Research Article

Large Deviations in Testing Squared Radial Ornstein-Uhlenbeck Model

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Received 12 May 2011; Accepted 10 July 2011

Academic Editor: Mohammad Fraiwan Al-Saleh

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We study the large deviations and moderate deviations of hypothesis testing for squared radial Ornstein-Uhlenbeck model. Large deviation principles for the log-likelihood ratio are obtained, by which we give negative regions in testing squared radial Ornstein-Uhlenbeck model and get the decay rates of the error probabilities.

1. Introduction

Let us consider the hypothesis testing for the following squared radial Ornstein-Uhlenbeck model:

$$dX_t = (\delta + 2\alpha X_t)dt + 2\sqrt{X_t}dW_t, \quad X_0 = 0, \quad (1.1)$$

where $\alpha < 0$ is the unknown parameter to be tested on the basis of continuous observation of the process $\{X_t, t \geq 0\}$ on the time interval $[0, T]$, W is a standard Brownian motion and, $\delta > 0$ is known. We denote the distribution of the solution (1.1) by P_α^δ .

We decide the two hypothesis:

$$H_0 : \alpha = \alpha_0, \quad H_1 : \alpha = \alpha_1, \quad (1.2)$$

where $\alpha_0, \alpha_1 < 0$. The hypothesis testing is based on a partition of

$$\Omega_T = \left\{ \frac{dP_{\alpha_1}^\delta}{dP_{\alpha_0}^\delta} \bigg|_{\mathcal{F}_{[0,T]}} \in \mathbb{R} \right\} \quad (1.3)$$

of the outcome process on $[0, T]$ into two (decision) regions B_T and its complement B_T^c , and we decide that H_0 is true or false according to the outcome $X \in B_T$ or $X \in B_T^c$.

The probability $e_1(T)$ of accepting H_1 when H_0 is actually true is called the error probability of the first kind. The probability $e_2(T)$ of accepting H_0 when H_1 is actually true is called the error probability of the second kind. That is,

$$e_1(T) = P_{\alpha_0}^\delta(B_T), \quad e_2(T) = P_{\alpha_1}^\delta(B_T^c). \quad (1.4)$$

By the Neyman-Pearson lemma (cf. [1]), the optional decision region B_T has the following form:

$$\left\{ \frac{1}{T} \log \frac{dP_{\alpha_1}^\delta}{dP_{\alpha_0}^\delta} \bigg|_{\mathcal{F}_{[0,T]}} \geq c \right\}, \quad (1.5)$$

where $\mathcal{F}_{[0,T]}$ is the σ -algebra generated by the outcome process on $[0, T]$.

The research of hypothesis testing problem has started in the 1930s (cf. [1]). Since the optional decision region B_T has the above form, we are interested in the calculation or approximation of the constant c , and the hypothesis testing problem can be studied by large deviations (cf. [2–6]). In those papers, some large deviation estimates of the error probabilities for some i.i.d. sequences, Markov chains, stationary Gaussian processes, stationary diffusion processes, Ornstein-Uhlenbeck processes are obtained. In this paper, we study the large deviations and moderate deviations for the hypothesis testing problem of squared radial Ornstein-Uhlenbeck model; by large deviation principle, we obtain that the decay of the error probability of the second kind approaches to 0 or 1 exponentially fast depending on the fixed exponent of the decay of the error probability of the first kind; we also give negative regions and get the decay rates of the error probabilities by moderate deviation principle. The large and moderate deviations for parameter estimators of squared radial Ornstein-Uhlenbeck model were studied in [7, 8].

2. Main Results

In this section, we state our main results.

Theorem 2.1. *Let $a(T)$ be a positive function satisfying*

$$\frac{a(T)}{T} \rightarrow 0, \quad \frac{a(T)}{\sqrt{T}} \rightarrow \infty \quad \text{as } T \rightarrow \infty. \quad (2.1)$$

For any $a > 0$, set

$$B_T = \left\{ \frac{1}{a(T)} \log \frac{dP_{\alpha_1}^\delta}{dP_{\alpha_0}^\delta} \bigg|_{\mathcal{F}_{[0,T]}} \geq \frac{T}{4a(T)} \frac{(\alpha_1 - \alpha_0)^2 \delta}{\alpha_0} - \frac{|\alpha_1^2 - \alpha_0^2|}{2\alpha_0} \sqrt{\frac{a\delta}{-\alpha_0}} \right\}. \quad (2.2)$$

Then

$$\begin{aligned}\lim_{T \rightarrow \infty} \frac{T}{a^2(T)} \log P_{\alpha_0}^\delta(B_T) &= -a, \\ \lim_{T \rightarrow \infty} \frac{T}{a^2(T)} \log P_{\alpha_1}^\delta(B_T^c) &= -\infty.\end{aligned}\tag{2.3}$$

Theorem 2.2. If $\alpha_1 < \alpha_0 < 0$, then for each $a > 0$, there exists a $\xi(a) \in \mathbb{R}$, such that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \log P_{\alpha_0}^\delta \left(\frac{1}{T} \log \frac{dP_{\alpha_1}^\delta}{dP_{\alpha_0}^\delta} \Big|_{\mathcal{F}_{[0,T]}} \geq \xi(a) \right) = -a,\tag{2.4}$$

and when $a \in (0, z_{\alpha_1})$,

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \log P_{\alpha_1}^\delta \left(\frac{1}{T} \log \frac{dP_{\alpha_1}^\delta}{dP_{\alpha_0}^\delta} \Big|_{\mathcal{F}_{[0,T]}} < \xi(a) \right) \leq -I_{\alpha_1}(\xi(a)),\tag{2.5}$$

when $a \in (z_{\alpha_1}, +\infty)$,

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \log \left(1 - P_{\alpha_1}^\delta \left(\frac{1}{T} \log \frac{dP_{\alpha_1}^\delta}{dP_{\alpha_0}^\delta} \Big|_{\mathcal{F}_{[0,T]}} < \xi(a) \right) \right) \leq -I_{\alpha_1}(\xi(a)),\tag{2.6}$$

where

$$\begin{aligned}z_{\alpha_1} &= -\frac{(\alpha_0 - \alpha_1)^2 \delta}{4\alpha_1}, \\ I_{\alpha_1}(z) &= \begin{cases} \frac{\alpha_0^2 - \alpha_1^2}{z + (\alpha_1 - \alpha_0)\delta/2} \left(\frac{\delta}{4} - \frac{\alpha_1(z + (\alpha_1 - \alpha_0)\delta/2)}{\alpha_1^2 - \alpha_0^2} \right)^2, & z < \frac{(\alpha_0 - \alpha_1)\delta}{2}, \\ +\infty, & \text{otherwise.} \end{cases}\end{aligned}\tag{2.7}$$

Theorem 2.3. If $\alpha_0 < \alpha_1 < 0$, then for each $a > 0$, there exists a $\tilde{\xi}(a) \in \mathbb{R}$, such that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \log P_{\alpha_0}^\delta \left(\frac{1}{T} \log \frac{dP_{\alpha_1}^\delta}{dP_{\alpha_0}^\delta} \Big|_{\mathcal{F}_{[0,T]}} \geq \tilde{\xi}(a) \right) = -a,\tag{2.8}$$

and when $a \in (0, \hat{z}_{\alpha_1})$,

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \log P_{\alpha_1}^\delta \left(\frac{1}{T} \log \frac{dP_{\alpha_1}^\delta}{dP_{\alpha_0}^\delta} \Big|_{\mathcal{F}_{[0,T]}} < \tilde{\xi}(a) \right) \leq -\hat{I}_{\alpha_1}(\tilde{\xi}(a)),\tag{2.9}$$

when $a \in (\hat{z}_{\alpha_1}, +\infty)$,

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \log \left(1 - P_{\alpha_1}^\delta \left(\frac{1}{T} \log \frac{dP_{\alpha_1}^\delta}{dP_{\alpha_0}^\delta} \bigg|_{\mathcal{F}_{[0,T]}} < \tilde{\xi}(a) \right) \right) \leq -\hat{I}_{\alpha_1}(\tilde{\xi}(a)), \quad (2.10)$$

where

$$\begin{aligned} \hat{z}_{\alpha_1} &= -\frac{(\alpha_0 - \alpha_1)^2 \delta}{4\alpha_1}, \\ \hat{I}_{\alpha_1}(z) &= \begin{cases} \frac{\alpha_0^2 - \alpha_1^2}{z + (\alpha_1 - \alpha_0)\delta/2} \left(\frac{\delta}{4} - \frac{\alpha_1(z + (\alpha_1 - \alpha_0)\delta/2)}{\alpha_1^2 - \alpha_0^2} \right)^2, & z > \frac{(\alpha_0 - \alpha_1)\delta}{2}, \\ +\infty, & \text{otherwise.} \end{cases} \end{aligned} \quad (2.11)$$

3. Moderate Deviations in Testing Squared Radial Ornstein-Uhlenbeck Model

In this section, we will prove Theorem 2.1. Let us introduce the log-likelihood ratio process of squared radial Ornstein-Uhlenbeck model and study the moderate deviations of the log-likelihood ratio process.

By [7], the log-likelihood ratio process has the representation

$$\log \frac{dP_{\alpha_1}^\delta}{dP_{\alpha_0}^\delta} \bigg|_{\mathcal{F}_{[0,T]}} = \frac{1}{2}(\alpha_1 - \alpha_0)(X_t - \delta t) - \frac{\alpha_1^2 - \alpha_0^2}{2} \int_0^t X_s^2 ds. \quad (3.1)$$

The following Lemma (cf. [9]) plays an important role in this paper.

Lemma 3.1. *The law of X_t under P_α^δ is $\gamma(\delta/2, \alpha/e^{2t\alpha})$, where $\gamma(a, b)$ denotes the Gamma distribution:*

$$\gamma(a, b)(dx) = \frac{b^a x^{a-1}}{\Gamma(a)} e^{-bx}(dx), \quad x > 0. \quad (3.2)$$

Moreover, for any $\theta \in \mathbb{R}$,

$$\mathbb{E}_\alpha^\delta(e^{\theta X_t}) = \left(1 - \frac{\theta}{\alpha}(e^{2t\alpha} - 1) \right)^{-\delta/2}. \quad (3.3)$$

Lemma 3.2. *For any closed subset $F \subset \mathbb{R}$,*

$$\limsup_{T \rightarrow \infty} \frac{T}{a^2(T)} \log P_{\alpha_0}^\delta \left(\frac{1}{a(T)} \left(\log \frac{dP_{\alpha_1}^\delta}{dP_{\alpha_0}^\delta} \bigg|_{\mathcal{F}_{[0,T]}} - \frac{(\alpha_1 - \alpha_0)^2 \delta T}{4\alpha_0} \right) \in F \right) \leq -\inf_{z \in F} \frac{-4\alpha_0^3 z^2}{(\alpha_1^2 - \alpha_0^2)^2 \delta}, \quad (3.4)$$

and for any open subset $G \subset \mathbb{R}$,

$$\liminf_{T \rightarrow \infty} \frac{T}{a^2(T)} \log P_{\alpha_0}^\delta \left(\frac{1}{a(T)} \left(\log \frac{dP_{\alpha_1}^\delta}{dP_{\alpha_0}^\delta} \Big|_{\mathcal{F}_{[0,T]}} - \frac{(\alpha_1 - \alpha_0)^2 \delta T}{4\alpha_0} \right) \in G \right) \geq - \inf_{z \in G} \frac{-4\alpha_0^3 z^2}{(\alpha_1^2 - \alpha_0^2)^2 \delta}. \quad (3.5)$$

Proof. Let

$$\Lambda_T(y) = \log \mathbb{E}_{\alpha_0}^\delta \exp \left\{ \frac{a(T)y}{T} \left(\log \frac{dP_{\alpha_1}^\delta}{dP_{\alpha_0}^\delta} \Big|_{\mathcal{F}_{[0,T]}} - \frac{(\alpha_1 - \alpha_0)^2 \delta T}{4\alpha_0} \right) \right\}. \quad (3.6)$$

By (3.1), for any $\varphi < 0$, we have

$$\begin{aligned} \Lambda_T(y) &= -\frac{a(T)y\delta(\alpha_1 - \alpha_0)^2}{4\alpha_0} + \log \mathbb{E}_{\alpha_0}^\delta \exp \left\{ \lambda(X_T - \delta T) + u \int_0^T X_t^2 ds \right\} \\ &= -\frac{a(T)y\delta(\alpha_1 - \alpha_0)^2}{4\alpha_0} + \log \mathbb{E}_\varphi^\delta \left[\frac{dP_{\alpha_0}^\delta}{dP_\varphi^\delta} \exp \left\{ \lambda(X_T - \delta T) + u \int_0^T X_t^2 ds \right\} \right] \\ &= -\frac{a(T)y\delta(\alpha_1^2 - \alpha_0^2)}{4\alpha_0} + \log \mathbb{E}_\varphi^\delta \left[\exp \left\{ \left(\lambda + \frac{\alpha_0 - \varphi}{2} \right) X_T - \frac{\alpha_0 - \varphi}{2} \delta T \right. \right. \\ &\quad \left. \left. + \left(u - \frac{1}{2}\alpha_0^2 + \frac{1}{2}\varphi^2 \right) \int_0^T X_s^2 ds \right\} \right], \end{aligned} \quad (3.7)$$

where

$$\lambda = \frac{a(T)y(\alpha_1 - \alpha_0)}{2T}, \quad u = -\frac{a(T)y(\alpha_1^2 - \alpha_0^2)}{2T}. \quad (3.8)$$

For T large enough, $\alpha_0^2 - 2u > 0$, we can choose $\varphi = -\sqrt{\alpha_0^2 - 2u}$, then $\varphi < 0$ and

$$\Lambda_T(y) = -\frac{a(T)y\delta(\alpha_1^2 - \alpha_0^2)}{4\alpha_0} - \frac{\alpha_0 - \varphi}{2} \delta T + \log \mathbb{E}_\varphi^\delta \left[\exp \left\{ \left(\lambda + \frac{\alpha_0 - \varphi}{2} \right) X_T \right\} \right]. \quad (3.9)$$

By Lemma 3.1, we have

$$\begin{aligned} &\lim_{T \rightarrow \infty} \frac{T}{a^2(T)} \log \mathbb{E}_\varphi^\delta \left[\exp \left\{ \left(\lambda + \frac{\alpha_0 - \varphi}{2} \right) X_T \right\} \right] \\ &= \lim_{T \rightarrow \infty} -\frac{T\delta}{2a^2(T)} \log \left(1 - \frac{1}{\varphi} \left(\lambda + \frac{\alpha_0 - \varphi}{2} \right) (e^{2T\varphi} - 1) \right) = 0. \end{aligned} \quad (3.10)$$

Therefore,

$$\begin{aligned}
\Lambda(y) &:= \lim_{T \rightarrow \infty} \frac{T}{a^2(T)} \Lambda_T(y) \\
&= \lim_{T \rightarrow \infty} \frac{T}{a^2(T)} \left(-\frac{a(T)y\delta(\alpha_1^2 - \alpha_0^2)}{4\alpha_0} - \frac{\alpha_0 - \varphi}{2} \delta T \right) \\
&= \lim_{T \rightarrow \infty} \frac{T}{a^2(T)} \left(-\frac{a(T)y\delta(\alpha_1^2 - \alpha_0^2)}{4\alpha_0} - \frac{\alpha_0 \delta T}{2} \left(1 - \sqrt{1 + \frac{a(T)y(\alpha_1^2 - \alpha_0^2)}{T\alpha_0^2}} \right) \right) \\
&= \frac{y^2 (\alpha_1^2 - \alpha_0^2)^2 \delta}{16 - \alpha_0^3}.
\end{aligned} \tag{3.11}$$

Finally, the Gärtner-Ellis theorem (cf. [10]) implies the conclusion of Lemma 3.2. $\square$

Noting that

$$\log \frac{dP_{\alpha_1}^\delta}{dP_{\alpha_0}^\delta} \Big|_{\mathcal{F}_{[0,T]}} = -\log \frac{dP_{\alpha_0}^\delta}{dP_{\alpha_1}^\delta} \Big|_{\mathcal{F}_{[0,T]}}, \tag{3.12}$$

we also have the following result.

Lemma 3.3. *For any closed subset $F \subset \mathbb{R}$,*

$$\limsup_{T \rightarrow \infty} \frac{T}{a^2(T)} \log P_{\alpha_1}^\delta \left(\frac{1}{a(T)} \left(\log \frac{dP_{\alpha_1}^\delta}{dP_{\alpha_0}^\delta} \Big|_{\mathcal{F}_{[0,T]}} + \frac{(\alpha_1 - \alpha_0)^2 \delta T}{4\alpha_1} \right) \in F \right) \leq -\inf_{z \in F} \frac{-4\alpha_1^3 z^2}{(\alpha_1^2 - \alpha_0^2)^2 \delta}, \tag{3.13}$$

and for any open subset $G \subset \mathbb{R}$,

$$\liminf_{T \rightarrow \infty} \frac{T}{a^2(T)} \log P_{\alpha_1}^\delta \left(\frac{1}{a(T)} \left(\log \frac{dP_{\alpha_1}^\delta}{dP_{\alpha_0}^\delta} \Big|_{\mathcal{F}_{[0,T]}} + \frac{(\alpha_1 - \alpha_0)^2 \delta T}{4\alpha_1} \right) \in G \right) \geq -\inf_{z \in G} \frac{-4\alpha_1^3 z^2}{(\alpha_1^2 - \alpha_0^2)^2 \delta}. \tag{3.14}$$

Proof of Theorem 2.1. The first claim is a direct conclusion of Lemma 3.2. Since

$$\frac{T}{a(T)} \frac{(\alpha_1 - \alpha_0)^2 \delta}{\alpha_0} + \frac{T}{a(T)} \frac{(\alpha_1 - \alpha_0)^2 \delta}{\alpha_1} \longrightarrow -\infty, \quad \text{as } T \longrightarrow \infty, \tag{3.15}$$

by Lemma 3.3, we see that the second one also holds. $\square$

4. Large Deviations in Testing Fractional Ornstein-Uhlenbeck Model

In this section, we will prove Theorems 2.2 and 2.3. We first study the large deviations of the log-likelihood ratio process.

Lemma 4.1. Assume $\alpha_1 < \alpha_0 < 0$. Then for any closed subset $F \subset \mathbb{R}$,

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log P_{\alpha_0}^\delta \left(\frac{1}{T} \log \frac{dP_{\alpha_1}^\delta}{dP_{\alpha_0}^\delta} \bigg|_{\mathcal{F}_{[0,T]}} \in F \right) \leq -\inf_{z \in F} I_{\alpha_0}(z), \quad (4.1)$$

and for any open subset $G \subset \mathbb{R}$,

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \log P_{\alpha_0}^\delta \left(\frac{1}{T} \log \frac{dP_{\alpha_1}^\delta}{dP_{\alpha_0}^\delta} \bigg|_{\mathcal{F}_{[0,T]}} \in G \right) \geq -\inf_{z \in G} I_{\alpha_0}(z), \quad (4.2)$$

where

$$I_{\alpha_0}(z) = \begin{cases} \frac{\alpha_0^2 - \alpha_1^2}{z + (\alpha_1 - \alpha_0)\delta/2} \left(\frac{\delta}{4} - \frac{\alpha_0(z + (\alpha_1 - \alpha_0)\delta/2)}{\alpha_1^2 - \alpha_0^2} \right)^2, & z < \frac{(\alpha_0 - \alpha_1)\delta}{2}, \\ +\infty, & \text{otherwise.} \end{cases} \quad (4.3)$$

Proof. Let

$$\Lambda_T(y) = \log \mathbb{E}_{\alpha_0}^\delta \exp \left\{ y \log \frac{dP_{\alpha_1}^\delta}{dP_{\alpha_0}^\delta} \bigg|_{\mathcal{F}_{[0,T]}} \right\}. \quad (4.4)$$

Then for $\varphi < 0$, we have

$$\begin{aligned} \Lambda_T(y) &= \log \mathbb{E}_{\alpha_0}^\delta \exp \left\{ \lambda(X_T - \delta T) + u \int_0^T X_t^2 ds \right\} \\ &= \log \mathbb{E}_\varphi^\delta \left[\frac{dP_{\alpha_0}^\delta}{dP_\varphi^\delta} \exp \left\{ \lambda(X_T - \delta T) + u \int_0^T X_t^2 ds \right\} \right] \\ &= \log \mathbb{E}_\varphi^\delta \left[\exp \left\{ \left(\lambda + \frac{\alpha_0 - \varphi}{2} \right) (X_T - \delta T) + \left(u - \frac{1}{2} \alpha_0^2 + \frac{1}{2} \varphi^2 \right) \int_0^T X_s^2 ds \right\} \right], \end{aligned} \quad (4.5)$$

where $\lambda = (\alpha_1 - \alpha_0)y/2$, $u = y(\alpha_0^2 - \alpha_1^2)/2$.

Since $\alpha_0^2 - 2u > 0$, for $y > (\alpha_0^2/\alpha_0^2 - \alpha_1^2)$, we can choose $\varphi = -\sqrt{\alpha_0^2 - 2u}$, for each $y > (\alpha_0^2/\alpha_0^2 - \alpha_1^2)$; then $\varphi < 0$ and

$$\Lambda_T(y) = -\delta T \left(\lambda + \frac{\alpha_0 - \varphi}{2} \right) + \log \mathbb{E}_\varphi^\delta \left[e^{(\lambda + (\alpha_0 - \varphi)/2)X_T} \right]. \quad (4.6)$$

By Lemma 3.1, we get

$$\lim_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_\varphi^\delta \left[\exp \left\{ \left(\lambda + \frac{\alpha_0 - \varphi}{2} \right) X_T \right\} \right] = 0. \quad (4.7)$$

Therefore,

$$\Lambda(y) =: \lim_{T \rightarrow \infty} \frac{1}{T} \Lambda_T(y) = -\frac{\delta}{2} \left((\alpha_1 - \alpha_0)y + \alpha_0 + \sqrt{\alpha_0^2 + (\alpha_1^2 - \alpha_0^2)y} \right). \quad (4.8)$$

Since $\Lambda(y)$ is a strictly convex differentiable function on $\mathfrak{D}_\Lambda = (\alpha_-, +\infty)$ with

$$\begin{aligned} \alpha_- &= \frac{\alpha_0^2}{\alpha_0^2 - \alpha_1^2} < 0, \\ \lim_{y \rightarrow \alpha_-} \Lambda'(y) &= +\infty, \end{aligned} \quad (4.9)$$

where $\mathfrak{D}_\Lambda$ is the effective domain of Λ , we see that $\Lambda(y)$ is steep. Finally, by

$$\sup_{y \in \mathbb{R}} \{zy - \Lambda(y)\} = \begin{cases} \frac{\alpha_0^2 - \alpha_1^2}{z + (\alpha_1 - \alpha_0)\delta/2} \left(\frac{\delta}{4} - \frac{\alpha_0(z + (\alpha_1 - \alpha_0)\delta/2)}{\alpha_1^2 - \alpha_0^2} \right)^2, & z < \frac{(\alpha_0 - \alpha_1)\delta}{2}, \\ +\infty, & \text{otherwise,} \end{cases} \quad (4.10)$$

and Gärtner-Ellis theorem, we complete the proof of this lemma. $\square$

Similarly, when $\alpha_0 < \alpha_1 < 0$, we have

$$\Lambda(y) =: \lim_{T \rightarrow \infty} \frac{1}{T} \Lambda_T(y) = -\frac{\delta}{2} \left((\alpha_1 - \alpha_0)y + \alpha_0 + \sqrt{\alpha_0^2 + (\alpha_1^2 - \alpha_0^2)y} \right). \quad (4.11)$$

Since $\Lambda(y)$ is a strictly convex differentiable function on $\mathfrak{D}_\Lambda = (-\infty, \alpha_+)$ with

$$\alpha_+ = \frac{\alpha_0^2}{\alpha_0^2 - \alpha_1^2} > 0, \quad (4.12)$$

and $\lim_{y \rightarrow \alpha_+} \Lambda'(y) = +\infty$, we can see that $\Lambda(y)$ is steep. By Gärtner-Ellis theorem, we also have the following result.

Lemma 4.2. Assume $\alpha_0 < \alpha_1 < 0$. Then for any closed subset $F \subset \mathbb{R}$,

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log P_{\alpha_0}^\delta \left(\frac{1}{T} \log \frac{dP_{\alpha_1}^\delta}{dP_{\alpha_0}^\delta} \Big|_{\mathcal{F}_{[0,T]}} \in F \right) \leq -\inf_{z \in F} \hat{I}_{\alpha_0}(z), \quad (4.13)$$

and for any open subset $G \subset \mathbb{R}$,

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \log P_{\alpha_0}^\delta \left(\frac{1}{T} \log \frac{dP_{\alpha_1}^\delta}{dP_{\alpha_0}^\delta} \Big|_{\mathcal{F}_{[0,T]}} \in G \right) \geq -\inf_{z \in G} \hat{I}_{\alpha_0}(z), \quad (4.14)$$

where

$$\hat{I}_{\alpha_0}(z) = \begin{cases} \frac{\alpha_0^2 - \alpha_1^2}{z + ((\alpha_1 - \alpha_0)\delta/2)} \left(\frac{\delta}{4} - \frac{\alpha_0(z + (\alpha_1 - \alpha_0)\delta/2)}{\alpha_1^2 - \alpha_0^2} \right)^2, & z > \frac{(\alpha_0 - \alpha_1)\delta}{2}, \\ +\infty, & \text{otherwise.} \end{cases} \quad (4.15)$$

Note that

$$\log \frac{dP_{\alpha_1}^\delta}{dP_{\alpha_0}^\delta} \Big|_{\mathcal{F}_{[0,T]}} = -\log \frac{dP_{\alpha_0}^\delta}{dP_{\alpha_1}^\delta} \Big|_{\mathcal{F}_{[0,T]}}. \quad (4.16)$$

Then we have the following Lemma.

Lemma 4.3. Assume $\alpha_1 < \alpha_0 < 0$. Then for any closed subset $F \subset \mathbb{R}$,

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log P_{\alpha_1}^\delta \left(\frac{1}{T} \log \frac{dP_{\alpha_1}^\delta}{dP_{\alpha_0}^\delta} \Big|_{\mathcal{F}_{[0,T]}} \in F \right) \leq -\inf_{z \in F} \hat{I}_{\alpha_1}(z), \quad (4.17)$$

and for any open subset $G \subset \mathbb{R}$,

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \log P_{\alpha_1}^\delta \left(\frac{1}{T} \log \frac{dP_{\alpha_1}^\delta}{dP_{\alpha_0}^\delta} \Big|_{\mathcal{F}_{[0,T]}} \in G \right) \geq -\inf_{z \in G} \hat{I}_{\alpha_1}(z), \quad (4.18)$$

where

$$I_{\alpha_1}(z) = \begin{cases} \frac{\alpha_0^2 - \alpha_1^2}{z + (\alpha_1 - \alpha_0)\delta/2} \left(\frac{\delta}{4} - \frac{\alpha_1(z + (\alpha_1 - \alpha_0)\delta/2)}{\alpha_1^2 - \alpha_0^2} \right)^2, & z < \frac{(\alpha_0 - \alpha_1)\delta}{2}, \\ +\infty, & \text{otherwise.} \end{cases} \quad (4.19)$$

Lemma 4.4. Assume $\alpha_0 < \alpha_1 < 0$. Then for any closed subset $F \subset \mathbb{R}$,

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log P_{\alpha_1}^\delta \left(\frac{1}{T} \log \frac{dP_{\alpha_1}^\delta}{dP_{\alpha_0}^\delta} \Big|_{\mathcal{F}_{[0,T]}} \in F \right) \leq -\inf_{z \in F} I_{\alpha_1}(z), \quad (4.20)$$

and for any open subset $G \subset \mathbb{R}$,

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \log P_{\alpha_1}^\delta \left(\frac{1}{T} \log \frac{dP_{\alpha_1}^\delta}{dP_{\alpha_0}^\delta} \Big|_{\mathcal{F}_{[0,T]}} \in G \right) \geq -\inf_{z \in G} I_{\alpha_1}(z), \quad (4.21)$$

where

$$\hat{I}_{\alpha_1}(z) = \begin{cases} \frac{\alpha_0^2 - \alpha_1^2}{z + (\alpha_1 - \alpha_0)\delta/2} \left(\frac{\delta}{4} - \frac{\alpha_1(z + (\alpha_1 - \alpha_0)\delta/2)}{\alpha_1^2 - \alpha_0^2} \right)^2, & z > \frac{(\alpha_0 - \alpha_1)\delta}{2}, \\ +\infty, & \text{otherwise.} \end{cases} \quad (4.22)$$

By the expression of $I_{\alpha_0}(z)$, $I_{\alpha_1}(z)$, $\hat{I}_{\alpha_0}(z)$, and $\hat{I}_{\alpha_1}(z)$, the following lemma is.

Lemma 4.5. (i)

$$\begin{aligned} I_{\alpha_0}(z) = 0 & \quad \text{iff } z_{\alpha_0} = \frac{(\alpha_0 - \alpha_1)^2 \delta}{4\alpha_0}, \\ I_{\alpha_1}(z) = 0 & \quad \text{iff } z_{\alpha_1} = -\frac{(\alpha_1 - \alpha_0)^2 \delta}{4\alpha_1}, \end{aligned} \quad (4.23)$$

for all $z < (\alpha_0 - \alpha_1)\delta/2$,

$$I_{\alpha_0}(z) = I_{\alpha_1}(z) + z; \quad (4.24)$$

(ii)

$$\begin{aligned} \hat{I}_{\alpha_0}(z) = 0 & \quad \text{iff } \hat{z}_{\alpha_0} = \frac{(\alpha_0 - \alpha_1)^2 \delta}{4\alpha_0}, \\ \hat{I}_{\alpha_1}(z) = 0 & \quad \text{iff } \hat{z}_{\alpha_1} = -\frac{(\alpha_1 - \alpha_0)^2 \delta}{4\alpha_1}, \end{aligned} \quad (4.25)$$

for all $z > (\alpha_0 - \alpha_1)\delta/2$,

$$\hat{I}_{\alpha_0}(z) = \hat{I}_{\alpha_1}(z) + z. \quad (4.26)$$

Proof of Theorems 2.2 and 2.3. Since the proofs of the two theorems are similar, we only prove Theorem 2.2. Since $I_{\alpha_0}(z)$ is increasing on $(z_{\alpha_0}, (\alpha_0 - \alpha_1)\delta/2)$ and $I_{\alpha_0}(z_{\alpha_0}) = 0$, Therefore, for $a > 0$, by Lemma 4.1, we can choose a $\xi(a) \in (z_{\alpha_0}, (\alpha_0 - \alpha_1)\delta/2)$ such that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \log P_{\alpha_0}^\delta \left(\frac{1}{T} \log \frac{dP_{\alpha_1}^\delta}{dP_{\alpha_0}^\delta} \Big|_{\mathcal{F}_{[0,T]}} \geq \xi(a) \right) = -a. \quad (4.27)$$

It is clear that $\xi(a)$ is increasing for $a > 0$, and by Lemma 4.5, we get $I_{\alpha_0}(z_{\alpha_1}) = z_{\alpha_1}$, which implies $\xi(z_{\alpha_1}) = z_{\alpha_1}$. Hence for $a \in (0, z_{\alpha_1})$, we have $\xi(a) \in (0, z_{\alpha_1})$, and since $I_{\alpha_1}(z)$ is nonincreasing for $z \leq \xi(a)$, therefore we get

$$I_{\alpha_1}(\xi(a)) = \inf\{I_{\alpha_1}(z) : z \leq \xi(a)\}. \quad (4.28)$$

Similarly, for $a \in (z_{\alpha_1}, +\infty)$, we have $\xi(a) \in (z_{\alpha_1}, (\alpha_0 - \alpha_1)\delta/2)$, and since $I_{\alpha_1}(z)$ is non-decreasing for $z \geq \xi(a)$, therefore we get

$$I_{\alpha_1}(\xi(a)) = \inf\{I_{\alpha_1}(z) : z \geq \xi(a)\}, \quad (4.29)$$

which complete the proof of Theorem 2.2. $\square$

Acknowledgments

The authors would like to express their gratitude to Professor F. Q. Gao and the reviewer for their valuable comments.

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