Research Article

On Tightness of the Skew Random Walks

Youngsoo Seol

Department of Mathematics, Iowa State University, Ames, IA 50010, USA

Correspondence should be addressed to Youngsoo Seol, yseol@iastate.edu

Received 8 August 2011; Accepted 5 December 2011

Academic Editor: Nikolaos E. Limnios

Copyright © 2012 Youngsoo Seol. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The primary purpose of this paper is to prove a tightness of α -skew random walks. The tightness result implies, in particular, that the α -skew Brownian motion can be constructed as the scaling limit of such random walks. Our proof of tightness is based on a fourth-order moment method.

1. Introduction and Statement of the Main Result

Skew Brownian motion was introduced by Itô and McKean [1] to furnish a construction of certain stochastic processes related to Feller's classification of second-order differential operators associated with diffusion processes (see also Section 4.2 in [2]). For $\alpha \in (0, 1)$, the α -skew Brownian motion is defined as a one-dimensional Markov process with the same transition mechanism as of the usual Brownian motion, with the only exception that the excursions away from zero are assigned a positive sign with probability α and a negative sign with probability $1 - \alpha$. The signs form an i.i.d. sequence and are chosen independently of the past history of the process. If $\alpha = 1/2$, the process is the usual Brownian motion.

Formally, the α -skew random walk on $\mathbb{Z}$ starting at 0 is defined as the birth-death Markov chain $S^{(\alpha)} = \{S_k^{(\alpha)}; k \geq 0\}$ with $S_0^\alpha = 0$ and one-step transition probabilities given by

$$P(S_{k+1}^{(\alpha)} = m + 1 \mid S_k^{(\alpha)} = m) = \begin{cases} \alpha & \text{if } m = 0, \\ \frac{1}{2}, & \text{otherwise,} \end{cases} \quad (1.1)$$

$$P(S_{k+1}^{(\alpha)} = m - 1 \mid S_k^{(\alpha)} = m) = \begin{cases} 1 - \alpha & \text{if } m = 0, \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

In the special case $\alpha = 1/2$, $S^{(1/2)}$ is a simple symmetric random walk on $\mathbb{Z}$. Notice that when $\alpha \neq 1/2$, the jumps (in general, increments) of the random walk are not independent.

Harrison and Shepp [3] asserted (without proof) that the functional central limit theorem (FCLT, for short) for reflecting Brownian motion can be used to construct skew Brownian motion as the limiting process of a suitably modified symmetric random walk on the integer lattice. This result has served as a foundation for numerical algorithms tracking moving particle in a highly heterogeneous porous media; see, for instance, [4–7]. In [5] it was suggested that tightness could be obtained based on second moments; however this is not possible even in the case of simple symmetric random walk. The lack of statistical independence of the increments makes a fourth moment proof all the more challenging. Although proofs of FCLTs in more general frameworks have subsequently been obtained by other methods, for example, by Skorokhod embedding in [8], a self-contained simple proof of tightness for simple skew random walk has not been available in the literature.

The main goal of this paper is to prove the following result. Let $C(\mathbb{R}_+, \mathbb{R})$ be the space of continuous functions from $\mathbb{R}_+ = [0, \infty)$ into $\mathbb{R}$, equipped with the topology of uniform convergence on compact sets. For $n \in \mathbb{N}$, let $X_n^{(\alpha)} \in C(\mathbb{R}_+, \mathbb{R})$ denote the following linear interpolation of $S_{[nt]}^{(\alpha)}$:

$$X_n^{(\alpha)}(t) = \frac{1}{\sqrt{n}} \left(S_{[nt]}^{(\alpha)} + (nt - [nt]) \cdot S_{[nt]+1}^{(\alpha)} \right). \quad (1.2)$$

Here and henceforth $[x]$ denotes the integer part of a real number x .

Theorem 1.1. *For any $\alpha \in (0, 1)$, there exists a constant $C > 0$, such that the inequality*

$$E \left| X_n^{(\alpha)}(t) - X_n^{(\alpha)}(s) \right|^4 \leq C |s - t|^2, \quad (1.3)$$

holds uniformly for all $s, t > 0$, and $n \in \mathbb{N}$.

The results stated above implies the following (see, for instance, [9, page 98]).

Corollary 1.2. *The family of processes $X_n^{(\alpha)}$, $n \in \mathbb{N}$, is tight in $C(\mathbb{R}_+, \mathbb{R})$.*

2. Proof of Theorem 1.1

In this section we complete the proof of our main result, Theorem 1.1. In what follows we will use S to denote the simple symmetric random walk $S^{(1/2)}$. The following observations can be found in [3].

Proposition 2.1. (a) $|S^{(\alpha)}|$ has the same distribution as $|S|$ on $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. That is, $|S^{(\alpha)}|$ is a simple symmetric random walk on $\mathbb{Z}_+$, reflected at 0.

(b) The processes $-S^{(\alpha)}$ and $S^{(1-\alpha)}$ have the same distribution.

The next statement describes n -step transition probabilities of the skew random walks by relating them to those of S (see, for instance, [5, page 436]).

Proposition 2.2. For $m \in \mathbb{Z}$, $k > 0$

$$P(S_k^{(\alpha)} = m) = \begin{cases} \alpha \cdot P(|S_k| = m) & \text{if } m > 0 \\ (1 - \alpha) \cdot P(|S_k| = -m) & \text{if } m < 0 \\ P(|S_k^{(\alpha)}| = 0) = P(|S_k| = 0) & \text{if } m = 0. \end{cases} \quad (2.1)$$

The following observation is evident from the explicit form of the distribution function of $S_k^{(\alpha)}$, given in Proposition 2.2.

Proposition 2.3. With probability one,

$$\begin{aligned} E(S_{j+1}^{(\alpha)} - S_j^{(\alpha)} \mid S_j^{(\alpha)}) &= (2\alpha - 1)\mathbf{1}_{\{S_j^{(\alpha)}=0\}}, \\ E\left[\left(S_{i+1}^{(\alpha)} - S_i^{(\alpha)}\right)^2 \mid S_i^{(\alpha)}\right] &= 1. \end{aligned} \quad (2.2)$$

To show the result of Theorem 1.1, we will need a corollary to Karamata's Tauberian theorem, which we are going now to state. For a measure μ on $[0, \infty)$, denote by $\hat{\mu}(\lambda) := \int_0^\infty e^{-\lambda x} \mu(dx)$ the Laplace transform of μ . The transform is well defined for $\lambda \in (c, \infty)$, where $c > 0$ is a nonnegative constant, possibly $+\infty$. If μ and ν are measures on $[0, \infty)$ such that $\hat{\mu}(\lambda)$ and $\hat{\nu}(\lambda)$ both exist for all $\lambda > 0$, then the convolution $\gamma = \mu * \nu$ has the Laplace transform $\hat{\gamma}(\lambda) = \hat{\mu}(\lambda)\hat{\nu}(\lambda)$ for $\lambda > 0$. If μ is a discrete measure concentrated on $\mathbb{Z}_+$, one can identify μ with a sequence μ_n of its values on $n \in \mathbb{Z}_+$. For such discrete measures, we have the following. (see, e.g., Corollary 8.10 in [10, page 118]).

Proposition 2.4. Let $\tilde{\mu}(t) = \sum_{n=0}^\infty \mu_n t^n$, $0 \leq t < 1$, where $\{\mu_n\}_{n=0}^\infty$ is a sequence of nonnegative numbers. For L slowly varying at infinity and $0 \leq \theta < \infty$ one has

$$\tilde{\mu}(t) \sim (1-t)^{-\theta} L\left(\frac{1}{1-t}\right) \quad \text{as } t \uparrow 1 \quad (2.3)$$

if and only if

$$\sum_{j=0}^n \mu_j \sim \frac{1}{\Gamma(\theta)} n^\theta L(n) \quad \text{as } n \rightarrow \infty. \quad (2.4)$$

Here and henceforth, $a_n \sim b_n$ for two sequence of real numbers $\{a_n\}_{n \in \mathbb{N}}$ and $\{b_n\}_{n \in \mathbb{N}}$ means $\lim_{n \rightarrow \infty} a_n/b_n = 1$.

We are now in a position to prove the following key proposition. Define a sequence $\{q(k)\}_{k \in \mathbb{Z}_+}$ as follows

$$g(k) = \begin{cases} 0 & \text{if } k \in \mathbb{N} \text{ is odd} \\ \binom{2i}{i} 2^{-2i} & \text{if } k = 2i \in \mathbb{N} \text{ is even.} \end{cases} \quad (2.5)$$

Note that in view of Proposition 2.2,

$$g(k) = P(S_k = 0) = P(|S_k| = 0) = P\left(\left|S_k^{(\alpha)}\right| = 0\right) = P\left(S_k^{(\alpha)} = 0\right). \quad (2.6)$$

Proposition 2.5.

- (a) If $\mu(j) = g * g(j)$ then $\sum_{j=0}^m \mu(j) \sim m$.
- (b) If $\nu(j) = g * g * g * g(j)$ then $\sum_{j=0}^m \nu(j) \sim m^2$.

Proof. For $t \in (0, 1)$, let $\tilde{g}(t) = \sum_{k=0}^{\infty} g(k)t^k$. Notice that $\tilde{g}(t)$ is well defined since $g(k) = P(S_k = 0) < 1$ for $k \geq 0$. Since $g(2j) = \binom{2j}{j} 2^{-2j} = (-1)^j \binom{-1/2}{j}$, we have

$$\begin{aligned} \tilde{g}(t) &= \sum_{k=0}^{\infty} g(k)t^k = \sum_{j=0}^{\infty} \binom{2j}{j} 2^{-2j} t^{2j} = \sum_{j=0}^{\infty} (-1)^j \binom{-\frac{1}{2}}{j} t^{2j} \\ &= \sum_{j=0}^{\infty} \binom{-\frac{1}{2}}{j} (-t^2)^j = (1 - t^2)^{-1/2}. \end{aligned} \quad (2.7)$$

Notice that, using the notation of Proposition 2.4, $\tilde{g}(t) = \hat{g}(\lambda)$ if $t = e^{-\lambda}$. Therefore, $\tilde{\mu}(t) = \tilde{g}^2(t) = (1 - t^2)^{-1}$ while $\tilde{\nu}(t) = \tilde{g}^4(t) = (1 - t^2)^{-2}$. Thus claims (a) and (b) of the proposition follow from Proposition 2.4 applied, respectively, with $\theta = 1, L = 1$ for μ and with $\theta = 2, L = 1$ for ν . $\square$

The last technical lemma we need is the following claim.

Lemma 2.6. For integers $0 < i_1 < i_2 < i_3 < i_4$ define

$$\begin{aligned} A(i_1, i_2, i_3) &:= E\left(S_{i_3+1}^{(\alpha)} - S_{i_3}^{(\alpha)}\right)^2 \left(S_{i_2+1}^{(\alpha)} - S_{i_2}^{(\alpha)}\right) \left(S_{i_1+1}^{(\alpha)} - S_{i_1}^{(\alpha)}\right), \\ B(i_1, i_2, i_3, i_4) &:= E\left(S_{i_4+1}^{(\alpha)} - S_{i_4}^{(\alpha)}\right) \left(S_{i_3+1}^{(\alpha)} - S_{i_3}^{(\alpha)}\right) \left(S_{i_2+1}^{(\alpha)} - S_{i_2}^{(\alpha)}\right) \left(S_{i_1+1}^{(\alpha)} - S_{i_1}^{(\alpha)}\right). \end{aligned} \quad (2.8)$$

Then there is a constant $C > 0$ such that

$$\begin{aligned} \sum_{1 \leq i_1 < i_2 < i_3 \leq k-j} A(i_1, i_2, i_3) &\leq C|k-j|^2, \\ \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq k-j} B(i_1, i_2, i_3, i_4) &\leq C|k-j|^2. \end{aligned} \quad (2.9)$$

Proof. Using Proposition 2.3, the Markov property, and the fact the excursions of $S^{(\alpha)}$ away from zero are the same as excursions of the simple symmetric random walk S , we obtain

$$\begin{aligned} A(i_1, i_2, i_3) &= E\left(S_{i_3+1}^{(\alpha)} - S_{i_3}^{(\alpha)}\right)^2 \left(S_{i_2+1}^{(\alpha)} - S_{i_2}^{(\alpha)}\right) \left(S_{i_1+1}^{(\alpha)} - S_{i_1}^{(\alpha)}\right) \mathbf{1}_{\{S_{i_1}^{(\alpha)}=0\}} \mathbf{1}_{\{S_{i_2}^{(\alpha)}=0\}} \\ &= P(S_{i_1} = 0) \cdot (2\alpha - 1) \cdot P(S_{i_2} = 0 \mid S_{i_1} = 0) \cdot (2\alpha - 1) \\ &= (2\alpha - 1)^2 g(i_1) g(i_2 - i_1). \end{aligned} \quad (2.10)$$

Therefore,

$$\sum_{1 \leq i_1 < i_2 < i_3 \leq k-j} A(i_1, i_2, i_3) \leq \sum_{i_3=0}^{[k-j]} \sum_{i_2=0}^{i_3-1} \sum_{i_1=0}^{i_2-1} g(i_2 - i_1) g(i_1). \quad (2.11)$$

Using Proposition 2.5, we obtain

$$\begin{aligned} \sum_{i_3=0}^{[k-j]} \sum_{i_2=0}^{i_3-1} \sum_{i_1=0}^{i_2-1} g(i_2 - i_1) g(i_1) &= \sum_{i_3=0}^{[k-j]} \sum_{i_2=0}^{i_3-1} g * g(i_2) \leq \sum_{i_3=0}^{[k-j]} \sum_{i_2=0}^{[k-j]} g * g(i_2) \\ &\leq C_1 |k-j|^2, \end{aligned} \quad (2.12)$$

for some constant $C_1 > 0$ and any $k, j \in \mathbb{N}$.

Similarly,

$$\begin{aligned} B(i_1, i_2, i_3, i_4) &= (2\alpha - 1)^4 \cdot P(S_{i_1} = 0) \cdot \prod_{a=1}^3 P(S_{i_{a+1}} = 0 \mid S_{i_a} = 0) \\ &= (2\alpha - 1)^4 g(i_1) g(i_2 - i_1) g(i_3 - i_2) g(i_4 - i_3). \end{aligned} \quad (2.13)$$

Hence, using again Proposition 2.5,

$$\sum_{0 \leq i_1 < i_2 < i_3 < i_4} B(i_1, i_2, i_3, i_4) \leq \sum_{i_4=0}^{[k-j]} g * g * g * g(i_4) \leq C_2 |k-j|^2, \quad (2.14)$$

for some constant $C_2 > 0$ and any $k, j \in \mathbb{N}$.

To conclude the proof of the lemma, set $C := \max\{C_1, C_2\}$. □

We are now in a position to complete the proof of our main result.

Completion of the Proof of Theorem 1.1

First consider the case where $s = j/n < k/n = t$ are grid points. Then

$$\begin{aligned}
 E \left| \frac{S_{[nt]}^{(\alpha)}}{\sqrt{n}} - \frac{S_{[ns]}^{(\alpha)}}{\sqrt{n}} \right|^4 &= \frac{1}{n^2} E \left| S_k^{(\alpha)} - S_j^{(\alpha)} \right|^4 = \frac{1}{n^2} E \left| \sum_{i=j}^{k-1} (S_{i+1}^{(\alpha)} - S_i^{(\alpha)}) \right|^4 \\
 &= \frac{1}{n^2} \sum_{i=j}^{k-1} E (S_{i+1}^{(\alpha)} - S_i^{(\alpha)})^4 + \frac{1}{n^2} \sum_{i_1 < i_2 \leq k-j} E (S_{i_1+1}^{(\alpha)} - S_{i_1}^{(\alpha)})^2 (S_{i_2+1}^{(\alpha)} - S_{i_2}^{(\alpha)})^2 \\
 &\quad + \frac{1}{n^2} \sum_{i_1 < i_2 < i_3 \leq k-j} E (S_{i_3+1}^{(\alpha)} - S_{i_3}^{(\alpha)})^2 (S_{i_2+1}^{(\alpha)} - S_{i_2}^{(\alpha)}) (S_{i_1+1}^{(\alpha)} - S_{i_1}^{(\alpha)}) \\
 &\quad + \frac{1}{n^2} \sum_{i_1 < i_2 < i_3 < i_4 \leq k-j} E \left(\prod_{a=1}^4 S_{i_a+1}^{(\alpha)} - S_{i_a}^{(\alpha)} \right) \\
 &\leq \frac{1}{n^2} \sum_{i=j}^{k-1} 1 + \frac{1}{n^2} \binom{k-j}{2} \binom{k-j}{2} + \frac{1}{n^2} C_1 |k-j|^2 + \frac{1}{n^2} C_2 |k-j|^2 \\
 &\leq C_3 |t-s|^2,
 \end{aligned} \tag{2.15}$$

for a large enough constant $C_3 > 0$.

To conclude the proof of Theorem 1.1, it remains to observe that for nongrid points s and t one can use an approximation by neighbor grid points. In fact, the approximation argument given in [9, pages 100-101] for regular random walks goes through verbatim.

Acknowledgments

The author would like to thank Professor Edward C. Waymire for suggesting this problem and for helpful comments. He also wants to thank Professor Alexander Roitershtein for helpful suggestions and corrections.

References

- [1] K. Itô and H. P. McKean Jr., "Brownian motions on a half line," *Illinois Journal of Mathematics*, vol. 7, pp. 181–231, 1963.
- [2] K. Itô and H. P. McKean Jr., *Diffusion Processes and Their Sample Paths*, Springer, Berlin, Germany, 1974.
- [3] J. M. Harrison and L. A. Shepp, "On skew Brownian motion," *The Annals of Probability*, vol. 9, no. 2, pp. 309–313, 1981.
- [4] H. Hoteit, R. Mose, A. Younes, F. Lehmann, and Ph. Ackerer, "Three-dimensional modeling of mass transfer in porous media using the mixed hybrid finite elements and the random-walk methods," *Mathematical Geology*, vol. 34, no. 4, pp. 435–456, 2002.
- [5] A. Lejay, "On the constructions of the skew Brownian motion," *Probability Surveys*, vol. 3, pp. 413–466, 2006.

- [6] T. Appuhamillage, V. Bokil, E. Thomann, E. Waymire, and B. Wood, "Occupation and local times for skew Brownian motion with applications to dispersion across an interface," *The Annals of Applied Probability*, vol. 21, no. 1, pp. 183–214, 2011.
- [7] M. J. Saxton and K. Jacobson, "Single-particle tracking: applications to membrane dynamics," *Annual Review of Biophysics and Biomolecular Structure*, vol. 26, pp. 373–399, 1997.
- [8] A. S. Cherny, A. N. Shiryaev, and M. Yor, "Limit behaviour of the "horizontal-vertical" random walk and some extensions of the Donsker-Prokhorov invariance principle," *Theory of Probability and its Applications*, vol. 47, no. 3, pp. 498–517, 2002.
- [9] R. N. Bhattacharya and E. C. Waymire, *Stochastic Processes with Applications*, Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics, John Wiley & Sons, New York, NY, USA, 1990.
- [10] R. Bhattacharya and E. C. Waymire, *A Basic Course in Probability Theory*, Universitext, Springer, New York, NY, USA, 2007.