\right)^g\left(\frac{v}{4}\right)^h\left(\frac{w}{2}\right)^m\right). \tag{2.13}$$

Performing the variable changements $s = g+h$ and $t = 2g+2h+m$, with s and t as above, we get

$$\widetilde{F}_N(u,v,w) = 2^N\left(1+\sum_{t=1}^{N}\sum_{s=1}^{[t/2]}\binom{N-t+s}{s}\binom{t-s-1}{s-1}\left(\frac{u+v}{4}\right)^s\left(\frac{w}{2}\right)^{t-2s}\right). \tag{2.14}$$

In the last sum, we applied again the binomial formula

$$\sum_{g=0}^{s}\binom{s}{g}\left(\frac{u}{4}\right)^g\left(\frac{v}{4}\right)^{s-g} = \left(\frac{u+v}{4}\right)^s. \tag{2.15}$$

Therefore, we get formula (2.6) where only the indices s and t appear. The lemma is so proved. □

If we want to understand the appearance of CHDCs in probabilistic terms it is natural to assign each CHDC the same probability, so that the combinatorial frequency of particular

configurations will be proportional to their probability. For this, let us define a family of probability spaces $(\Omega_N, \mathcal{F}_N, P_N)_{N\in\mathbb{N}_{\geq 1}}$. We choose Ω_N, to be the set of all different CHDCs. The σ-algebra $\mathcal{F}_N$ is the power set of Ω_N and for P_N we take the probability measure having uniform distribution on Ω_N. Normalizing the function $\widetilde{F}_N$ by $\widetilde{F}_N|_{(u,v,w)=(1,1,1)}$ then just gives the joint probability generating function of the random variables n_b, n_r, n_{br}, defined on $(\Omega_N, \mathcal{F}_N, P_N)$, which count, for each hard dimer configuration, the number of blue, red dimers and the number of inner vertices, respectively.

The main result of this section is an explicit formula for the normalizing constant C_N of the probability measure P_N, that holds for any N, derived by evaluating the combinatorial generating function $\widetilde{F}_N$ at the point $u = v = w = 1$. Considering the change of variable $k = N - t + s$, we have

$$C_N = 2^N\left(1 + \sum_{k=1}^{N-1}\sum_{s=1}^{k}\binom{k}{s}\binom{N-k-1}{s-1}\frac{1}{2^{N-k}}\right) = 2^N\sum_{k=1}^{N}\binom{N-1}{k-1}\frac{1}{2^{N-k}} = 2\cdot 3^{N-1}. \quad (2.16)$$

Throughout the text, we use the convention that $\binom{a}{b} = 0$, whenever $a < b$, $a < 0$ or $b < 0$.

Remark 2.2. Note that, upon normalization, the factors $\binom{k}{s}\binom{N-k-1}{s-1}$ in (2.16) yield a hypergeometric distribution with respect to the variable s. Here, the sample size is $N - k$ and k (resp., $N-k-1$) are the total number of successes (resp. failures). Therefore, summing over s, we get the binomial coefficient $\binom{N-1}{k-1}$.

In this way, we have also found the joint mass function $\widetilde{P}_N$ related to the r.v.s $n_b + n_r$ and $n_b + n_r + \gamma_b + \gamma_r$, more precisely,

$$\widetilde{P}_N(s,k) = P_N(n_b + n_r = s; n_b + n_r + \gamma_b + \gamma_r = k) \quad (2.17)$$

is given by

$$\widetilde{P}_N(s,k) = \begin{cases} \left(\dfrac{2}{3}\right)^{N-1}, & \text{for } s = 0, \\ \dbinom{k}{s}\dbinom{N-k-1}{s-1}\left(\dfrac{2}{3}\right)^{k-1}\left(\dfrac{1}{3}\right)^{N-k}, & \text{otherwise.} \end{cases} \quad (2.18)$$

Remark 2.3. From (2.18), we deduce that the r.v. $n_b + n_r + \gamma_b + \gamma_r$ is binomial with parameters $N-1$ and $2/3$. Since $n_b + n_r + n_{br} = N - (n_b + n_r + \gamma_b + \gamma_r)$, it follows that the r.v. $n_b + n_r + n_{br}$ is also binomial with parameters $N-1$ and $1/3$.

3. Number of Dimers: The First Two Moments

When proving a CLT, we have to rescale the r.v.s by subtracting the means and dividing by the standard deviations in question. In the previous section, we have seen that $n_b + n_r + \gamma_b + \gamma_r$ is binomial, whose moments are known. Although the distribution of the r.v. $n_b + n_r$ is not of

common type, we are able to compute its mean and variance. For this, we rely on the fact that the conditional distribution of $n_b + n_r$, given $n_b + n_r + \gamma_b + \gamma_r$, is hypergeometric by Remark 2.2. We start with the computation of the mean.

Proposition 3.1. *For any N, the following formula*

$$E_N(n_b + n_r) = \frac{2N-1}{9} \tag{3.1}$$

holds.

By E_N, we indicate the mean with respect to the probability measure P_N.

Proof. By the properties of the conditional expectation and taking into account Remark 2.2, we have

$$\begin{aligned} E_N(n_b + n_r) &= \sum_{k=1}^{N} E_N(n_b + n_r \mid n_b + n_r + \gamma_b + \gamma_r = k) P_N(n_b + n_r + \gamma_b + \gamma_r = k) \\ &= \sum_{k=1}^{N} \frac{k(N-k)}{N-1} \binom{N-1}{k-1} \left(\frac{2}{3}\right)^{k-1} \left(\frac{1}{3}\right)^{N-k}. \end{aligned} \tag{3.2}$$

In fact, the expectation of our hypergeometric distribution is $k(N-k)/(N-1)$. Hence,

$$E_N(n_b + n_r) = \frac{1}{3} \sum_{k=1}^{N} k \binom{N-2}{k-1} \left(\frac{2}{3}\right)^{k-1} \left(\frac{1}{3}\right)^{N-1-k} = \frac{1}{3}\left[\frac{2}{3}(N-2) + 1\right] = \frac{2N-1}{9}. \tag{3.3}$$

In the last sum, we have used the decomposition $k = (k-1) + 1$ which gives the first and zeroth moment of the binomial distribution with parameters $N-2$ and $2/3$. □

Remark 3.2. By identity (2.1), Remark 2.3, and from (3.1), we are able to calculate the single point number's mean. In fact

$$\begin{aligned} E_N(\gamma_b + \gamma_r) &= N - E_N(n_b + n_r) - E_N(n_b + n_r + n_{br}) \\ &= N - \frac{2N-1}{9} - \frac{N-1}{3} = \frac{4}{9}(N+1). \end{aligned} \tag{3.4}$$

Note that, as $N \to \infty$,

$$E_N(\gamma_b + \gamma_r) \asymp \frac{4}{9} N \asymp 2E_N(n_b + n_r), \tag{3.5}$$

that is, for the present model, the expected number of single points is asymptotically twice the expected number of dimers. Moreover, fixing the number of single points, the conditional probability distribution of γ_b (resp., γ_r) is binomial and symmetric.

In order to find the variance of $n_b + n_r$, we apply the law of total variance involving the conditional expectation and the conditional variance, (see, e.g., [2])

$$\begin{aligned}\mathrm{Var}_N(n_b + n_r) = {} & E_N\left(\mathrm{Var}_N\left(n_b + n_r \mid n_b + n_r + \gamma_b + \gamma_r\right)\right) \\ & + \mathrm{Var}_N\left(E_N\left(n_b + n_r \mid n_b + n_r + \gamma_b + \gamma_r\right)\right).\end{aligned} \tag{3.6}$$

The symbol Var_N indicates the variance w.r.t. P_N. We recall that the conditional variance of a r.v. X given a r.v. Y is defined as

$$\mathrm{Var}(X \mid Y) = E\left(X^2 \mid Y\right) - (E(X \mid Y))^2. \tag{3.7}$$

Alternatively one can define $\mathrm{Var}(X \mid Y)$ as that function of Y, whose value at $Y = y$ is given by

$$\mathrm{Var}(X \mid Y = y) = E\left(X^2 \mid Y = y\right) - \left(E(X \mid Y = y)\right)^2. \tag{3.8}$$

In the next Lemmas, we find the exact expressions of the two terms in (3.6).

Lemma 3.3. *For all N, one has*

$$E_N\left(\mathrm{Var}_N\left(n_b + n_r \mid n_b + n_r + \gamma_b + \gamma_r\right)\right) = \frac{4}{81}(N-3). \tag{3.9}$$

Proof. As above, we use the fact that the variance of the hypergeometric distribution is known. In our case,

$$\mathrm{Var}_N\left(n_b + n_r \mid n_b + n_r + \gamma_b + \gamma_r = k\right) = \frac{(N-k)(N-k-1)k(k-1)}{(N-1)^2(N-2)}. \tag{3.10}$$

This entails that

$$\begin{aligned}& E_N\left(\mathrm{Var}_N\left(n_b + n_r \mid n_b + n_r + \gamma_b + \gamma_r\right)\right) \\ & \quad = \sum_{k=1}^{N} E_N\left(\mathrm{Var}_N\left(n_b + n_r \mid n_b + n_r + \gamma_b + \gamma_r = k\right)\right) P_N\left(n_b + n_r + \gamma_b + \gamma_r = k\right) \\ & \quad = \frac{1}{9}\sum_{k=1}^{N-2} \frac{k(k-1)}{N-1}\binom{N-3}{k-1}\left(\frac{2}{3}\right)^{k-1}\left(\frac{1}{3}\right)^{N-2-k} \\ & \quad = \frac{1}{9(N-1)}\left[\frac{4}{9}(N-3)(N-4) + \frac{4}{3}(N-3)\right] = \frac{4}{81}(N-3).\end{aligned} \tag{3.11}$$

In fact, we can write $k(k-1) = (k-1)(k-2) + 2(k-1)$, so that the factorial moments of the binomial distribution with parameters $N-3$ and $2/3$ appear. It is easy to check that its second factorial moment is $4/9(N-3)(N-4)$. □

Lemma 3.4. *For all N, it holds that*

$$\mathrm{Var}_N\left(E_N\left(n_b+n_r \mid n_b+n_r+\gamma_b+\gamma_r\right)\right)=\frac{2}{81}(N+7). \tag{3.12}$$

Proof. Using again the properties of the conditional expectation, we have

$$\begin{aligned}&\mathrm{Var}_N\left(E_N\left(n_b+n_r \mid n_b+n_r+\gamma_b+\gamma_r\right)\right)\\&\quad=E_N\left(E_N\left(n_b+n_r \mid n_b+n_r+\gamma_b+\gamma_r\right)^2\right)-\left(E_N(n_b+n_r)\right)^2.\end{aligned} \tag{3.13}$$

From Proposition 3.1, we know that $E_N(n_b+n_r)=(2N-1)/9$, so that it remains to compute the first term in (3.13). The latter is given by

$$E_N\left(E_N\left(n_b+n_r \mid n_b+n_r+\gamma_b+\gamma_r\right)^2\right)=\sum_{k=1}^{N}\frac{k^2(N-k)^2}{(N-1)^2}\binom{N-1}{k-1}\left(\frac{2}{3}\right)^{k-1}\left(\frac{1}{3}\right)^{N-k}. \tag{3.14}$$

As in the previous Lemma 3.3, it is useful to use the factorial moments of the binomial distribution. In achieving this, we simply write $(N-k)^2=(N-k)(N-k-1)+(N-k)$. According to this decomposition, the sum (3.14) splits into two terms, which we denote by A_1 and A_2, respectively. The first term A_1 is

$$\begin{aligned}A_1&=\frac{N-2}{9(N-1)}\sum_{k=1}^{N-2}k^2\binom{N-3}{k-1}\left(\frac{2}{3}\right)^{k-1}\left(\frac{1}{3}\right)^{N-2-k}=\frac{N-2}{9(N-1)}\left[\frac{4}{9}(N-3)(N-4)+2(N-3)+1\right],\\A_2&=\frac{1}{3(N-1)}\sum_{k=1}^{N-1}k^2\binom{N-2}{k-1}\left(\frac{2}{3}\right)^{k-1}\left(\frac{1}{3}\right)^{N-1-k}=\frac{1}{3(N-1)}\left[\frac{4}{9}(N-2)(N-3)+2(N-2)+1\right].\end{aligned} \tag{3.15}$$

In the above computations, we have used the equality $k^2=(k-1)(k-2)+3(k-1)+1$, so that the factorial moments come into play. From (3.13) and (3.15), it is easy to get (3.12). □

Finally, we are able to state the next proposition for the variance of n_b+n_r.

Proposition 3.5. *For all N, one has*

$$\mathrm{Var}_N(n_b+n_r)=\frac{2}{81}(3N+1). \tag{3.16}$$

Proof. This follows immediately from (3.6) and Lemmas 3.3 and 3.4. □

Remark 3.6. From the formulas for the mean and the variance of n_b+n_r ((3.1) and (3.16)), one can deduce that the distribution of n_b+n_r is asymptotically not binomial. In fact, $E_N(n_b+n_r) \asymp (2/9)N$ and $\mathrm{Var}_N(n_b+n_r) \asymp (2/27)N \neq (2/9)\cdot(7/9)N$. In the next section, we prove that it is asymptotically Gaussian, for large N, that is, a CLT holds.

4. Central Limit Theorem for the Dimers' Number

In the present section, we study the asymptotic distribution of the dimers' number, in particular, we prove a CLT for the total number of dimers plus single points $n_b + n_r + \gamma_b + \gamma_r$, analyzed in Section 2, and the number of dimers $n_b + n_r$. The limit distribution is a bivariate Gaussian distribution with correlation coefficient equal to $-1/\sqrt{3}$.

The following proof is a generalization of De Moivre-Laplace's Theorem.

Theorem 4.1. *A central limit theorem holds for the joint probability distribution. This means that for any $-\infty \le a < b \le \infty$ and $-\infty \le a' < b' \le \infty$, one has*

$$\lim_{N\to\infty} \sum_{\substack{s:x\in(a,b)\\ k:y\in(a',b')}} \widetilde{P}_N(s,k) = \frac{1}{2\pi\sqrt{2/3}} \int_a^b \int_{a'}^{b'} e^{-(3/4)(z^2+z'^2+(2\sqrt{3}/3)zz')} dz dz', \tag{4.1}$$

where

$$x = \frac{s-(2/9)N}{\sqrt{6N}/9}, \qquad y = \frac{k-(2/3)N}{\sqrt{2N}/3}. \tag{4.2}$$

Remark 4.2. In (4.2) above, we consider only the first order of the expectations and the variances with respect to N. It is easy to see that the result remains the same when including terms of zeroth order, as they do not contribute to the asymptotics.

Proof. We have the following.

Step 1. We shall first verify a local version of the CLT, that is,

$$\begin{aligned} &P_N\left(n_b+n_r=s; n_b+n_r+\gamma_b+\gamma_r=k\right) \\ &\quad= \frac{1}{\sqrt{(2\pi)^2((2/27)N)((2/9)N)(2/3)}} e^{-(3/4)(x^2+(2\sqrt{3}/3)xy+y^2)}\left(1+r_{x,y}(N)\right), \end{aligned} \tag{4.3}$$

where x, y are given by (4.2). Moreover, $\lim_{N\to\infty} r_{x,y}(N) = 0$, uniformly with respect to x and y, belonging to finite intervals (a,b) and (a',b'), respectively.

By the remark above and the fact that the indices s and k are both of order N, we can forget the constants in (2.18), that is, $k-1 \asymp k$ and $N-k-1 \asymp N-k$, as $N \to \infty$.

As in De Moivre-Laplace's Theorem, we apply Stirling's formula to the binomial coefficients in (2.18). In the present model, we have two binomial coefficients instead of one, so that the calculus becomes heavier than in De Moivre-Laplace's Theorem. We write

$$\binom{k}{s}\binom{N-k}{s}\left(\frac{2}{3}\right)^k\left(\frac{1}{3}\right)^{N-k} = \underbrace{e^{\lambda_{N,s,k}}}_{A_{N,s,k}} \cdot \underbrace{\frac{1}{\sqrt{2\pi(s(k-s)/k)}\sqrt{2\pi(s(N-k-s)/(N-k))}}}_{B_{N,s,k}}$$
$$\cdot \underbrace{\left(\frac{k}{3s}\right)^s\left(\frac{2k}{3(k-s)}\right)^{k-s}\left(\frac{2(N-k)}{3s}\right)^s\left(\frac{N-k}{3(N-k-s)}\right)^{N-k-s}}_{C_{N,s,k}}, \tag{4.4}$$

where

$$\lambda_{N,s,k} \equiv \lambda_k - \lambda_s - \lambda_{k-s} + \lambda_{N-k} - \lambda_s - \lambda_{N-k-s}, \qquad \frac{1}{12n+1} \le \lambda_n \le \frac{1}{12n}, \tag{4.5}$$

for every $n \in \mathbb{N}$.

Taking into account (4.2), we consider the first factor $A_{N,s,k}$ in (4.4) with respect to the variables x and y. It is easy to see that

$$A_{N,s,k} = 1 + r^{(1)}_{x,y}(N), \tag{4.6}$$

with $r^{(1)}_{x,y}(N) \to 0$, as $N \to \infty$, uniformly with respect to x and y. In fact, since x, y belong to bounded intervals, one can estimate $\lambda_{N,s,k}$ uniformly from above and from below with respect to x and y.

For the factor $B_{N,s,k}$ in (4.4), the following asymptotic holds

$$B_{N,s,k} \asymp \frac{1}{2\pi\left(\sqrt{6N}/9\right)\left(\sqrt{2N}/3\right)\sqrt{2/3}}, \tag{4.7}$$

as $N \to \infty$. In fact, from (4.2), we estimate

$$\frac{s(k-s)}{k} = \frac{4}{27}N\left(1 + r^{(2)}_{x,y}(N)\right), \tag{4.8}$$

with $r^{(2)}_{x,y}(N) \to 0$, as $N \to \infty$, uniformly with respect to x and y.

Analogously,

$$\frac{s(N-k-s)}{N-k} = \frac{2}{27}N\left(1 + r^{(3)}_{x,y}(N)\right), \tag{4.9}$$

with $r^{(3)}_{x,y}(N) \to 0$, as $N \to \infty$, uniformly with respect to x and y.

By (4.7), we see that the correlation coefficient can be $\pm 1/\sqrt{3}$. Its sign will be determined later.

Finally, we consider the logarithm of the last factor $C_{N,s,k}$ in (4.4),

$$\begin{aligned}\ln C_{N,s,k} = &-s\ln\left(\frac{3s}{k}\right) - (k-s)\ln\left(\frac{3(k-s)}{2k}\right) - s\ln\left(\frac{3s}{2(N-k)}\right)\\ &- (N-k-s)\ln\left(\frac{3(N-k-s)}{N-k}\right) \equiv \sum_{i=1}^{4} C_{N,s,k}^{i}.\end{aligned} \tag{4.10}$$

Now we express each term of the sum in (4.10) $C_{N,s,k}^{i}$, $i = 1,2,3,4$, in terms of x and y, defined in (4.2). We start with $C_{N,s,k}^{1}$,

$$C_{N,s,k}^{1} = -\frac{\sqrt{2N}}{9}\left(\sqrt{2N}+\sqrt{3}x\right)\ln\left(1+\frac{\sqrt{3}x-y}{\sqrt{2N}+y}\right). \tag{4.11}$$

Since the last logarithm above is of the form $\ln(1+z)$, with $z \to 0$, we can expand it around $z = 0$, $\ln(1+z) = z - (z^2/2) + o(z^2)$, as $z \to 0$. The same is true for each logarithm function present in any $C_{N,s,k}^{i}$, $i = 1,2,3,4$. So $C_{N,s,k}^{1}$ becomes, as $N \to \infty$,

$$C_{N,s,k}^{1} \asymp -\frac{\sqrt{2N}}{18}\frac{\left(\sqrt{2N}+\sqrt{3}x\right)\left(\sqrt{3}x-y\right)\left(2\sqrt{2N}-\sqrt{3}x+3y\right)}{\left(\sqrt{2N}+y\right)^2}. \tag{4.12}$$

Analogously, for the other $C_{N,s,k}^{i}$, $i = 2,3,4$, we find

$$\begin{aligned}C_{N,s,k}^{2} &\asymp \frac{\sqrt{2N}}{72}\frac{\left(2\sqrt{2N}-\sqrt{3}x+3y\right)\left(\sqrt{3}x-y\right)\left(4\sqrt{2N}+\sqrt{3}x+3y\right)}{\left(\sqrt{2N}+y\right)^2},\\ C_{N,s,k}^{3} &\asymp -\frac{\sqrt{2N}}{18}\frac{\left(\sqrt{2N}+\sqrt{3}x\right)\left(\sqrt{3}x+2y\right)\left(2\sqrt{2N}-\sqrt{3}x-6y\right)}{\left(\sqrt{2N}-2y\right)^2},\\ C_{N,s,k}^{4} &\asymp 2\frac{\sqrt{2N}}{9}\frac{\left(\sqrt{N/2}-\sqrt{3}x-3y\right)\left(\sqrt{3}x+2y\right)\left(\sqrt{2N}+\sqrt{3}x\right)}{\left(\sqrt{2N}-2y\right)^2}.\end{aligned} \tag{4.13}$$

From (4.12)-(4.13), we find the main contribution

$$\sum_{i=1}^{4} C_{N,s,k}^{i} \asymp -\frac{3}{4}\left[x^2 + 2\frac{\sqrt{3}}{3}xy + y^2\right]. \tag{4.14}$$

We have thus proven formula (4.3).

Remark 4.3. Note that the last term in (4.14) is of the form

$$-\frac{1}{2(1-\rho^2)}\left(x^2-2\rho xy+y^2\right), \tag{4.15}$$

with $\rho = -1/\sqrt{3}$. We get thus a bivariate Gaussian distribution with correlation coefficient equal to $-1/\sqrt{3}$, that is, the r.v.s $n_b + n_r$ and $n_b + n_r + \gamma_b + \gamma_r$ are negatively correlated.

Step 2. Now, we want to show formula (4.1), adapting the steps from the one-dimensional case, see [3]. For finite a, b and a', b', this follows from (4.3) which implies that the l.h.s. of (4.1) is just a Riemann sum approximation to the r.h.s.. To understand infinite boundaries, we consider, for example, the particular case where $a = a' = -\infty$ and $b, b' < \infty$. The other cases can be treated similarly. Let $f(z, z')$ be the joint density function of the integral in (4.1). Since

$$\int_{\mathbb{R}^2} f(z,z')dzdz' = 1, \tag{4.16}$$

there exist for all $\varepsilon > 0$ finite constants $A, A' > 0$ such that

$$\int_{-A}^{A}\int_{-A'}^{A'} f(z,z')dzdz' \geq 1-\varepsilon. \tag{4.17}$$

Moreover, for all $\varepsilon > 0$ $\exists\overline{N}$: for all $N \geq \overline{N}$

$$\left|\sum_{\substack{|x|\leq A\\ |y|\leq A'}} \widetilde{P}_N(s,k) - \int\int_{\substack{|z|\leq A\\ |z'|\leq A'}} f(z,z')dzdz'\right| \leq \varepsilon. \tag{4.18}$$

From (4.17) and (4.18), we deduce that

$$\sum_{\substack{|x|\leq A\\ |y|\leq A'}} \widetilde{P}_N(s,k) \geq 1-2\varepsilon. \tag{4.19}$$

for N large enough. Without loss of generality, we now assume that $A \geq b, A' \geq b'$. Then, it remains to verify that, for all $\varepsilon' > 0$,

$$\left|\sum_{\substack{-\infty<x\leq b\\ -\infty<y\leq b'}} \widetilde{P}_N(s,k) - \int\int_{\substack{-\infty<z\leq b\\ -\infty<z'\leq b'}} f(z,z')\ dzdz'\right| \leq \varepsilon', \tag{4.20}$$

for N large enough. In order to show this, one has to express the double sum and integral of (4.20) by means of the larger domains $\left\{\begin{smallmatrix}-\infty<x\leq A\\ -\infty<y\leq A'\end{smallmatrix}\right\}$ and $\left\{\begin{smallmatrix}-\infty<z\leq A\\ -\infty<z'\leq A'\end{smallmatrix}\right\}$ by subtraction of

appropriate terms. The resulting sums and integrals can be split over domains which may be finite or infinite. The sums over the finite domains are again Riemann approximations to the corresponding integrals. The sums and integrals over the infinite domains can be made arbitrarily small by employing estimates (4.17) and (4.19), which completes the proof. □

Choosing $a' = -\infty, b' = \infty$ in Theorem 4.1, we get the following result.

Corollary 4.4. *A CLT holds for the r.v. $n_b + n_r$, that is, for any $-\infty \leq a < b \leq \infty$, one has*

$$\lim_{N\to\infty} \sum_{s:x\in(a,b)} P_N(n_b + n_r = s) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-z^2/2} dz. \tag{4.21}$$

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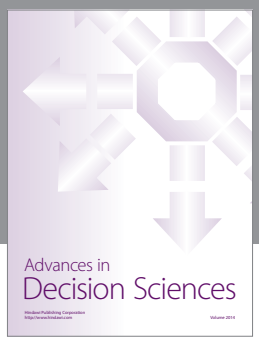

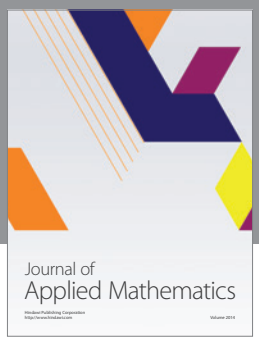

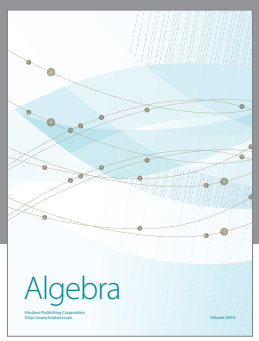

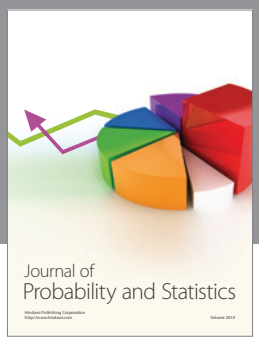

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