Research Article

On $k$-Gamma and $k$-Beta Distributions and Moment Generating Functions

Gauhar Rahman, Shahid Mubeen, Abdur Rehman, and Mammona Naz

Department of Mathematics, University of Sargodha, Sargodha 40000, Pakistan

Correspondence should be addressed to Gauhar Rahman; gauhar55uom@gmail.com

Received 10 February 2014; Revised 29 June 2014; Accepted 4 July 2014; Published 15 July 2014

Academic Editor: Chin-Shang Li

Copyright © 2014 Gauhar Rahman et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The main objective of the present paper is to define $k$-gamma and $k$-beta distributions and moments generating function for the said distributions in terms of a new parameter $k > 0$. Also, the authors prove some properties of these newly defined distributions.

1. Basic Definitions

In this section we give some definitions which provide a base for our main results. The definitions (1.1–1.3) are given in [1] while (1.4–1.6) are introduced in [2]. Also, we have taken some statistics related definitions (1.7–1.11) from [3–5].

1.1. Pochhammer Symbol. The factorial function is denoted and defined by

$$(a)_n = \begin{cases} a(a+1)(a+2)\cdots(a+n-1); & \text{for } n \geq 1, \ a \neq 0, \\ 1; & \text{if } n = 0. \end{cases}$$

From the relation (3), using integration by parts, we can easily show that

$$\Gamma(z+1) = z\Gamma(z).$$

(4)

The relation between Pochhammer symbol and gamma function is given by

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}.$$  

(5)

1.3. Beta Function. The beta function of two variables is defined as

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} \, dt, \quad \text{Re}(x), \text{Re}(y) > 0$$

and, in terms of gamma function, it is written as

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$  

(7)

1.4. Pochhammer $k$-Symbol. For $k > 0$, the Pochhammer $k$-symbol is denoted and defined by

$$(a)_{n,k} = \begin{cases} a(a+k)(a+2k)\cdots(a+(n-1)k); & \text{for } n \geq 1, \ a \neq 0, \\ 1; & \text{if } n = 0. \end{cases}$$

(8)
1.5. $k$-Gamma Function. For $k > 0$ and $z \in \mathbb{C}$, the $k$-gamma function is defined as

$$
\Gamma_k(z) = \lim_{n \to \infty} \frac{n^k k^k (nk)^{z/k-1}}{(z)_n k^k}
$$

and the integral representation of $k$-gamma function is

$$
\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-t/k} dt.
$$

1.6. $k$-Beta Function. For $\Re(x), \Re(y) > 0$, the $k$-beta function of two variables is defined by

$$
B_k(x, y) = \frac{\Gamma_k(x) \Gamma_k(y)}{\Gamma_k(x + y)}.
$$

and, in terms of $k$-gamma function, $k$-beta function is defined as

$$
B_k(x, y) = \frac{\Gamma_k(x + k) \Gamma_k(y + k)}{\Gamma_k(x + y + k)}.
$$

Also, the researchers [6–10] have worked on the generalized $k$-gamma and $k$-beta functions and discussed the following properties:

$$
\Gamma_k(x + k) = x \Gamma_k(x),
$$

$$
(x)_{n,k} = \frac{\Gamma_k(x + nk)}{\Gamma_k(x)}.
$$

Using the above relations, we see that, for $x, y > 0$ and $k > 0$, the following properties of $k$-beta function are satisfied by authors (see [6, 7, 11]):

$$
\beta_k(x + y) = \frac{x}{x + y} \beta_k(x, y),
$$

$$
\beta_k(x, y + k) = \frac{y}{x + y} \beta_k(x, y),
$$

$$
\beta_k(xk, yk) = \frac{1}{k} \beta_k(x, y),
$$

$$
\beta_k(x, k) = \frac{1}{x}, \quad \beta_k(k, y) = \frac{1}{y}.
$$

Note that when $k \to 1$, $\beta_k(x, y) \to \beta(x, y)$.

For more details about the theory of $k$-special functions like $k$-gamma function, $k$-beta function, $k$-hypergeometric functions, solutions of $k$-hypergeometric differential equations, contagious functions relations, inequalities with applications and integral representations with applications involving $k$-gamma and $k$-beta functions and so forth. (See [12–17].)

1.7. Probability Distribution and Expected Values. In a random experiment with $n$ outcomes, suppose a variable $X$ assumes the values $x_1, x_2, x_3, \ldots, x_n$ with corresponding probabilities $P_1, P_2, P_3, \ldots, P_n$; then this collection is called probability distribution and $\Sigma P_i = 1$ (in case of discrete distributions). Also, if $f(x)$ is a continuous probability distribution function defined on an interval $[a, b]$, then $\int_a^b f(x)dx = 1$.

In statistics, there are three types of moments which are (i) moments about any point $x = a$, (ii) moments about $x = 0$, and (iii) moments about mean position of the given data. Also, expected value of the variate is defined as the first moment of the probability distribution about $x = 0$ and the $r$th moment about mean of the probability distribution is defined as $E(x_i - \bar{x})^r$ where $\bar{x}$ is the mean of the distribution.

Also, $E(x)$ shows the expected value of the variate $x$ and is defined as the first moment of the probability distribution about $x = 0$; that is,

$$
\mu'_1 = E(x) = \int_a^b xf(x)dx.
$$

1.8. Gamma Distribution. A continuous random variable $Z$ is said to have a gamma distribution with parameter $m > 0$, if its probability distribution function is defined by

$$
f(z) = \begin{cases} 
\frac{1}{\Gamma(m)} z^{m-1} e^{-z}, & 0 \leq z < \infty, \\
0, & \text{elsewhere}
\end{cases}
$$

and its distribution function $F(z)$ is defined by

$$
F(z) = \begin{cases} 
\int_0^z \frac{1}{\Gamma(m)} z^{m-1} e^{-z} dz, & z \geq 0, \\
0, & z < 0,
\end{cases}
$$

which is also called the incomplete gamma function.

1.9. Moment Generating Function of Gamma Distribution. The moment generating function of $Z$ is defined by

$$
M_0(t) = E(e^{tZ}) = \int_0^\infty e^{tZ} f(z) dz
$$

$$
= \int_0^\infty \frac{1}{\Gamma(m)} z^{m-1} e^{-z(t-1)} dz.
$$

1.10. Beta Distribution of the First Kind. A continuous random variable $Z$ is said to have a beta distribution with two parameters $m$ and $n$, if its probability distribution function is defined by

$$
f(z) = \begin{cases} 
\frac{1}{B(m, n)} z^{m-1} (1-z)^{n-1}, & 0 \leq z \leq 1; \ m, n > 0 \\
0, & \text{elsewhere}
\end{cases}
$$

(24)
This distribution is known as a beta distribution of the first kind and a beta variable of the first kind is referred to as $\beta_1(m,n)$. Its distribution function $F(z)$ is given by

$$F(z) = \begin{cases} \frac{1}{B(m,n)} (1 - z)^{m-1} \frac{1}{z^{m+n}}, & 0 \leq z \leq 1; m,n > 0, \\ 0, & z > 1. \end{cases}$$

(25)

1.1. Beta Distribution of the Second Kind. A continuous random variable $Z$ is said to have a beta distribution of the second kind with parameters $m$ and $n$, if its probability distribution function is defined by

$$f(z) = \frac{1}{\beta(m,n) (1 + z)^{m+n}}, \quad 0 < z < \infty; \quad m, n > 0,$$

and its probability distribution function is given by

$$F(z) = \int_0^z \frac{1}{\beta(m,n) (1 + x)^{m+n}} dx, \quad 0 < z < \infty; \quad m, n > 0.$$  

(26)

(27)

2. Main Results: $k$-Gamma and $k$-Beta Distributions

In this section, we define gamma and beta distributions in terms of a new parameter $k > 0$ and discuss some properties of these distributions in terms of $k$.

**Definition 1.** Let $Z$ be a continuous random variable; then it is said to have a $k$-gamma distribution with parameters $m > 0$ and $k > 0$, if its probability density function is defined by

$$f_k(z) = \begin{cases} \frac{1}{\Gamma_k(m)} z^{m-1} e^{-z^{1/k}}, & 0 < z < \infty, k > 0, \\ 0, & \text{otherwise} \end{cases}$$

(28)

and its distribution function $F_k(z)$ is defined by

$$F_k(z) = \begin{cases} \frac{1}{\Gamma_k(m)} \int_0^z z^{m-1} e^{-t^{1/k}} dt, & z > 0, \\ 0, & z < 0. \end{cases}$$

(29)

**Proposition 2.** The newly defined $\Gamma_k(m)$ distribution satisfies the following properties.

(i) The $k$-gamma distribution is the probability distribution that is area under the curve is unity.

(ii) The mean of $k$-gamma distribution is equal to a parameter $m.$

(iii) The variance of $k$-gamma distribution is equal to the product of two parameters $mk.$

Proof of (i). Using the definition of $k$-gamma distribution along with the relation (10), we have

$$\int_0^\infty f_k(z) dz = \int_0^\infty \frac{1}{\Gamma_k(m)} z^{m-1} e^{-z^{1/k}} dz = \frac{\Gamma_k(m)}{\Gamma_k(m)} = 1.$$  

(30)

Proof of (ii). As mean of a distribution is the expected value of the variate, so the mean of the $k$-gamma distribution is given by

$$E_k(z) = \frac{1}{\Gamma_k(m)} \int_0^\infty z \cdot z^{m-1} e^{-z^{1/k}} dz.$$  

Using the definition of $k$-gamma function and the relation (13), we have

$$E_k(z) = \frac{1}{\Gamma_k(m)} \int_0^\infty z \cdot z^{m-1} e^{-z^{1/k}} dz = \frac{\Gamma_k(m + k)}{\Gamma_k(m)} = m \Gamma_k(m).$$  

(32)

Proof of (iii). As variance of a distribution is equal to $E(x^2) - (E(x))^2$, so the variance of $k$-gamma distribution is calculated as

$$Var_k(Z) = E_k(Z^2) - (E_k(Z))^2.$$  

(33)

Now, we have to find $E_k(Z^2)$, which is given by

$$E_k(Z^2) = \frac{1}{\Gamma_k(m)} \int_0^\infty z^2 \cdot z^{m-1} e^{-z^{1/k}} dz = \frac{\Gamma_k(m + 2k)}{\Gamma_k(m)} = (m + k) \Gamma_k(m).$$  

(34)

Thus we obtain the variance of $k$-gamma distribution as

$$\sigma_k^2 = m(m + k) - m^2 = mk,$$  

(35)

where $\sigma_k^2$ is the notation of variance present in the literature.

2.1. $k$-Beta Distribution of First Kind. Let $Z$ be a continuous random variable; then it is said to have a $k$-beta distribution of the first kind with two parameters $m$ and $n$, if its probability distribution function is defined by

$$f_k(z) = \begin{cases} \frac{1}{k \Gamma_k(m,n)} z^{m-1}(1 - z)^{n-1}, & 0 < z < 1; m, n, k > 0, \\ 0, & \text{otherwise}. \end{cases}$$

(36)
In the above distribution, the beta variable of the first kind is referred to as $\beta_{1,k}(m,n)$ and its distribution function $F_k(z)$ is given by

$$F_k(z) = \begin{cases} 0, & z < 0, \\ \int_0^z \frac{1}{kB_k(m,n)} z^{m/k-1}(1-z)^{n/k-1}dz, & 0 \leq z \leq 1; \ m, n > 0, \\ 0, & z > 1. \end{cases}$$  \hfill (37)

**Proposition 3.** The $k$-beta distribution $\beta_{1,k}(m,n)$ satisfies the following basic properties.

(i) $k$-beta distribution is the probability distribution that is the area of $\beta_{1,k}(m,n)$ under a curve $f_k(z)$ is unity.

(ii) The mean of this distribution is $m/(m+n)$.

(iii) The variance of $\beta_{1,k}(m,n)$ is $mnk/((m+n)^2(m+n+k))$.

**Proof of (i).** By using the above definition of $k$-beta distribution, we have

$$\int_0^z F_k(z)dz = \frac{1}{kB_k(m,n)} \int_0^z z^{m/k-1}(1-z)^{n/k-1}dz,$$

$$= \frac{B_k(m+n)}{B_k(m,n)} = 1.$$  \hfill (38)

By the relation (11), we get

$$\int_0^z F_k(z)dz = \int_0^z \frac{1}{kB_k(m,n)} z^{m/k-1}(1-z)^{n/k-1}dz$$

$$= \frac{B_k(m+n)}{B_k(m,n)} = 1.$$  \hfill (39)

**Proof of (ii).** The mean of the distribution, $\mu'_{1,k}$, is given by

$$\mu'_{1,k} = E_k(Z) = \int_0^\infty zF_k(z)dz$$

$$= \int_0^z \frac{1}{kB_k(m,n)} z \cdot z^{m/k-1}(1-z)^{n/k-1}dz,$$

$$= \frac{\Gamma_k(m+k)\Gamma_k(n)\Gamma_k(m+n)}{\Gamma_k(m)\Gamma_k(n)\Gamma_k(m+n+k)} = \frac{m}{m+n},$$  \hfill (40)

and

**Proof of (iii).** The variance of $\beta_{1,k}(m,n)$ is given by

$$\sigma_k^2 = (\text{Var})_k = E_k(\text{Var}(Z)) - (E_k(Z))^2,$$

$$E_k(\text{Var}(Z)) = \int_0^1 \frac{1}{kB_k(m,n)} z^{m/k-1}(1-z)^{n/k-1}dz$$

$$= \frac{B_k(m+2k,n)}{B_k(m,n)}$$

$$= \frac{\Gamma_k(m+k)\Gamma_k(n)\Gamma_k(m+n+2k)}{\Gamma_k(m)\Gamma_k(n)\Gamma_k(m+n+2k)}.$$  \hfill (43)

Thus substituting the values of $E_k(Z^2)$ and $E_k(Z)$ in (42) along with some algebraic calculations we have the desired result. \hfill \Box

2.2. $k$-Beta Distribution of the Second Kind. A continuous random variable $Z$ is said to have a $k$-beta distribution of the second kind with parameters $m$ and $n$, if its probability distribution function is defined by

$$f_k(z) = \begin{cases} \frac{1}{k\beta_k(m,n)(1+z)^{(m+n)/k}}, & 0 \leq z < \infty; \ m, n, k > 0, \\ 0, & \text{otherwise.} \end{cases}$$  \hfill (44)

**Theorem 4.** The $k$-beta function of the second kind represents a probability distribution function that is

$$\int_0^\infty f_k(z)dz = 1.$$  \hfill (45)

**Proof.** We observe that

$$\int_0^\infty f_k(z)dz = \int_0^\infty \frac{1}{k\beta_k(m,n)(1+z)^{(m+n)/k}}dz.$$  \hfill (46)

Let $1+z = 1/y$, so that $dz = -dy/y^2$; thus by using the relation (11), the above equation gives

$$= \frac{1}{\beta_k(m,n)} \frac{1}{k} \int_0^1 y^{n/k-1}(1-y)^{m/k-1}dy = \frac{\beta_k(m+n)}{\beta_k(m,n)} = 1.$$  \hfill (47)

\hfill \Box

3. Moment Generating Function of $k$-Gamma Distribution

In this section, we derive the moment generating function of continuous random variable $Z$ of newly defined $k$-gamma
distribution in terms of a new parameter $k > 0$, which is illustrated as

$$M_{0,k}(t) = E_k\left(e^{zt}\right) = \int_0^\infty \frac{1}{t^k} e^{-t^k} z^{m-1} e^{-z/k} dz = \frac{1}{t} \int_0^\infty \frac{z^{m-1} e^{-z/k}}{(1-t^{1/k})} dz.$$  

(48)

Let $u = z(1-kt)^{1/k}$, so that $z = u/(1-kt)^{1/k}$ and $dz = du/(1-kt)^{1/k}$. Then substituting these values in (48), we obtain

$$M_{0,k}(t) = \frac{1}{(1-kt)^{(m-1)/k} \Gamma_k(m)} \int_0^\infty \frac{u^{m-1} e^{-u/k}}{(1-kt)^{1/k}} du = \frac{\Gamma_k(m)}{(1-kt)^{m/k} \Gamma_k(m)} = (1-kt)^{-m/k}, \quad |kt| < 1.$$  

(49)

Now differentiating $r$ times $M_{0,k}(t)$ with respect to $t$ and putting $t = 0$, we get

$$\mu'_{r,k} = m(m+k)(m+2k)\cdots(m+(r-1)k).$$  

(50)

Thus when $r = 1$, we obtain $\mu'_{1,k} = m$, when $r = 2$, $\mu'_{2,k} = m(m+k)$, and hence $\mu_{2,k} = \mu'_{2,k} = mk = \text{variance of the k-gamma distribution proved in Proposition 2}.$

3.1. Higher Moment in terms of $k$. The $r$th moment in terms of $k$ is given by

$$\mu'_{r,k} = E(Z^r) = \frac{1}{kB_k(m,n)} \int_0^1 z^r \cdot z^{m/k-1} (1-z)^{n/k-1} \, dz.$$  

(51)

Changing the variables as $z = (1-y)/y \Rightarrow dz = (-1/y^2)dy$, the above equation becomes

$$= \frac{1}{kB_k(m,n)} \int_0^1 y^{n/r-k-1} (1-y)^{m/k+r-1} \, dy.$$  

(54)

Replacing $(1-y)$ by $t$, we have

$$\mu'_{r,k} = \frac{1}{B_k(m,n)} \frac{1}{k} \int_0^1 t^{m/k+r-1} (1-t)^{n/k-r-1} \, dt$$

(55)

Thus when $r = 1$, then $k$-gamma distribution and $k$-beta distribution tend to classical gamma and beta distribution.

(ii) The authors also conclude that the area of $k$-gamma distribution and $k$-beta distribution for each positive value of $k$ is one and their mean is equal to a parameter $m$ and $m/(m+n)$, respectively. The variance of $k$-gamma distribution for each positive value of $k$ is equal to $k$ times of the parameter $m$. In this case if $k = 1$, then it will be equal to variance of gamma distribution. The variance of $k$-beta distribution for each positive value of $k$ is also defined.

(iii) In this paper the authors introduced moments generating function and higher moments in terms of a new parameter $k > 0$.

4. Conclusion

In this paper the authors conclude that we have the following.

(i) If $k$ tends to 1, then $k$-gamma distribution and $k$-beta distribution tend to classical gamma and beta distribution.

(ii) The authors also conclude that the area of $k$-gamma distribution and $k$-beta distribution for each positive value of $k$ is one and their mean is equal to a parameter $m$ and $m/(m+n)$, respectively. The variance of $k$-gamma distribution for each positive value of $k$ is equal to $k$ times of the parameter $m$. In this case if $k = 1$, then it will be equal to variance of gamma distribution. The variance of $k$-beta distribution for each positive value of $k$ is also defined.

(iii) In this paper the authors introduced moments generating function and higher moments in terms of a new parameter $k > 0$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The authors would like to express profound gratitude to referees for deeper review of this paper and the referee’s useful suggestions that led to an improved presentation of the paper.

References


