Research Article

Complete Convergence for Weighted Sums of Sequences of Negatively Dependent Random Variables

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Received 30 September 2010; Accepted 21 January 2011

1. Introduction and Lemmas

Definition 1.1. Random variables \( X \) and \( Y \) are said to be negatively dependent (ND) if

\[
P(X \leq x, Y \leq y) \leq P(X \leq x)P(Y \leq y)
\]

(1.1)

for all \( x, y \in \mathbb{R} \). A collection of random variables is said to be pairwise negatively dependent (PND) if every pair of random variables in the collection satisfies (1.1).

It is important to note that (1.1) implies that

\[
P(X > x, Y > y) \leq P(X > x)P(Y > y)
\]

(1.2)

for all \( x, y \in \mathbb{R} \). Moreover, it follows that (1.2) implies (1.1), and, hence, (1.1) and (1.2) are equivalent. However, (1.1) and (1.2) are not equivalent for a collection of 3 or more random variables. Consequently, the following definition is needed to define sequences of negatively dependent random variables.
Definition 1.2. The random variables $X_1, \ldots, X_n$ are said to be negatively dependent (ND) if, for all real $x_1, \ldots, x_n,$

$$P\left(\bigcap_{j=1}^{n}(X_j \leq x_j)\right) \leq \prod_{j=1}^{n} P(X_j \leq x_j),$$

$$P\left(\bigcap_{j=1}^{n}(X_j > x_j)\right) \leq \prod_{j=1}^{n} P(X_j > x_j).$$

An infinite sequence of random variables $\{X_n; n \geq 1\}$ is said to be ND if every finite subset $X_1, \ldots, X_n$ is ND.

Definition 1.3. Random variables $X_1, X_2, \ldots, X_n, n \geq 2,$ are said to be negatively associated (NA) if, for every pair of disjoint subsets $A_1$ and $A_2$ of $\{1, 2, \ldots, n\},$

$$\text{cov}(f_1(X_i; i \in A_1), f_2(X_j; j \in A_2)) \leq 0,$$

where $f_1$ and $f_2$ are increasing in every variable (or decreasing in every variable), provided this covariance exists. A random variables sequence $\{X_n; n \geq 1\}$ is said to be NA if every finite subfamily is NA.

The definition of PND is given by Lehmann [1], the concept of ND is given by Bozorgnia et al. [2], and the definition of NA is introduced by Joag-Dev and Proschan [3]. These concepts of dependence random variables have been very useful in reliability theory and applications.

First, note that by letting $f_1(X_1, X_2, \ldots, X_{n-1}) = I_{(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_{n-1} \leq x_{n-1})}$ and $f_2(X_n) = I_{(X_n \leq x_n)}$ and $f_1(X_1, X_2, \ldots, X_{n-1}) = I_{(X_1 > x_1, X_2 > x_2, \ldots, X_{n-1} > x_{n-1})}$ and $f_2(X_n) = I_{(X_n > x_n)}$, separately, it is easy to see that NA implies (1.3). Hence, NA implies ND. But there are many examples which are ND but are not NA. We list the following two examples.

Example 1.4. Let $X_i$ be a binary random variable such that $P(X_i = 1) = P(X_i = 0) = 0.5$ for $i = 1, 2, 3.$ Let $(X_1, X_2, X_3)$ take the values $(0, 0, 1), (0, 1, 0), (1, 0, 0),$ and $(1, 1, 1),$ each with probability $1/4.$

It can be verified that all the ND conditions hold. However,

$$P(X_1 + X_3 \leq 1, X_2 \leq 0) = \frac{4}{8} \neq P(X_1 + X_3 \leq 1)P(X_2 \leq 0) = \frac{3}{8}.$$

Hence, $X_1, X_2,$ and $X_3$ are not NA.

In the next example $X = (X_1, X_2, X_3, X_4)$ possesses ND, but does not possess NA obtained by Joag-Dev and Proschan [3].

Example 1.5. Let $X_i$ be a binary random variable such that $P(X_i = 1) = .5$ for $i = 1, 2, 3, 4.$ Let $(X_1, X_2)$ and $(X_3, X_4)$ have the same bivariate distributions, and let $X = (X_1, X_2, X_3, X_4)$ have joint distribution as shown in Table 1.
It can be verified that all the ND conditions hold. However,

$$P(X_i = 1, i = 1, 2, 3, 4) > P(X_1 = X_2 = 1)P(X_3 = X_4 = 1), \quad (1.6)$$

violating NA.

From the above examples, it is shown that ND does not imply NA and ND is much weaker than NA. In the papers listed earlier, a number of well-known multivariate distributions are shown to possess the ND properties, such as (a) multinomial, (b) convolution of unlike multinomials, (c) multivariate hypergeometric, (d) Dirichlet, (e) Dirichlet compound multinomial, and (f) multinomials having certain covariance matrices. Because of the wide applications of ND random variables, the notions of ND random variables have received more and more attention recently. A series of useful results have been established (cf. Bozorgnia et al. [2], Amini [4], Fakoor and Azarnoosh [5], Nili Sani et al. [6], Klesov et al. [7], and Wu and Jiang [8]). Hence, the extending of the limit properties of independent or NA random variables to the case of ND random variables is highly desirable and of considerable significance in the theory and application. In this paper we study and obtain some probability inequalities and some complete convergence theorems for weighted sums of sequences of negatively dependent random variables.

In the following, let $a_n \ll b_n (a_n \gg b_n)$ denote that there exists a constant $c > 0$ such that $a_n \leq cb_n$ $(a_n \geq cb_n)$ for sufficiently large $n$, and let $a_n \approx b_n$ mean $a_n \ll b_n$ and $a_n \gg b_n$. Also, let $\log x$ denote $\ln(\max(e, x))$ and $S_n = \sum_{j=1}^{n} X_j$.

**Lemma 1.6** (see [2]). Let $X_1, \ldots, X_n$ be ND random variables and $\{f_n; n \geq 1\}$ a sequence of Borel functions all of which are monotone increasing (or all are monotone decreasing). Then $\{f_n(X_n); n \geq 1\}$ is still a sequence of ND r. v. ’s.

**Lemma 1.7** (see [2]). Let $X_1, \ldots, X_n$ be nonnegative r. v. ’s which are ND. Then

$$E\left(\prod_{j=1}^{n} X_j\right) \leq \prod_{j=1}^{n} EX_j. \quad (1.7)$$
In particular, let \(X_1, \ldots, X_n\) be ND, and let \(t_1, \ldots, t_n\) be all nonnegative (or non-positive) real numbers. Then

\[
E \left( \exp \left( \sum_{j=1}^{n} t_j X_j \right) \right) \leq \prod_{j=1}^{n} E(\exp(t_j X_j)).
\]  

(1.8)

**Lemma 1.8.** Let \(\{X_n; n \geq 1\}\) be an ND sequence with \(E X_n = 0\) and \(E|X_n|^p < \infty\), \(p \geq 2\). Then for \(B_n = \sum_{i=1}^{n} EX_i^2\),

\[
E|S_n|^p \leq c_p \left\{ \sum_{i=1}^{n} E|X_i|^p + B_n^{p/2} \right\},
\]

(1.9)

\[
E \left( \max_{1 \leq i \leq n} |S_i|^p \right) \leq c_p \log^n n \left\{ \sum_{i=1}^{n} E|X_i|^p + B_n^{p/2} \right\},
\]

(1.10)

where \(c_p > 0\) depends only on \(p\).

**Remark 1.9.** If \(\{X_n; n \geq 1\}\) is a sequence of independent random variables, then (1.9) is the classic Rosenthal inequality [9]. Therefore, (1.9) is a generalization of the Rosenthal inequality.

**Proof of Lemma 1.8.** Let \(a > 0\), \(X_i' = \min(X_i, a)\), and \(S_n' = \sum_{i=1}^{n} X_i'\). It is easy to show that \(\{X_i'; i \geq 1\}\) is a negatively dependent sequence by Lemma 1.6. Noting that \((e^x - 1 - x)/x^2\) is a nondecreasing function of \(x\) on \(\mathbb{R}\) and that \(EX_i' \leq EX_i \leq 0\), \(tX_i' \leq ta\), we have

\[
E \left( e^{X_i} \right) = 1 + tEX_i' + E \left( \frac{e^{tX_i'} - 1 - tX_i'}{t^2 X_i'^2} \right)
\]

\[
\leq 1 + (e^{ta} - 1 - ta) a^{-2} EX_i^2
\]

\[
\leq 1 + (e^{ta} - 1 - ta) a^{-2} EX_i^2
\]

\[
\leq \exp \left\{ (e^{ta} - 1 - ta) a^{-2} EX_i^2 \right\}.
\]

(1.11)

Here the last inequality follows from \(1 + x \leq e^x\), for all \(x \in \mathbb{R}\).

Note that \(B_n = \sum_{i=1}^{n} EX_i^2\) and \(\{X_i'; i \geq 1\}\) is ND, we conclude from the above inequality and Lemma 1.7 that, for any \(x > 0\) and \(h > 0\), we get

\[
e^{-hx} E \left( e^{hx} S_n \right) = e^{-hx} E \left( \prod_{i=1}^{n} e^{hx} X_i' \right) \leq e^{-hx} \prod_{i=1}^{n} E \left( e^{hx} X_i' \right)
\]

\[
\leq \exp \left\{ -hx + \left( e^{ha} - 1 - ha \right) a^{-2} B_n \right\}.
\]

(1.12)
Letting \( h = \ln((xa)/B_n + 1)/a > 0 \), we get

\[
(e^{ha} - 1 - ha) a^{-2} B_n = \frac{x}{a} - \frac{B_n}{a^2} \ln \left( \frac{xa}{B_n} + 1 \right) \leq \frac{x}{a}.
\]

(1.13)

Putting this one into (1.12), we get furthermore

\[
e^{-hx} E(e^{hS_n}) \leq \exp \left\{ \frac{x}{a} - \frac{x}{a} \ln \left( \frac{xa}{B_n} + 1 \right) \right\}.
\]

(1.14)

Putting \( x/a = t \) into the above inequality, we get

\[
P(S_n \geq x) \leq \sum_{i=1}^{n} P(X_i > a) + P(S'_n \geq x)
\]

\[
\leq \sum_{i=1}^{n} P(X_i > a) + e^{-hx} Ee^{hS_n}
\]

\[
\leq \sum_{i=1}^{n} P(X_i > \frac{x}{t}) + \exp \left\{ t - t \ln \left( \frac{x^2}{tB_n} + 1 \right) \right\}
\]

\[
= \sum_{i=1}^{n} P(X_i > \frac{x}{t}) + e^t \left( 1 + \frac{x^2}{tB_n} \right)^{-t}.
\]

(1.15)

Letting \(-X_i\) take the place of \(X_i\) in the above inequality, we can get

\[
P(-S_n \geq x) = P(S_n \leq -x) \leq \sum_{i=1}^{n} P(-X_i > \frac{x}{t}) + e^t \left( 1 + \frac{x^2}{tB_n} \right)^{-t}
\]

\[
= \sum_{i=1}^{n} P(X_i < -\frac{x}{t}) + e^t \left( 1 + \frac{x^2}{tB_n} \right)^{-t}.
\]

(1.16)

Thus

\[
P(|S_n| \geq x) = P(S_n \geq x) + P(S_n \leq -x) \leq \sum_{i=1}^{n} P(|X_i| < \frac{x}{t}) + 2e^t \left( 1 + \frac{x^2}{tB_n} \right)^{-t}.
\]

(1.17)

Multiplying (1.17) by \(px^{p-1}\), letting \( t = p \), and integrating over \( 0 < x < +\infty \), according to

\[
E|X|^p = p \int_0^{+\infty} x^{p-1} P(|X| \geq x) dx,
\]

(1.18)
we obtain

\[ E|S_n|^p = p \int_0^{+\infty} x^{p-1} P(|S_n| > x) \, dx \]

\[ \leq p \sum_{i=1}^n \int_0^{+\infty} x^{p-1} P\left( |X_i| > \frac{x}{p} \right) \, dx + 2pe^p \int_0^{+\infty} x^{p-1} \left( 1 + \frac{x^2}{pB_n} \right)^{-p} \, dx \]

\[ = p^{p+1} \sum_{i=1}^n E|X_i|^p + pe^p (pB_n)^{p/2} \int_0^{+\infty} \frac{u^{p/2-1}}{(1+u)^p} \, du \]

\[ = p^{p+1} \sum_{i=1}^n E|X_i|^p + pe^p B\left( \frac{p}{2}, \frac{p}{2} \right) B_n^{p/2}, \tag{1.19} \]

where \( B(\alpha, \beta) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} \, dx = \int_0^{+\infty} x^{\alpha-1} (1+x)^{-(\alpha+\beta)} \, dx, \alpha, \beta > 0 \) is Beta function. Letting \( c_p = \max(p^{p+1}, p^{1+p/2}e^p B(p/2, p/2)) \), we can deduce (1.9) from (1.19). From (1.9), we can prove (1.10) by a similar way of Stout’s paper [10, Theorem 2.3.1].

**Lemma 1.10.** Let \( \{X_n; n \geq 1\} \) be a sequence of ND random variables. Then there exists a positive constant \( c \) such that, for any \( x \geq 0 \) and all \( n \geq 1, \)

\[ \left( 1 - P\left( \max_{1 \leq k \leq n} |X_k| > x \right) \right)^2 \sum_{k=1}^n P(|X_k| > x) \leq cP\left( \max_{1 \leq k \leq n} |X_k| > x \right). \tag{1.20} \]

**Proof.** Let \( A_k = (|X_k| > x) \) and \( a_n = 1 - P(\bigcup_{k=1}^n A_k) = 1 - P(\max_{1 \leq k \leq n} |X_k| > x). \) Without loss of generality, assume that \( a_n > 0 \). Note that \( I_{(X_k>x)} - EI_{(X_k>x)}; k \geq 1 \) and \( I_{(X_k<x)} - EI_{(X_k<x)}; k \geq 1 \) are still ND by Lemma 1.6. Using (1.9), we get

\[ E \left( \sum_{k=1}^n (I_{A_k} - EI_{A_k}) \right)^2 = E \left( \sum_{k=1}^n (I_{(X_k>x)} - EI_{(X_k>x)}) + (I_{(X_k<x)} - EI_{(X_k<x)}) \right)^2 \]

\[ \leq 2E \left( \sum_{k=1}^n (I_{(X_k>x)} - EI_{(X_k>x)}) \right)^2 + 2E \left( \sum_{k=1}^n (I_{(X_k<x)} - EI_{(X_k<x)}) \right)^2 \]

\[ \leq c \sum_{k=1}^n P(A_k). \tag{1.21} \]

Combining with the Cauchy-Schwarz inequality, we obtain

\[ \sum_{k=1}^n P(A_k) = \sum_{k=1}^n P\left( A_k, \bigcup_{j=1}^n A_j \right) = \sum_{k=1}^n E\left( I_{A_k} I_{\bigcup_{j=1}^n A_j} \right) \]

\[ = E\left( \sum_{k=1}^n (I_{A_k} - EI_{A_k}) \bigcup_{j=1}^n A_j \right) + \sum_{k=1}^n P(A_k)P\left( \bigcup_{j=1}^n A_j \right) \]
Theorem 2.1. Let \( \{X_n; n \geq 1\} \) be a sequence of identically distributed ND random variables and \( \{a_{nk}; 1 \leq k \leq n, \ n \geq 1\} \) an array of real numbers, and let \( r > 1, p > 2 \) if, for some \( 2 \leq q < p, \)

\[
N(n, m + 1) = \| \{ k \geq 1; |a_{nk}| \geq (m + 1)^{-1/p} \} \| \approx m^{r(r-1)/p}, \quad n, m \geq 1,
\]

\[
EX = 0 \quad \text{for} \ 1 \leq q(r-1),
\]

\[
\sum_{k=1}^{n} a_{nk}^2 \ll n^\delta \quad \text{for} \ 2 \leq q(r-1) \quad \text{and some} \ 0 < \delta < \frac{2}{p},
\]

Thus

\[
\alpha_n^2 \sum_{k=1}^{n} P(A_k) \leq c(1 - \alpha_n),
\]

that is,

\[
(1 - P(\max_{1 \leq k \leq n} |X_k| > x))^2 \sum_{k=1}^{n} P(|X_k| > x) \leq cP(\max_{1 \leq k \leq n} |X_k| > x).
\]
then, for \( r \geq 2 \),

\[
E|X|^{p(r-1)} < \infty
\tag{2.4}
\]

if and only if

\[
\sum_{n=1}^{\infty} n^{r-2} P\left( \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} a_{n_k}X_i \right| > \varepsilon n^{1/p} \right) < \infty, \quad \forall \varepsilon > 0.
\tag{2.5}
\]

For \( 1 < r < 2 \), (2.4) implies (2.5), conversely, and (2.5) and \( n^{r-2} P(\max_{1 \leq k \leq n}|a_{n_k}X_k| > n^{1/p}) \) decreasing on \( n \) imply (2.4).

For \( p = 2, q = 2 \), we have the following theorem.

**Theorem 2.2.** Let \( \{X, X_n; n \geq 1\} \) be a sequence of identically distributed ND random variables and \( \{a_{n_k}; 1 \leq k \leq n, n \geq 1\} \) an array of real numbers, and let \( r > 1 \). If

\[
N(n, m + 1) \equiv \#\left\{ k; |a_{n_k}| \geq (m + 1)^{-1/2} \right\} \approx m^{r-1}, \quad n, m \geq 1,
\tag{2.6}
\]

\[
EX = 0, \quad 1 \leq 2(r - 1),
\tag{2.7}
\]

\[
\sum_{k=1}^{n} |a_{n_k}|^{2(r-1)} = O(1),
\]

then, for \( r \geq 2 \),

\[
E|X|^{2(r-1) \log|X|} < \infty
\tag{2.8}
\]

if and only if

\[
\sum_{n=1}^{\infty} n^{r-2} P\left( \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} a_{n_k}X_i \right| > \varepsilon n^{1/2} \right) < \infty, \quad \forall \varepsilon > 0.
\tag{2.9}
\]

For \( 1 < r < 2 \), (2.8) implies (2.9), conversely, and (2.9) and \( n^{r-2} P(\max_{1 \leq k \leq n}|a_{n_k}X_k| > n^{1/2}) \) decreasing on \( n \) imply (2.8).

**Remark 2.3.** Since NA random variables are a special case of ND r. v. ’s, Theorems 2.1 and 2.2 extend the work of Liang and Su [14, Theorem 2.1].

**Remark 2.4.** Since, for some \( 2 \leq q \leq p \), \( \sum_{k \in N} |a_{n_k}|^{q(r-1)} \ll 1 \) as \( n \to \infty \) implies that

\[
N(n, m + 1) \equiv \#\left\{ k \geq 1; |a_{n_k}| \geq (m + 1)^{-1/p} \right\} \ll m^{q(r-1)/p} \quad \text{as} \quad n \to \infty,
\tag{2.10}
\]
taking \( r = 2 \), then conditions (2.1) and (2.6) are weaker than conditions (2.13) and (2.9) in Li et al. [13]. Therefore, Theorems 2.1 and 2.2 not only promote and improve the work of Li et al. [13, Theorem 2.2] for i.i.d. random variables to an ND setting but also obtain their necessities and relax the range of \( r \).

**Proof of Theorem 2.1.** Equation (2.4)⇒(2.5). To prove (2.5) it suffices to show that

\[
\sum_{n=1}^{\infty} n^{r-2} P\left( \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} a_{ni}^\pm X_i \right| > \varepsilon n^{1/p} \right) < \infty, \quad \forall \varepsilon > 0,
\]

(2.11)

where \( a_{ni}^+ = \max(a_{ni}, 0) \) and \( a_{ni}^- = \max(-a_{ni}, 0) \). Thus, without loss of generality, we can assume that \( a_{ni} > 0 \) for all \( n \geq 1, i \leq n \). For \( 0 < \alpha < 1/p \) small enough and sufficiently large integer \( K \), which will be determined later, let

\[
X_n^{(1)} = -n^\alpha I_{(a_{ni}X_i < -n^\alpha)} + a_{ni}X_i I_{(a_{ni}|X_i| \leq n^\alpha)} + n^\alpha I_{(a_{ni}X_i > n^\alpha)},
\]

\[
X_n^{(2)} = (a_{ni}X_i - n^\alpha) I_{(n^\alpha < a_{ni}X_i < en^{1/p}/K)},
\]

\[
X_n^{(3)} = (a_{ni}X_i + n^\alpha) I_{(-en^{1/p}/K < a_{ni}X_i < -n^\alpha)},
\]

\[
X_n^{(4)} = a_{ni}X_n - X_n^{(1)} - X_n^{(2)} - X_n^{(3)} = (a_{ni}X_i + n^\alpha) I_{(a_{ni}X_i \leq en^{1/p}/K)} + (a_{ni}X_i - n^\alpha) I_{(a_{ni}X_i \geq en^{1/p}/K)},
\]

\[
S_{nk}^{(j)} = \sum_{i=1}^{k} X_n^{(j)}, \quad j = 1, 2, 3, 4; 1 \leq k \leq n, \ n \geq 1.
\]

Thus \( S_{nk} = \sum_{i=1}^{k} a_{ni}X_i = \sum_{j=1}^{4} S_{nk}^{(j)} \). Note that

\[
\left( \max_{1 \leq k \leq n} |S_{nk}| > 4\varepsilon n^{1/p} \right) \subseteq \bigcup_{j=1}^{4} \left( \max_{1 \leq k \leq n} |S_{nk}^{(j)}| > \varepsilon n^{1/p} \right).
\]

(2.13)

So, to prove (2.5) it suffices to show that

\[
I_j = \sum_{n=1}^{\infty} n^{r-2} P\left( \max_{1 \leq k \leq n} |S_{nk}^{(j)}| > \varepsilon n^{1/p} \right) < \infty, \quad j = 1, 2, 3, 4.
\]

(2.14)
For any \( q' > q \),

\[
\sum_{i=1}^{n} a_{ni}'^{(r-1)} = \sum_{j=1}^{\infty} \sum_{(j+1)^{-1} \leq a_{nj}' < j^{-1}} a_{ni}'^{(r-1)} \leq \sum_{j=1}^{\infty} \sum_{(j+1)^{-1} \leq a_{n_j} < j^{-1}} j^{-q'(r-1)/p} \ll \sum_{j=1}^{\infty} (N(n, j + 1) - N(n, j)) j^{-q'(r-1)/p}
\]

\[
\ll \sum_{j=1}^{\infty} N(n, j) \left( j^{-q'(r-1)/p} - (j + 1)^{-q'(r-1)/p} \right)
\]

\[
\ll \sum_{j=1}^{\infty} j^{-1-(q'-q)(r-1)/p} < \infty.
\]

Now, we prove that

\[
n^{-1/p} \max_{1 \leq k \leq n} |E_{nk}^{(1)}| \to 0, \quad n \to \infty. \tag{2.16}
\]

(i) For \( 0 < q(r-1) < 1 \), taking \( q < q' < p \) such that \( 0 < q'(r-1) < 1 \), by (2.4) and (2.15), we get

\[
n^{-1/p} \max_{1 \leq k \leq n} |E_{nk}^{(1)}| \leq n^{-1/p} \sum_{i=1}^{n} \left( E[|a_{ni}X_i| I(|a_{ni}X_i| \leq n^q)] + n^a P(|a_{ni}X_i| > n^a) \right)
\]

\[
\leq n^{-1/p} \left( \sum_{i=1}^{n} E[|a_{ni}X_i| q'(r-1)|a_{ni}X_i|^{1-q'(r-1)} I(|a_{ni}X_i| \leq n^q)] + n^a a'^{(r-1)} \sum_{i=1}^{n} E[|a_{ni}X_i| q'(r-1)] \right)
\]

\[
\ll n^{-1/p+q'-q(r-1)} \to 0, \quad n \to \infty.
\]

(ii) For \( 1 \leq q(r-1) \), letting \( q < q' < p \), by (2.2), (2.4), and (2.15), we get

\[
n^{-1/p} \max_{1 \leq k \leq n} |E_{nk}^{(1)}| \leq n^{-1/p} \sum_{i=1}^{n} \left( E[|a_{ni}X_i| I(|a_{ni}X_i| > n^q)] + n^a P(|a_{ni}X_i| > n^a) \right)
\]

\[
\leq n^{-1/p} \sum_{i=1}^{n} \left( E[|a_{ni}X_i| \left( \frac{|a_{ni}X_i|}{n^a} \right)^{q'(r-1)-1} I(|a_{ni}X_i| \leq n^q)] + n^a a' a'^{(r-1)} E[|a_{ni}X_i| q'(r-1)] \right)
\]

\[
\ll n^{-1/p+q'-q(r-1)} \to 0.
\]
Hence, (2.16) holds. Therefore, to prove $I_1 < \infty$ it suffices to prove that

$$\tilde{I}_1 = \sum_{n=1}^{\infty} n^{r-2} P\left( \max_{1 \leq k \leq n} S^{(1)}_{n_k} - E S^{(1)}_{n_k} > \varepsilon n^{1/p} \right) < \infty, \quad \forall \varepsilon > 0. \quad (2.19)$$

Note that $\{X^{(1)}_{ni}; 1 \leq i \leq n, n \geq 1\}$ is still ND by the definition of $X^{(1)}_{ni}$ and Lemma 1.6. Using the Markov inequality and Lemma 1.8, we get for a suitably large $M$, which will be determined later,

$$\tilde{I}_1 \leq \sum_{n=1}^{\infty} n^{r-2-M/p} \log n \left( \sum_{i=1}^{n} E|X^{(1)}_{ni}|^M + \left( \sum_{i=1}^{n} E\left( X^{(1)}_{ni} \right)^2 \right)^{M/2} \right) \quad (2.20)$$

Taking $M > \max(2/p(r - 1)(1 - aq)/(1 - ap))$, then $r - 2 - M/p + aM - aq(r - 1) < -1$, and, by (2.15), we get

$$\tilde{I}_{11} \leq \sum_{n=1}^{\infty} n^{r-2-M/p} \log n \left( \sum_{i=1}^{n} E|a_m X_i|^M I_{(|a_m X_i| \leq n^p)} + n^{M/2} P(|a_m X_i| > n^p) \right) \leq \sum_{n=1}^{\infty} n^{r-2-M/p} \log n \left( \sum_{i=1}^{n} E|a_m X_i|^{q(r-1)} n^{a(M-q(r-1))} + n^{a(M-q(r-1))} E|a_m X_i|^{q(r-1)} \right) \quad (2.21)$$

$$\quad \leq \sum_{n=1}^{\infty} n^{r-2-M/p+\alpha M-\alpha q(r-1)} \log n \quad < \infty.$$  

(i) For $q(r-1) < 2$, taking $q < q' < p$ such that $q'(r-1) < 2$ and taking $M > \max(2/p(r - 1)/2 - 2ap + apq(r - 1))$, from (2.15) and $r - 2 - M/p + aM - Maq'(r - 1)/2 < -1$, we have

$$\tilde{I}_{12} \leq \sum_{n=1}^{\infty} n^{r-2-M/p} \log n \left( \sum_{i=1}^{n} E|a_m X_i|^{q(r-1)} n^{a(2-q(r-1))} I_{(|a_m X_i| \leq n^p)} \right. \left. + n^{2a-\alpha q'(r-1)} E|a_m X_i|^{q'(r-1)} \right)^{M/2} \quad (2.22)$$

$$\quad \leq \sum_{n=1}^{\infty} n^{r-2-M/p+\alpha M-Maq'(r-1)/2} \log n \quad < \infty.$$
(ii) For $q(r-1) ≥ 2$, taking $q < q' < p$ and $M > \max(2, \frac{2p(r-1)}{(2-pδ)})$, where $δ$ is defined by (2.3), we get, from (2.3), (2.4), (2.15), and $r - 2 - \frac{M}{p} + δM/2 < -1,$

\[
\tilde{I}_{12} \ll \sum_{n=1}^{\infty} n^{r-2-M/p} \log^M n \left[ \sum_{i=1}^{n} a_{ni}^2 + n^{2a-αq(r-1)} E|a_{ni}X_i|^q(r-1) \right]^{M/2} \\
\ll \sum_{n=1}^{\infty} n^{r-2-M/p+δM/2} \log^M n \\
< \infty.
\]

Since

\[
\left( \sum_{i=1}^{n} X_{ni}^{(2)} > \varepsilon n^{1/p} \right) = \left( \sum_{i=1}^{n} (a_{ni}X_i - n^α) I_{(n^α < a_{ni}X_i < \varepsilon n^{1/r}/K)} > \varepsilon n^{1/p} \right) \\
\subseteq (\text{there at least exist } K \text{ indices } k \text{ such that } a_{nk}X_k > n^α),
\]

we have

\[
P\left( \sum_{i=1}^{n} X_{ni}^{(2)} > \varepsilon n^{1/p} \right) \leq \sum_{1 \leq i_1 < i_2 < \cdots < i_K \leq n} P(a_{ni_1}X_{i_1} > n^α, a_{ni_2}X_{i_2} > n^α, \ldots, a_{ni_k}X_{i_k} > n^α).
\]

(2.25)

By Lemma 1.6, $\{a_{ni}X_i; 1 \leq i \leq n, n \geq 1\}$ is still ND. Hence, for $q < q' < p$ we conclude that

\[
P\left( \sum_{i=1}^{n} X_{ni}^{(2)} > \varepsilon n^{1/p} \right) \leq \sum_{1 \leq i_1 < i_2 < \cdots < i_K \leq n} K \prod_{j=1}^{K} P(a_{ni_j}X_{i_j} > n^α) \\
\leq \left( \sum_{i=1}^{n} P(|a_{ni}X_i| > n^α) \right)^K \\
\leq \left( \sum_{i=1}^{n} n^{-αq(r-1)} E|a_{ni}X_i|^q(r-1) \right)^K \\
\ll n^{-αq(r-1)K},
\]

via (2.4) and (2.15). $X_{ni}^{(2)} > 0$ from the definition of $X_{ni}^{(2)}$. Hence by (2.26) and by taking $α > 0$ and $K$ such that $r - 2 - αKq(r-1) < -1,$ we have

\[
I_2 = \sum_{n=1}^{\infty} n^{r-2} P\left( \sum_{i=1}^{n} X_{ni}^{(2)} > \varepsilon n^{1/p} \right) \ll \sum_{n=1}^{\infty} n^{r-2-αq(r-1)K} < \infty.
\]

(2.27)

Similarly, we have $X_{ni}^{(3)} < 0$ and $I_3 < \infty$. 

Last, we prove that $I_4 < \infty$. Let $Y = KX/\varepsilon$. By the definition of $X_{n_i}^{(4)}$ and (2.1), we have

$$P\left( \max_{1 \leq k \leq n} |S_{nk}^{(4)}| > \varepsilon n^{1/p} \right) \leq P\left( \sum_{i=1}^{n} |X_{ni}^{(4)}| > \varepsilon n^{1/p} \right)$$

$$\leq P\left( \bigcup_{i=1}^{n} \left( a_n |X_i| > \frac{\varepsilon n^{1/p}}{K} \right) \right)$$

$$\leq \sum_{i=1}^{n} P\left( a_n |X_i| > \frac{\varepsilon n^{1/p}}{K} \right)$$

$$= \sum_{j=1}^{\infty} \sum_{(j+1)^{-1} \leq n_i < j} P\left( |Y| > (nj)^{1/p} \right)$$

$$= \sum_{j=1}^{\infty} \left( N(n, j + 1) - N(n, j) \right) \sum_{l=1}^{\infty} P(l \leq |Y|^p < l + 1)$$

$$= \sum_{l=1}^{[l/n]} \sum_{j=1}^{\infty} \left( N(n, j + 1) - N(n, j) \right) P(l \leq |Y|^p < l + 1)$$

$$\approx \sum_{l=1}^{\infty} \left( \frac{1}{n} \right)^{q(r-1)/p} P(l \leq |Y|^p < l + 1).$$

Combining with (2.15),

$$I_4 \approx \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \left( \frac{1}{n} \right)^{q(r-1)/p} P(l \leq |Y|^p < l + 1)$$

$$= \sum_{l=1}^{\infty} \sum_{n=1}^{l} n^{r-2-q(r-1)/p} P(l \leq |Y|^p < l + 1)$$

$$\approx \sum_{l=1}^{\infty} l^{r-1} P(l \leq |Y|^p < l + 1)$$

$$\approx E|Y|^{p(r-1)} \approx E|X|^{p(r-1)} < \infty.$$

Now we prove (2.5)⇒(2.4). Since

$$\max_{1 \leq j \leq n} |a_{nj}X_j| \leq \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} a_{ni}X_i \right| + \max_{1 \leq j \leq n} \left| \sum_{i=j+1}^{n} a_{ni}X_i \right|,$$

then from (2.5) we have

$$\sum_{n=1}^{\infty} n^{r-2} P\left( \max_{1 \leq j \leq n} |a_{nj}X_j| > n^{1/p} \right) < \infty.$$
Combining with the hypotheses of Theorem 2.1,

\[
P\left(\max_{1 \leq j \leq n}|a_{nj}X_j| > n^{1/p}\right) \rightarrow 0, \quad n \rightarrow \infty. \tag{2.32}
\]

Thus, for sufficiently large \(n\),

\[
P\left(\max_{1 \leq j \leq n}|a_{nj}X_j| > n^{1/p}\right) < \frac{1}{2}. \tag{2.33}
\]

By Lemma 1.6, \(\{a_{nj}X_j; 1 \leq j \leq n, n \geq 1\}\) is still ND. By applying Lemma 1.10 and (2.1), we obtain

\[
\sum_{k=1}^{n} P\left(\left|a_{nk}X_k\right| > n^{1/p}\right) \leq 4CP\left(\max_{1 \leq k \leq n}\left|a_{nk}X_k\right| > n^{1/p}\right). \tag{2.34}
\]

Substituting the above inequality in (2.5), we get

\[
\sum_{n=1}^{\infty} n^{r-2} \sum_{k=1}^{n} P\left(\left|a_{nk}X_k\right| > n^{1/p}\right) < \infty. \tag{2.35}
\]

So, by the process of proof of \(I_4 < \infty\),

\[
E[X]^p(\rho-1) \approx \sum_{n=1}^{\infty} n^{r-2} \sum_{k=1}^{n} P\left(\left|a_{nk}X_k\right| > n^{1/p}\right) < \infty. \tag{2.36}
\]

\textit{Proof of Theorem 2.2.} Let \(p = 2, \alpha < 1/p = 1/2,\) and \(K > 1/(2\alpha)\). Using the same notations and method of Theorem 2.1, we need only to give the different parts.

Letting (2.7) take the place of (2.15), similarly to the proof of (2.19) and (2.26), we obtain

\[
n^{-1/2} \max_{1 \leq k \leq n} \left|ES_n^{(1)}\right| \ll n^{-1/2+\alpha-2\alpha(\rho-1)} \rightarrow 0, \quad n \rightarrow \infty. \tag{2.37}
\]

Taking \(M > \max(2, 2(r-1))\), we have

\[
\bar{I}_{11} \ll \sum_{n=1}^{\infty} n^{-1-(1-2\alpha)(M/2-(\rho-1))} \log^M n < \infty. \tag{2.38}
\]

For \(r - 1 \leq 1\), taking \(M > \max(2, 2(r-1)/(1 - 2\alpha + 2\alpha(r-1)))\), we get

\[
\bar{I}_{12} \ll \sum_{n=1}^{\infty} n^{-1-(1-2\alpha(r-1)-2\alpha)M/2+(\rho-1)} \log^M n < \infty. \tag{2.39}
\]
For $r - 1 > 1$, $EX^2_{ni} < \infty$ from (2.8). Letting $M > 2(r - 1)^2$, by the Hölder inequality,

\[
\tilde{I}_{12} \ll \sum_{n=1}^{\infty} n^{r-2-M/2} \log^M n \left[ \sum_{i=1}^{n} a_{ni}^2 + n^{2r-2(r-1)} E(a_{ni}X_i)^2 \right]^{M/2} \\
\ll \sum_{n=1}^{\infty} n^{r-2-M/2} \log^M n \left[ \left( \sum_{i=1}^{n} a_{ni}^2 \right)^{1/(r-1)} \left( \sum_{i=1}^{n} \right)^{r-2/(r-1)} \right]^{M/2} \\
\ll \sum_{n=1}^{\infty} n^{-1-M/2(r-1)+(r-1)} \log^M n < \infty.
\]

By the definition of $K$,

\[
I_2 \ll \sum_{n=1}^{\infty} n^{-1-(r-1)(2\alpha K-1)} < \infty.
\] (2.41)

Similarly to the proof (2.31), we have

\[
I_4 \ll \sum_{l=1}^{\infty} \sum_{n=1}^{l} n^{-1} l^{-1} P\left( l \leq |Y|^2 < l + 1 \right) \\
= \sum_{l=1}^{\infty} l^{-1} \log l P\left( l \leq |Y|^2 < l + 1 \right) \\
\approx E\left( |Y|^{2(r-1)} \log |Y| \right) \\
\approx E\left( |X|^{2(r-1)} \log |X| \right) \\
< \infty.
\] (2.42)

Equation (2.9) $\Rightarrow$ (2.8) Using the same method of the necessary part of Theorem 2.1, we can easily get

\[
E\left( |X|^{2(r-1)} \log |X| \right) \approx \sum_{n=1}^{\infty} n^{r-2} \sum_{k=1}^{n} P\left( |a_{nk}X_k| > n^{1/2} \right) < \infty.
\] (2.43)

\[\square\]

**Acknowledgments**

The author is very grateful to the referees and the editors for their valuable comments and some helpful suggestions that improved the clarity and readability of the paper. This work was supported by the National Natural Science Foundation of China (11061012), the Support Program the New Century Guangxi China Ten-Hundred-Thousand Talents Project (2005214), and the Guangxi China Science Foundation (2010GXNSFA013120). Professor Dr. Qunying Wu engages in probability and statistics.
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