Research Article

Estimating the Reliability Function for a Family of Exponentiated Distributions

Ajit Chaturvedi and Anupam Pathak

Department of Statistics, University of Delhi, Delhi 110007, India

Correspondence should be addressed to Anupam Pathak; pathakanupam24@gmail.com

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A family of exponentiated distributions is proposed. The problems of estimating the reliability function are considered. Uniformly minimum variance unbiased estimators and maximum likelihood estimators are derived. A comparative study of the two methods of estimation is done. Simulation study is preformed.

1. Introduction

The reliability function $R(t)$ is defined as the probability of failure-free operation until time t . Thus, if the random variable (rv) X denotes the lifetime of an item, then $R(t) = P(X > t)$. Another measure of reliability under stress-strength setup is the probability $P = P(X > Y)$, which represents the reliability of an item of random strength X subject to random stress Y .

A lot of work has been done in the literature to deal with inferential problems related to various exponentiated distributions. In particular, Mudholkar and Srivastava [1] considered exponentiated Weibull family for analyzing bathtub failure-real data. The exponentiated Weibull family was also applied to the bus-motor-failure data in Davis [2], to head-and-neck cancer clinical trial data in Efron [3] [see [4]], and in analyzing flood data [see [5]]. Jiang and Murthy [6] considered exponentiated Weibull family and illustrated some of its properties by using graphical approach. Nassar and Eissa [7, 8] studied a two-parameter exponentiated Weibull distribution and they gave some of its properties and estimated the parameters by using the maximum likelihood and Bayes methods based on type II censored data. They used the squared-error and linear in exponential (LINEX) loss functions and an informative prior to obtain the Bayes estimates. Pal et al. [9] showed that the failure rate of exponentiated Weibull behaves more like the failure rate of the Weibull distribution than that of the

Gamma distribution. They also obtained maximum likelihood estimators, Fisher's information matrix, and confidence intervals for related parameters. Gupta et al. [10] introduced exponentiated exponential distribution in a series of papers. Gupta and Kundu [11–17], Gupta et al. [18], and Kundu et al. [19] concentrated on the study of the exponentiated exponential distribution. Kundu and Gupta [20] stated that exponentiated exponential distribution is an alternative to the well-known two-parameter Gamma, two-parameter Weibull or two-parameter lognormal distributions. Kundu and Gupta [20] estimated the reliability of the stress-strength model $P(X < Y)$, when X and Y are independent exponentiated exponential random variables. Raqab and Ahsanullah [21] estimated the parameters of the exponentiated exponential based on ordered statistics. Inference for exponentiated Weibull distribution based on record values was made by Raqab [22]. Abdel-Hamid and AL-Hussaini [23] obtained the maximum likelihood estimates of the parameters when step-stress accelerated life testing is applied for exponentiated exponential distribution. AL-Hussaini and Hussein [24, 25] fitted the exponentiated Burr model to the data of breaking strength of single carbon fibers of particular length (data in [26]) in the complete sample case and when type II censoring is imposed on data. SELF and LINEX loss functions are used in the Bayes estimation. Gupta et al. [10] introduced a new distribution, called the exponentiated Pareto distribution. Shawky and Abu-Zinadah [27] obtained maximum likelihood estimators of the different parameters of the

exponentiated Pareto distribution. They also considered five other estimation procedures and compared their performances through numerical simulations.

The purpose of the present paper is manifold. We propose a family of exponentiated distributions, which covers as many as ten exponentiated distributions as specific cases. We consider the problems of estimating $R(t)$ and “ P .” Uniformly minimum variance unbiased estimators (UMVUES) and maximum likelihood estimators (MLES) are derived. In order to obtain these estimators, the major role is played by the estimator of the probability density function (pdf) at a specified point, and the functional forms of the parametric functions to be estimated are not needed. It is worth mentioning here that, in order to estimate “ P ,” in the literature, the authors have considered the case when X and Y follow the same distributions, may be with different parameters. We have generalized the result to the case when X and Y may follow any distribution from the proposed exponentiated family of distributions. Simulation study is carried out to investigate the performance of the estimators.

In Section 2, we introduce the family of exponentiated distributions. In Section 3, we derive the UMVUES of $R(t)$ and “ P .” In Section 4, we obtain the MLES of $R(t)$ and “ P .” In Section 5, analysis of a simulated data has been presented for illustrative purposes. Finally, in Section 6, conclusions have been provided.

2. The Family of Exponentiated Distributions

Let the rv X follow the distribution having the pdf

$$f(x; a, \alpha, \lambda, \underline{\theta}) = \alpha \lambda g'(x; a, \underline{\theta}) e^{-\lambda g(x; a, \underline{\theta})} (1 - e^{-\lambda g(x; a, \underline{\theta})})^{\alpha-1};$$

$$x > a \geq 0, \alpha, \lambda > 0. \quad (1)$$

Here, $g(x; a, \underline{\theta})$ is a function of X and may also depend on (may be vector-valued) parameter $\underline{\theta}$. Moreover, $g(x; a, \underline{\theta})$ is a real-valued, monotonically increasing function of X with $g(a; a, \underline{\theta}) = 0$, $g(\infty; a, \underline{\theta}) = \infty$, and $g'(x; a, \underline{\theta})$ denotes the derivative of $g(x; a, \underline{\theta})$ with respect to x . Our case is a special model of the beta extended Weibull family proposed by Cordeiro et al. [28] and can be obtained from equation (7) when $b = 0$. We note that (1) represents a family of exponentiated distributions, as it covers the following exponentiated distributions as special cases.

- (i) For $g(x; a, \underline{\theta}) = x$ and $a = 0$, we get the exponentiated exponential distribution [see [10]].
- (ii) For $g(x; a, \underline{\theta}) = x^2$ and $a = 0$, it gives exponentiated Rayleigh distribution [see [29]].
- (iii) For $g(x; a, \underline{\theta}) = x^\delta$, $\delta > 0$ and $a = 0$, we get the exponentiated Weibull distribution [see [1]].
- (iv) For $g(x; a, \underline{\theta}) = \log(1+x^\delta)$, $\delta > 0$ and $a = 0$, it leads us to pdf of exponentiated Burr distribution [see [24]].
- (v) For $g(x; a, \underline{\theta}) = \log(x/a)$, it turns out to be exponentiated Pareto distribution [see [10]].

- (vi) For $g(x; a, \underline{\theta}) = (1 + (x/\xi))$, $\xi > 0$ and $a = 0$, it gives us exponentiated Lomax distribution [see [30]].
- (vii) For $g(x; a, \underline{\theta}) = (1 + (x^\xi/\nu))$, $\xi > 0$, $\nu > 0$ and $a = 0$, it turns out to be exponentiated Burr distribution with scale parameter ν [see [31]].
- (viii) For $g(x; a, \underline{\theta}) = x^\gamma \exp(\nu x)$, $\gamma > 0$, $\nu > 0$ and $a = 0$, it gives the exponentiated form of modified Weibull distribution of Lai et al. [32].
- (ix) For $g(x; a, \underline{\theta}) = \gamma \exp[(x/\gamma)^\nu - 1]$, $\gamma > 0$, $\nu > 0$ and $a = 0$, we get the exponentiated form of a modification of Weibull distribution [see [33]].
- (x) For $g(x; a, \underline{\theta}) = (x - a) + (\nu/\lambda) \log((x + \nu)/(a + \nu))$, $\nu > 0$, it gives exponentiated form of the generalized Pareto distribution [see [34]].

3. UMVUES of $R(t)$ and “ P ”

Throughout this section, we assume that α is unknown, but “ a ”, λ , and $\underline{\theta}$ are known. Let $X_1, X_2, \dots, X_n$ be a random sample of size n from (1).

Lemma 1. Let $T = \sum_{i=1}^n \log(1 - e^{-\lambda g(X_i; a, \underline{\theta})})$. Then, T is complete and sufficient for the family of distributions given in (1). Moreover, the pdf of T is

$$g(t; \alpha, \lambda, \underline{\theta}) = \frac{\alpha^n}{\Gamma(n)} (-t)^{n-1} e^{\alpha t}; \quad -\infty < t < 0. \quad (2)$$

Proof. Denoting by $h(x_1, x_2, \dots, x_n; a, \alpha, \lambda, \underline{\theta})$ the joint pdf of $X_1, X_2, \dots, X_n$, we have

$$h(x_1, x_2, \dots, x_n; a, \alpha, \lambda, \underline{\theta})$$

$$= (\alpha \lambda)^n \prod_{i=1}^n g'(x_i; a, \underline{\theta}) e^{-\lambda \sum_{i=1}^n g(x_i; a, \underline{\theta})} \quad (3)$$

$$\times e^{(\alpha-1) \sum_{i=1}^n \log(1 - e^{-\lambda g(x_i; a, \underline{\theta})})}.$$

It follows from (3) and Fishers-Neyman factorization theorem [see [35, p. 341]] that T is sufficient for the family of distributions $f(x; a, \alpha, \lambda, \underline{\theta})$. It is easy to see that the rv $V = -2\alpha \log(1 - e^{-\lambda g(x; a, \underline{\theta})})$ follows $\chi^2_{(2)}$ distribution. Thus, from the well-known additive property of Chi-square distribution [see Johnson and Kotz, [36, p. 170]], $-2\alpha T = -2\alpha \sum_{i=1}^n \log(1 - e^{-\lambda g(x_i; a, \underline{\theta})})$ is an $\chi^2_{(2n)}$ rv and the result follows. Since the distribution of T belongs to exponential family, it is also complete [see [35, p. 341]]. $\square$

The following lemma provides the UMVUES of the powers of α .

Lemma 2. For $q \in (-\infty, \infty)$, the UMVUE of α^q is

$$\hat{\alpha}^q = \begin{cases} \frac{\Gamma(n)}{\Gamma(n-q)} (-T)^{-q}; & q < n, \\ 0; & \text{otherwise.} \end{cases} \quad (4)$$

Proof. The result follows from Lemma 1, Lehmann-Scheffé theorem [see [35, p. 357]], and the fact that

$$E(-2\alpha T)^{-q} = \frac{\Gamma(n-q)}{2^q \Gamma(n)}; \quad q < n. \quad (5)$$

□

In the following lemma, we provide the UMVUE of the sampled pdf (1) at a specified point “ x ”

Lemma 3. The UMVUE of $f(x; a, \alpha, \lambda, \underline{\theta})$ at a specified point “ x ” is

$$\begin{aligned} & \hat{f}(x; a, \alpha, \lambda, \underline{\theta}) \\ &= \begin{cases} -(n-1) \lambda T^{-1} (1 - e^{-\lambda g(x; a, \underline{\theta})})^{-1} \\ \quad \times g'(x; a, \underline{\theta}) e^{-\lambda g(x; a, \underline{\theta})} \\ \quad \times [1 - T^{-1} \log(1 - e^{-\lambda g(x; a, \underline{\theta})})]^{n-2}; & T < \log\{1 - e^{-\lambda g(x; a, \underline{\theta})}\}, \\ 0; & \text{otherwise.} \end{cases} \end{aligned} \quad (6)$$

Proof. Since T is complete and sufficient for the family of distributions $f(x; a, \alpha, \lambda, \underline{\theta})$, any function $H(T)$ of T satisfying $E[H(T)] = f(x; a, \alpha, \lambda, \underline{\theta})$ will be the UMVUE of $f(x; a, \alpha, \lambda, \underline{\theta})$. To this end, from (1) and Lemma 1,

$$\begin{aligned} & \int_{-\infty}^0 H(t) \frac{\alpha^n}{\Gamma(n)} (-t)^{n-1} e^{\alpha t} dt \\ &= \alpha \lambda g'(x; a, \underline{\theta}) e^{-\lambda g(x; a, \underline{\theta})} (1 - e^{-\lambda g(x; a, \underline{\theta})})^{\alpha-1}, \end{aligned} \quad (7)$$

or

$$\begin{aligned} & -\frac{\alpha^{n-1}}{\Gamma(n)} \int_{\log(1-e^{-\lambda g(x; a, \underline{\theta})})}^{\infty} H[-u + \log\{1 - e^{-\lambda g(x; a, \underline{\theta})}\}] \\ & \quad \times [-u + \log(1 - e^{-\lambda g(x; a, \underline{\theta})})]^{n-1} e^{-\alpha u} du \\ &= \lambda (1 - e^{-\lambda g(x; a, \underline{\theta})})^{-1} g'(x; a, \underline{\theta}) e^{-\lambda g(x; a, \underline{\theta})}. \end{aligned} \quad (8)$$

Equation (8) is satisfied if we choose

$$\begin{aligned} & H[-u + \log(1 - e^{-\lambda g(x; a, \underline{\theta})})] \\ &= \begin{cases} \left(\begin{aligned} & -(n-1) \lambda g'(x; a, \underline{\theta}) \\ & \times e^{-\lambda g(x; a, \underline{\theta})} (-u)^{n-2} \\ & \times (1 - e^{-\lambda g(x; a, \underline{\theta})}) \\ & \times [-u + \log(1 - e^{-\lambda g(x; a, \underline{\theta})})]^{n-1} \end{aligned} \right)^{-1}; & u > 0, \\ 0; & \text{otherwise,} \end{cases} \end{aligned} \quad (9)$$

and the lemma holds. □

Remark 4. We can write (1) as

$$\begin{aligned} f(x; a, \alpha, \lambda, \underline{\theta}) &= \lambda (1 - e^{-\lambda g(x; a, \underline{\theta})})^{-1} \\ & \quad \times g'(x; a, \underline{\theta}) e^{-\lambda g(x; a, \underline{\theta})} \alpha \\ & \quad \times \sum_{i=0}^{\infty} \frac{[\log(1 - e^{-\lambda g(x; a, \underline{\theta})})]^i \alpha^i}{i!}. \end{aligned} \quad (10)$$

Using Lemma 1 of Chaturvedi and Tomer [37] and Lemma 2, the UMVUE of $f(x; a, \alpha, \lambda, \underline{\theta})$ at a specified point “ x ” is

$$\begin{aligned} \hat{f}(x; a, \alpha, \lambda, \underline{\theta}) &= \lambda (1 - e^{-\lambda g(x; a, \underline{\theta})})^{-1} \\ & \quad \times g'(x; a, \underline{\theta}) e^{-\lambda g(x; a, \underline{\theta})} \\ & \quad \times \sum_{i=0}^{\infty} \frac{[\log(1 - e^{-\lambda g(x; a, \underline{\theta})})]^i}{i!} \hat{\alpha}^{i+1} \\ &= \lambda (1 - e^{-\lambda g(x; a, \underline{\theta})})^{-1} g'(x; a, \underline{\theta}) e^{-\lambda g(x; a, \underline{\theta})} \\ & \quad \times (-T)^{-1} \sum_{i=0}^{n-2} \frac{\Gamma(n-1)}{\Gamma(n-i-1)} (-T)^i \\ & \quad \times \frac{[\log(1 - e^{-\lambda g(x; a, \underline{\theta})})]^i}{i!}, \end{aligned} \quad (11)$$

or

$$\begin{aligned} \hat{f}(x; a, \alpha, \lambda, \underline{\theta}) &= \begin{cases} -(n-1) \lambda T^{-1} (1 - e^{-\lambda g(x; a, \underline{\theta})})^{-1} \\ \quad \times g'(x; a, \underline{\theta}) e^{-\lambda g(x; a, \underline{\theta})} \\ \quad \times [1 - T^{-1} \log(1 - e^{-\lambda g(x; a, \underline{\theta})})]^{n-2}; & T < \log\{1 - e^{-\lambda g(x; a, \underline{\theta})}\}, \\ 0; & \text{otherwise,} \end{cases} \end{aligned} \quad (12)$$

which coincides with Lemma 3. Thus, the UMVUE of the power of α can be used to derive the UMVUE of $\hat{f}(x; a, \alpha, \lambda, \underline{\theta})$ at a specified point.

In the following theorem, we obtain the UMVUE of $R(t)$.

Theorem 5. The UMVUE of $R(t)$ is given by

$$\hat{R}(t) = \begin{cases} 1 - [1 - T^{-1} \log\{1 - e^{-\lambda g(t; a, \underline{\theta})}\}]^{n-1}; \\ \quad T < \log(1 - e^{-\lambda g(t; a, \underline{\theta})}), \\ 0; & \text{otherwise.} \end{cases} \quad (13)$$

Proof. Let us consider the expected value of the integral $\int_t^{\infty} \hat{f}(x; a, \alpha, \lambda, \underline{\theta}) dx$ with respect to T ; that is,

$$\begin{aligned} & \int_0^{\infty} \left\{ \int_t^{\infty} \hat{f}(x; a, \alpha, \lambda, \underline{\theta}) dx \right\} h(t; \alpha, \lambda, \underline{\theta}) dt \\ &= \int_t^{\infty} [E_T \{\hat{f}(x; a, \alpha, \lambda, \underline{\theta})\}] dx \\ &= \int_t^{\infty} \{f(x; a, \alpha, \lambda, \underline{\theta})\} dx = R(t). \end{aligned} \quad (14)$$

We conclude from (14) that the UMVUE of $R(t)$ can be obtained simply integrating $\hat{f}(x; a, \alpha, \lambda, \underline{\theta})$ from t to ∞ . Thus,

$$\begin{aligned}\hat{R}(t) &= -(n-1)T^{-1} \int_t^{g^{-1}\{-\lambda^{-1} \log(1-e^T)\}} \lambda \left(1 - e^{-\lambda g(x; a, \underline{\theta})}\right)^{-1} g'(x; a, \underline{\theta}) e^{-\lambda g(x; a, \underline{\theta})} \\ &\quad \times \left[1 - T^{-1} \log \left(1 - e^{-\lambda g(x; a, \underline{\theta})}\right)\right]^{n-2} dx; \\ T < \log \left\{1 - e^{-\lambda g(x; a, \underline{\theta})}\right\} &= -(n-1) \int_{T^{-1} \log(1-e^{-\lambda g(t; a, \underline{\theta})})}^1 (1-y)^{n-2} dy,\end{aligned}\tag{15}$$

and the theorem follows. $\square$

Let X and Y be two independent rvs following the families of distributions $f_1(x; a_1, \alpha_1, \lambda_1, \underline{\theta}_1)$ and $f_2(y; a_2, \alpha_2, \lambda_2, \underline{\theta}_2)$, respectively. We assume that α_1 and α_2 are unknown but $a_1, a_2, \lambda_1, \lambda_2, \underline{\theta}_1$, and $\underline{\theta}_2$ are known. Let $X_1, X_2, \dots, X_n$ be a random sample of size n from

$f_1(x; a_1, \alpha_1, \lambda_1, \underline{\theta}_1)$ and let $Y_1, Y_2, \dots, Y_m$ be a random sample of size m from $f_2(y; a_2, \alpha_2, \lambda_2, \underline{\theta}_2)$. Let us denote by $T = \sum_{i=1}^n \log(1 - e^{-\lambda_1 g(x_i; a_1, \underline{\theta}_1)})$ and $S = \sum_{j=1}^m \log(1 - e^{-\lambda_2 h(y_j; a_2, \underline{\theta}_2)})$. In what follows, we obtain the UMVUE of “ P ”.

Theorem 6. The UMVUE of “ P ” is given by

$$\hat{P} = \begin{cases} 1 + (m-1) \int_1^0 \left[1 - T^{-1} \log \left\{1 - e^{-\lambda_1 g(h^{-1}(-(1/\lambda_2) \log(1-e^S)))}\right\}\right]^{n-1} (1-\nu)^{m-2} d\nu; \\ \quad h^{-1}\{-\lambda_2^{-1} \log(1-e^S)\} > g^{-1}\{-\lambda_1^{-1} \log(1-e^T)\}, \\ 1 + (m-1) \int_{\log\{1-S^{-1}e^{\lambda_2 h(g^{-1}(-(1/\lambda_1) \log(1-e^T)))}\}}^0 \left[1 - T^{-1} \log \left\{1 - e^{-\lambda_1 g(h^{-1}(-(1/\lambda_2) \log(1-e^S)))}\right\}\right]^{n-1} \\ \quad \times (1-\nu)^{m-2} d\nu; \\ \quad h^{-1}\{-\lambda_2^{-1} \log(1-e^S)\} < g^{-1}\{-\lambda_1^{-1} \log(1-e^T)\}. \end{cases}\tag{16}$$

Proof. From the arguments similar to those adopted in proving Theorem 5, it can be shown that the UMVUE of “ P ” is given by

$$\hat{P} = \int_{y=a_2}^{\infty} \int_{x=y}^{\infty} \hat{f}_1(x; a_1, \alpha_1, \lambda_1, \underline{\theta}_1) \hat{f}_2(y; a_2, \alpha_2, \lambda_2, \underline{\theta}_2) dx dy.\tag{17}$$

Thus, using Lemma 3,

$$\begin{aligned}\hat{P} &= - \int_{y=a_2}^{\infty} \left[\int_{x=y}^{\infty} (n-1) \lambda_1 \left(1 - e^{-\lambda_1 g(x; a_1, \underline{\theta}_1)}\right)^{-1} g'(x; a_1, \underline{\theta}_1) e^{-\lambda_1 g(x; a_1, \underline{\theta}_1)} \right. \\ &\quad \cdot \left. \left(1 - T^{-1} \log \left(1 - e^{-\lambda_1 g(x; a_1, \underline{\theta}_1)}\right)\right)^{n-2} dx \right] \cdot \hat{f}_2(y; a_2, \alpha_2, \lambda_2, \underline{\theta}_2) dy \\ &= 1 + (m-1) \lambda_2 S^{-1} \int_{y=\max\{h^{-1}\{-\lambda_2^{-1} \log(1-e^S)\}, g^{-1}\{-\lambda_1^{-1} \log(1-e^T)\}\}}^{\infty} \left\{1 - T^{-1} \log \left(1 - e^{-\lambda_1 g(y; a_1, \underline{\theta}_1)}\right)\right\}^{n-1} \\ &\quad \cdot \left(1 - e^{-\lambda_2 h(y; a_2, \underline{\theta}_2)}\right)^{-1} h'(y; a_2, \underline{\theta}_2) e^{-\lambda_2 h(y; a_2, \underline{\theta}_2)} \\ &\quad \cdot \left[1 - S^{-1} \log \left(1 - e^{-\lambda_2 h(y; a_2, \underline{\theta}_2)}\right)\right]^{m-2} dy.\end{aligned}\tag{18}$$

Let us first consider the case when $h^{-1}\{-\lambda_2^{-1} \log(1-e^S)\} > g^{-1}\{-\lambda_1^{-1} \log(1-e^T)\}$. In this case, from (18),

$$\hat{P} = 1 + (m-1) \int_1^0 \left[1 - T^{-1} \log \left\{1 - e^{-\lambda_1 g(h^{-1}(-(1/\lambda_2) \log(1-e^S)))}\right\}\right]^{n-1} (1-\nu)^{m-2} d\nu.\tag{19}$$

Now we consider the case when $h^{-1}\{-\lambda_2^{-1}\log(1 - e^S)\} < g^{-1}\{-\lambda_1^{-1}\log(1 - e^T)\}$. In this case, from (18),

$$\hat{P} = 1 + (m-1) \int_{\log\{1-S^{-1}e^{\lambda_2 h(g^{-1}(-(1/\lambda_1)\log(1-e^T)))}\}}^0 \left[1 - T^{-1} \log\left\{1 - e^{-\lambda_1 g(h^{-1}(-(1/\lambda_2)\log(1-e^S)))}\right\}\right]^{n-1} (1-v)^{m-2} dv. \quad (20)$$

The theorem now follows on combining (19) and (20). $\square$

Corollary 7. For the case when $a_1 = a_2 = a$, say, $\lambda_1 = \lambda_2 = \lambda$, say, $\underline{\theta}_1 = \underline{\theta}_2 = \underline{\theta}$, say, and $g(x; a, \underline{\theta}) = h(y; a, \underline{\theta})$,

$$\hat{P} = \begin{cases} 1 - \sum_{i=0}^{n-1} (-1)^i \frac{(n-1)!(m-1)!}{(n-i-1)!(m+i-1)!} \left(\frac{S}{T}\right)^i; & T > S, \\ 1 - \sum_{i=0}^{m-2} (-1)^i \frac{(n-1)!(m-1)!}{(n+i)!(m-i-2)!} \left(\frac{T}{S}\right)^{i+1}, & T < S. \end{cases} \quad (21)$$

Remarks 1.

- (i) In the literature, the researchers have dealt with the estimation of $R(t)$ and “ P ” separately. If we look at the proofs of Theorem 5 and Theorem 6, we observe that the UMVUE of the sampled pdf is used to obtain the UMVUES of $R(t)$ and “ P .” Thus, we have established interrelationship between the two estimation problems.
- (ii) In the literature, the researchers have derived the UMVUES of “ P ” for the case when X and Y follow the same distribution (may be with different parameters). We have obtained UMVUES of “ P ” for all the three situations, when X and Y follow the same distribution having all the parameters same other than α 's, when X and Y have the same distribution with different parameters and when X and Y have different distributions.
- (iii) It follows from Lemma 2 that $\text{Var}(\hat{\alpha}) = \alpha^2/(n-1) \rightarrow 0$ as $n \rightarrow \infty$. Thus, $\hat{\alpha}$ is a consistent estimator of α . Since $\hat{f}(x; a, \alpha, \lambda, \underline{\theta})$, $\hat{R}(t)$, and $\hat{P}$ are continuous functions of consistent estimators, they are also consistent estimators of $f(x; a, \alpha, \lambda, \underline{\theta})$, $R(t)$, and “ P ” respectively.

4. MLES of $R(t)$ and “ P ” When All the Parameters Are Unknown

Looking at the distributions belonging to the family (1), we notice that, except for exponentiated Pareto and generalized exponentiated Pareto distributions [(V) and (X)], $a = 0$; that is, “ a ” is known. For these two distributions $\underline{\theta}$ contains “ a ”

Irrespective of the distribution, the MLE of a is $\tilde{a} = X_{(1)} = \min_{1 \leq i \leq n} X_i$. From (3), the log-likelihood function is

$$\begin{aligned} L(a, \alpha, \lambda, \underline{\theta} | \underline{x}) &= n \log \alpha + n \log \lambda + \sum_{i=1}^n \log g'(x_i; a, \underline{\theta}) \\ &\quad - \lambda \sum_{i=1}^n g(x_i; a, \underline{\theta}) + (\alpha - 1) \\ &\quad \times \sum_{i=1}^n \log(1 - e^{-\lambda g(x_i; a, \underline{\theta})}). \end{aligned} \quad (22)$$

First of all, we replace “ a ” by $X_{(1)}$ in (22); then we differentiate (22) with respect to all the unknown parameters and equate these differential equations to zero. The MLES of unknown parameters are obtained on solving these equations simultaneously. Let $\tilde{\alpha}$, $\tilde{\lambda}$, and $\tilde{\underline{\theta}}$ be the maximum likelihood estimators of α , λ , and $\underline{\theta}$, respectively.

The following lemma provides The MLE of $f(x; a, \alpha, \lambda, \underline{\theta})$ at a specified point “ x ”.

Lemma 8. The MLE of $f(x; a, \alpha, \lambda, \underline{\theta})$ at a specified point “ x ” is

$$\tilde{f}(x; a, \alpha, \lambda, \underline{\theta}) = \tilde{\alpha} \tilde{\lambda} g'(x; \tilde{a}, \tilde{\underline{\theta}}) e^{-\tilde{\lambda} g(x; \tilde{a}, \tilde{\underline{\theta}})} \left(1 - e^{-\tilde{\lambda} g(x; \tilde{a}, \tilde{\underline{\theta}})}\right)^{\tilde{\alpha}-1}. \quad (23)$$

Proof. The proof follows from (1) and the one-to-one property of the MLE. $\square$

In the following theorem, we derive the MLE of $R(t)$.

Theorem 9. The MLE of $R(t)$ is given by

$$\tilde{R}(t) = 1 - \left(1 - e^{-\tilde{\lambda} g(t; \tilde{a}, \tilde{\underline{\theta}})}\right)^{\tilde{\alpha}}. \quad (24)$$

Proof. Using Lemma 8 and invariance property of the MLES,

$$\begin{aligned} \tilde{R}(t) &= \int_t^\infty \tilde{f}(x; a, \alpha, \lambda, \underline{\theta}) dx \\ &= \tilde{\alpha} \tilde{\lambda} \int_t^\infty g'(x; \tilde{a}, \tilde{\underline{\theta}}) e^{-\tilde{\lambda} g(x; \tilde{a}, \tilde{\underline{\theta}})} \left(1 - e^{-\tilde{\lambda} g(x; \tilde{a}, \tilde{\underline{\theta}})}\right)^{\tilde{\alpha}-1} dx \\ &= \tilde{\alpha} \int_{(1-e^{-\tilde{\lambda} g(t; \tilde{a}, \tilde{\underline{\theta}})})}^1 (1-u)^{\tilde{\alpha}-1} du, \end{aligned} \quad (25)$$

and the theorem follows. $\square$

In the following theorem, we obtain the maximum likelihood estimator of “ P ”

Theorem 10. The MLE of “P” is given by

$$\bar{P} = \tilde{\alpha}_1 \int_0^1 \left[1 - \exp \left\{ -\lambda_2 h \left(g^{-1} \left\{ -\lambda_1^{-1} \log(1-u) \right\} \right) \right\} \right]^{\tilde{\alpha}_2} \times u^{\tilde{\alpha}_1 - 1} du. \quad (26)$$

Proof. Using Lemma 8 and the one-to-one property of the MLES

$$\begin{aligned} \bar{P} &= \int_{y=y_{(1)}}^{\infty} \int_{x=y}^{\infty} \tilde{f}_1(x; a_1, \alpha_1, \lambda_1, \underline{\theta}_1) \\ &\quad \times \tilde{f}_2(y; a_2, \alpha_2, \lambda_2, \underline{\theta}_2) dx dy \\ &= \int_{y=y_{(1)}}^{\infty} [1 - \tilde{R}_2(x)] \tilde{f}_1(x; a_1, \alpha_1, \lambda_1, \underline{\theta}_1) dx \\ &= \int_{y=y_{(1)}}^{\infty} [1 - \exp \{-\tilde{\lambda}_2 h(x; \tilde{a}_2, \tilde{\theta}_2)\}]^{\tilde{\alpha}_2} \\ &\quad \times \tilde{\alpha}_1 \tilde{\lambda}_1 g'(x; \tilde{a}_1, \tilde{\theta}_1) e^{-\tilde{\lambda}_1 g'(x; \tilde{a}_1, \tilde{\theta}_1)} \\ &\quad \times \left(1 - e^{-\tilde{\lambda}_1 g'(x; \tilde{a}_1, \tilde{\theta}_1)} \right)^{\tilde{\alpha}_1 - 1} dx, \end{aligned} \quad (27)$$

and the result follows. $\square$

Corollary 11. For the case when $a_1 = a_2 = a$, say, $\lambda_1 = \lambda_2 = \lambda$, say, $\underline{\theta}_1 = \underline{\theta}_2 = \underline{\theta}$, say, and $g(x; a, \underline{\theta}) = h(y; a, \underline{\theta})$,

$$\bar{P} = \frac{\tilde{\alpha}_1}{\tilde{\alpha}_1 + \tilde{\alpha}_2}. \quad (28)$$

Remarks 2.

- (i) All the comments made under Remarks 1 for UMVUES are tenable for MLES.
- (ii) In the present approaches of obtaining UMVUES and MLES, one does not need the expressions of $R(t)$ and “P”.
- (iii) Since the UMVUES and MLES of powers of α are obtained under the same conditions, we compare their performances. For $q = -1$ the UMVUE and MLE of α are, respectively, $\hat{\alpha} = (n-1)(-T)^{-1}$ and $\tilde{\alpha} = (n)(-T)^{-1}$. For these estimators

$$V(\hat{\alpha}) = \frac{\alpha^2}{(n-2)}, \quad V(\tilde{\alpha}) = \frac{n^2}{(n-1)^2(n-2)}. \quad (29)$$

Hence,

$$V(\tilde{\alpha}) - V(\hat{\alpha}) = \frac{(2n-1)}{(n-1)(n-2)} \alpha^2 > 0. \quad (30)$$

Thus, the UMVUE of α is more efficient than its MLE. Similarly, we can compare the performances of these estimators for other powers of α .

FIGURE 1: Cures of $\hat{f}(x; \alpha, \lambda)$ and $f(x; \alpha, \lambda)$.

5. Simulation Studies

In order to verify the consistency of estimators, we have drawn samples of sizes $n = 5, 20$ and 30 from (1) with $g(x; a, \underline{\theta}) = x$, $a = 0$, $\alpha = 3$, and $\lambda = 1$. In Figure 1, we have plotted $\hat{f}(x; \alpha, \lambda)$ corresponding to these samples. We conclude that as the sample size increases, the curves of $\hat{f}(x; \alpha, \lambda)$ come closer to the curve of $f(x; \alpha, \lambda)$, which is plotted in the same figure only for $n = 30$. For $n = 30$, the curves overlap. A similar pattern follows for the curves of $\tilde{f}(x; \alpha, \lambda)$.

For the case when α is unknown but other parameters are known, we have conducted simulation experiments using bootstrap resampling technique for sample sizes $n = 5, 10, 20$, and 50 . The samples are generated from (1), with $g(x; a, \underline{\theta}) = x$, $a = 0$, $\alpha = 3$, and $\lambda = 1$. For different values of t , we have computed $\hat{R}(t)$, $\tilde{R}(t)$, their corresponding bias, variance, 95% confidence length, and corresponding coverage percentage. All the computations are based on 500 bootstrap replications and results are reported in Table 1.

In order to estimate “P,” for the case when α_1 and α_2 are unknown but other parameters are known, we have conducted simulation experiments using bootstrap resampling technique for sample sizes $(n, m) = (5, 5), (10, 10), (15, 15), (25, 25)$, and $(30, 30)$. The samples are generated from (1), with $g(x; a_1, \underline{\theta}_1) = x$, $h(y; a_2, \underline{\theta}_2) = y$, $a_1 = a_2 = 0$, $\lambda_1 = \lambda_2 = 1$, $\alpha_1 = 2$, and $\alpha_2 = 0.5(0.25)1.25$. The computations are based on 500 bootstrap replications. We have computed $\hat{P}$, $\tilde{P}$, bias, variance, 95% confidence length, and corresponding coverage percentage. The results are presented in Table 2.

In order to demonstrate the application of the theory developed in Section 4, we consider the following example.

TABLE 1: Simulation results for $R(t)$.

t	$R(t)$	$n = 5$		$n = 10$		$n = 20$		$n = 50$	
		$\hat{R}(t)$	$\tilde{R}(t)$	$\hat{R}(t)$	$\tilde{R}(t)$	$\hat{R}(t)$	$\tilde{R}(t)$	$\hat{R}(t)$	$\tilde{R}(t)$
0.40	0.9642	0.9628	0.9429	0.9726	0.9638	0.9639	0.9598	0.9656	0.964
		-0.0014	-0.0213	0.0084	-4e - 04	-2e - 04	-0.0044	0.0014	-1e - 04
		0.001071326	0.001390097	0.0003646232	0.0003069262	0.0002067265	0.0002223408	0.0001467748	0.0001442984
		0.1211	0.1182	0.0647	0.0668	0.0555	0.0552	0.0477	0.0476
		86.6741	92.6193	91.428	93.112	93.5019	93.8349	94.0056	94.1243
0.50	0.9391	0.8576	0.8546	0.8789	0.8769	0.9282	0.9252	0.9399	0.9386
		-0.0815	-0.0845	-0.0601	-0.0622	-0.0109	-0.0139	8e - 04	-5e - 04
		0.009229477	0.008767263	0.007167494	0.006830551	0.001421963	0.001408017	0.0005898798	0.0005745804
		0.1912	0.1524	0.2262	0.2086	0.1412	0.1371	0.0958	0.0947
		93.9509	94.0809	94.1651	94.4641	94.2804	94.6018	94.5567	94.6561
0.80	0.8330	0.8314	0.8338	0.8355	0.8374	0.8408	0.8417	0.8319	0.8325
		-0.0016	7e - 04	0.0025	0.0044	0.0077	0.0087	-0.0011	-5e - 04
		0.004528085	0.002894221	0.002485577	0.002035995	0.002223323	0.002035827	0.001419582	0.001364689
		0.2552	0.2057	0.191	0.1725	0.1796	0.1711	0.1474	0.1446
		94.1964	94.3942	94.4812	94.5182	94.6424	94.6596	94.952	94.9534
1.00	0.7474	0.729	0.7499	0.7936	0.7994	0.7279	0.7338	0.7548	0.7568
		-0.0184	0.0025	0.0461	0.052	-0.0195	-0.0137	0.0074	0.0093
		0.008217478	0.005415503	0.005849085	0.005767851	0.004407818	0.003903546	0.002428885	0.002383391
		0.3395	0.2812	0.2359	0.2142	0.2505	0.2407	0.1915	0.1884
		94.0502	94.2394	94.5787	94.6276	94.9352	94.9729	95.0586	95.0578
1.50	0.5311	0.4442	0.4933	0.5094	0.532	0.5283	0.5392	0.5256	0.53
		-0.087	-0.0378	-0.0218	9e - 04	-0.0028	0.0081	-0.0055	-0.0011
		0.01260328	0.00618332	0.00520108	0.004460047	0.004379108	0.004317541	0.001922681	0.001875105
		0.2803	0.2719	8 0.2721	0.2648	0.26	0.2564	0.1747	0.1738
		94.2755	94.5729	94.464	94.653	94.7342	94.7791	95.0374	95.0517
2.00	0.3535	0.4971	0.5442	0.3349	0.3573	0.3429	0.354	0.3485	0.3529
		0.1435	0.1906	-0.0187	0.0037	-0.0106	5e - 04	-0.005	-6e - 04
		0.02667172	0.0416806	0.004913515	0.004757415	0.00239676	0.002330965	0.001298761	0.001284863
		0.3125	0.2928	0.2712	0.2765	0.1923	0.1943	0.1442	0.1448
		94.3591	94.7634	94.5513	94.7147	94.7592	94.8151	95.0453	95.0583
2.50	0.2266	0.2396	0.2813	0.2699	0.2906	0.2047	0.2131	0.226	0.2295
		0.013	0.0547	0.0433	0.064	-0.0219	-0.0135	-6e - 04	0.0029
		0.001733384	0.004863281	0.002956535	0.005265369	0.00116556	0.0009073532	0.0004148542	0.0004315035
		0.1576	0.1725	0.1312	0.1364	0.1045	0.1075	0.0809	0.0817
		94.0777	94.3206	94.2939	94.3724	94.7371	94.7634	95.0835	95.0883

Here, the first row indicates the estimate, the second row indicates the bias, the third row indicates variance, the fourth row indicates 95% bootstrap confidence length, and the fifth row indicates the coverage percentage.

Let the rv X follow exponentiated exponential distribution with pdf

$$f(x; \alpha, \lambda, \delta) = \alpha \lambda \delta x^{\delta-1} e^{-\lambda x^\delta} (1 - e^{-\lambda x^\delta})^{\alpha-1}; \quad x, \alpha, \lambda, \delta > 0, \quad (31)$$

where α, λ , and δ are unknown. Denoting, $L_1(\alpha, \lambda, \delta | x)$, the likelihood, the log-likelihood is given by

$$\begin{aligned} \log L_1(\alpha, \lambda, \delta x) &= n \log \alpha + n \log \lambda + n \log \delta \\ &+ (\delta - 1) \sum_{i=1}^n \log x_i - \lambda \sum_{i=1}^n x_i^\delta \\ &+ (\alpha - 1) \sum_{i=1}^n \log (1 - e^{-\lambda x_i^\delta}). \end{aligned} \quad (32)$$

Considering negative log-likelihood, then differentiating it with respect to all unknown parameters, and equating these differential coefficients to zero, from (32),

$$\begin{aligned} -\frac{n}{\alpha} - \sum_{i=1}^n \log (1 - e^{-\lambda x_i^\delta}) &= 0, \\ -\frac{n}{\lambda} + \sum_{i=1}^n x_i^\delta - (\alpha - 1) \sum_{i=1}^n \frac{x_i^\delta e^{-\lambda x_i^\delta}}{(1 - e^{-\lambda x_i^\delta})} &= 0, \\ -\frac{n}{\delta} - \sum_{i=1}^n \log x_i + \lambda \sum_{i=1}^n x_i^\delta \log x_i - (\delta - 1) \lambda \\ &\times \sum_{i=1}^n \frac{x_i^\delta e^{-\lambda x_i^\delta} \log x_i}{(1 - e^{-\lambda x_i^\delta})} = 0. \end{aligned} \quad (33)$$

TABLE 2: Simulation results for “P.”

(α_1, α_2)	(1, 0.50)		(1, 0.75)		(1, 1.00)		(1, 1.25)	
P	0.6667000		0.5714286		0.5000000		0.4444000	
(n, m)	$\tilde{P}$	$\hat{P}$	$\tilde{P}$	$\hat{P}$	$\tilde{P}$	$\hat{P}$	$\tilde{P}$	$\hat{P}$
(5, 5)	0.4160	0.4071	0.5196	0.5198	0.4937	0.4925	0.4503	0.4445
	−0.2507	−0.2595	−0.0518	−0.0516	−0.0063	−0.0075	0.0058	1e − 04
	0.08305567	0.09161194	0.04060813	0.04711423	0.01738566	0.02128741	0.03447488	0.04041304
	0.5225	0.5686	0.6963	0.7375	0.4996	0.5469	0.7179	0.7577
	92.2467	91.7696	92.6155	91.9685	93.9423	93.6839	93.2283	92.5227
(10, 10)	0.4219	0.4179	0.5324	0.5340	0.5028	0.5029	0.4477	0.4451
	−0.2448	−0.2487	−0.0391	−0.0374	0.0028	0.0029	0.0033	6e − 04
	0.06910139	0.07194417	0.01062704	0.01145625	0.01023848	0.01128198	0.01168588	0.01284482
	0.3560	0.3727	0.3524	0.3698	0.3863	0.4050	0.4024	0.4213
	93.4513	93.3550	93.5220	93.4724	94.1638	94.1400	93.2875	93.2019
(15, 15)	0.4507	0.4491	0.5601	0.5620	0.5015	0.5016	0.4499	0.4481
	−0.2159	−0.2176	−0.0113	−0.0095	0.0015	0.0016	0.0054	0.0037
	0.05200412	0.05309595	0.01157199	0.0122409	0.00717329	0.007664479	0.002471958	0.002628193
	0.2739	0.2826	0.4016	0.4130	0.3234	0.3340	0.1911	0.1976
	93.2657	93.2813	93.8776	93.8404	94.2206	94.1996	94.6752	94.6696
(25, 25)	0.4676	0.4670	0.5625	0.5637	0.4994	0.4994	0.4466	0.4455
	−0.1990	−0.1997	−0.0090	−0.0077	−6e − 04	−6e − 04	0.0021	0.001
	0.04415965	0.0446212	0.002974049	0.003065954	0.004487199	0.004669894	0.004399884	0.004568917
	0.2529	0.2578	0.2048	0.2088	0.2579	0.2630	0.266	0.271
	93.8839	93.87700	94.2551	94.2501	94.3152	94.3066	95.5079	95.4975
(30, 30)	0.4869	0.4867	0.5740	0.5753	0.4999	0.4999	0.447	0.4461
	−0.1798	−0.1800	0.0026	0.0038	−1e − 04	−1e − 04	0.0025	0.0017
	0.03621819	0.03642065	0.003972225	0.004105375	0.003633788	0.003756843	0.00493606	0.005091428
	0.2470	0.2511	0.243	0.2468	0.2391	0.243	0.2797	0.2841
	95.2267	95.2211	94.5145	94.5023	95.1531	95.1467	95.3624	95.3519

Here, the first row indicates the estimate, the second row indicates the bias, the third row indicates variance, the fourth row indicates 95% bootstrap confidence length, and the fifth row indicates the coverage percentage.

The following sample of size 50 is generated from (31), for $\alpha = 3.5$, $\lambda = 1$, and $\delta = 1$ as follows.

Sample 1. 1.4904, 0.5808, 2.2326, 0.9756, 2.3062, 1.4692, 1.0433, 2.9527, 0.9241, 2.5901, 1.2021, 1.3022, 3.6532, 0.8146, 1.7083, 0.3126, 2.4925, 0.9462, 0.4557, 2.0120, 1.6554, 1.1651, 0.9703, 1.0305, 0.4618, 0.7465, 1.9249, 1.5763, 1.9127, 0.9742, 2.7401, 1.3686, 1.8546, 2.3296, 1.4526, 1.1719, 4.4671, 1.1218, 1.1901, 0.8248, 3.0091, 0.9511, 1.3308, 2.2145, 2.0216, 1.6740, 1.5620, 2.0339, 0.6882, 3.1335.

Assuming that the data represents life spans of items in hours, $R(0.35) = 0.986$, and solving (33) simultaneously, we get $\tilde{\alpha} = 3.711821$, $\tilde{\lambda} = 1.170381$, $\tilde{\delta} = 1.098677$, and $\tilde{R}(0.35) = 0.9872$.

In order to obtain the maximum likelihood estimator of “P,” we have generated one more sample of size 50 from (31), for $\alpha = 3$, $\lambda = 1$, and $\delta = 1$ as follows.

Sample 2. 1.2271, 0.9467, 1.9800, 0.6613, 0.4526, 1.5701, 1.0414, 1.9008, 1.2378, 3.7137, 1.4012, 0.5640, 1.6704, 2.0615, 1.4360, 1.6696, 0.7443, 0.9568, 1.6306, 0.2327, 1.0024, 0.8178, 0.7704, 2.1587, 0.8460, 2.3833, 1.0702, 1.1584, 3.1261, 3.7417, 3.0401, 2.4465, 0.6874, 3.2289, 2.4881,

1.6636, 0.4313, 2.4856, 1.2428, 1.5750, 1.1564, 1.4370, 3.1483, 1.1472, 1.4889, 0.8209, 0.6693, 1.2684, 3.2650, 3.2285.

For this sample, we have $\tilde{\alpha} = 2.932394$, $\tilde{\lambda} = 1.044041$, and $\tilde{\delta} = 1.106363$. Using these results (obtained from Samples 1 and 2) and Theorem 10, we get $P = 0.5384615$ and $\tilde{P} = 0.5586546$.

6. Conclusions

We propose a class of distributions, which covers as many as ten exponentiated distributions as special cases. The problems of estimating $R(t)$ and “P” are considered. UMVUES and MLES are derived. A comparative study of the two methods of estimation is done. The estimators of “P” are derived, which cover the cases when X and Y may follow the same, as well as, different distributions. By estimating the sampled pdf to obtain the estimators of $R(t)$ and “P,” an interrelationship between the two estimation problems is established. Simulation study is performed, and a real-data example is presented.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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