Research Article

Statistical Tests for the Reciprocal of a Normal Mean with a Known Coefficient of Variation

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An asymptotic test and an approximate test for the reciprocal of a normal mean with a known coefficient of variation were proposed in this paper. The asymptotic test was based on the expectation and variance of the estimator of the reciprocal of a normal mean. The approximate test used the approximate expectation and variance of the estimator by Taylor series expansion. A Monte Carlo simulation study was conducted to compare the performance of the two statistical tests. Simulation results showed that the two proposed tests performed well in terms of empirical type I errors and power. Nevertheless, the approximate test was easier to compute than the asymptotic test.

1. Introduction

The reciprocal of a normal mean has been the subject of research in the areas of nuclear physics, agriculture, and economics. For example, Lamanna et al. [1] studied charged particle momentum, $p = \mu^{-1}$, where μ is the track curvature of a particle. The reciprocal of a normal mean is given by

$$\theta = \mu^{-1}, \quad (1)$$

where μ is the population mean. A variety of researchers have studied the reciprocal of a normal mean. For instance, Zaman [2] discussed the estimators without moments in the case of the reciprocal of a normal mean. The maximum likelihood estimate of the reciprocal of a normal mean with a class of zero-one loss functions was proposed by Zaman [3]. Withers and Nadarajah [4] presented a theorem to construct the point estimators for the inverse powers of a normal mean.

Suppose we have prior information for the coefficient of variation; $\tau = \sigma/\mu$, where σ is the standard deviation of a population. This phenomenon arises in area of agricultural, biological, environmental, and physical sciences. For instance, in environmental science, Bhat and Rao [5] explain that there are some situations that show the standard deviation of a pollutant is directly related to the mean, which means τ is known. In clinical chemistry, Bhat and Rao [5] also

state that “when the batches of some substance (chemicals) are to be analyzed, if sufficient batches of the substances are analyzed, their coefficients of variation will be known.” Furthermore, in medical, biological, and chemical studies, Brazauskas and Ghorai [6] provide some examples showing problems concerning coefficients of variation that are known in practice. Many statistical problems are due to the study of the mean of a normal distribution with a known coefficient of variation (see, e.g., Searls [7], Khan [8], Arnholt and Hebert [9], and Srisodaphol and Tongmol [10] and the references cited in the mentioned papers).

The estimation and testing of a normal mean with a known coefficient of variation are not equivalent to the case of known variance since the population mean is unknown. Furthermore, let $X_1, \dots, X_n$ be a random sample of size n from a normal distribution. The estimator of θ is $\hat{\theta} = \bar{X}^{-1}$ where $\bar{X}$ is the sample mean. The distribution of $\hat{\theta}$ is not a normal distribution. Therefore, we cannot construct a confidence interval for a normal mean and then transform the confidence interval for the reciprocal of a normal mean. Similarly, the hypothesis testing for a normal mean is not equivalent to the hypothesis testing for the reciprocal of a normal mean because the testing is developed based on the distribution of a sample mean.

Two confidence intervals for the reciprocal of a normal mean with a known coefficient of variation were proposed by Wongkhao et al. [11]. Their confidence intervals can be applied when the coefficient of variation of a control group is known. One of their confidence intervals was developed based on an asymptotic normality of the pivotal statistic Z , where Z follows the standard normal distribution. The other confidence interval was constructed based on the generalized confidence interval [12]. Simulation results showed that the coverage probabilities of the two confidence intervals were not significantly different. The limits of the asymptotic confidence interval are difficult to compute since they depend on an infinite summation. However, there has not yet been a study using a statistical test for the reciprocal of a normal mean with a known coefficient of variation. Therefore, we were motivated to propose two statistical tests for the reciprocal of a normal mean with a known coefficient of variation. One of the proposed statistical tests was based on an asymptotic method. The other statistical test was developed using the simple approximate expression for the expectation of the estimator of $\hat{\theta}$. In addition, we also compared the empirical probability of type I errors and the empirical power of the test using a Monte Carlo simulation.

The structure of this paper is as follows: Section 2 provides the theorem and corollary, which were used for constructing the asymptotic test. An approximate test is proposed in Section 3. The performance of the two proposed statistical tests for θ is investigated through a Monte Carlo simulation study in Section 4. We then conclude this paper in Section 5.

2. Asymptotic Test for the Reciprocal of a Normal Mean with a Known Coefficient of Variation

The null hypothesis of interest is $H_0 : \theta = \theta_0$. The theorem and corollary concerning the expectation of $\hat{\theta} = \bar{X}^{-1}$ and $\text{var}(\hat{\theta})$ proposed by Wongkhao et al. [11] were used to construct the asymptotic test as reviewed below.

Theorem 1 (Wongkhao et al. [11]). *Let $X_1, \dots, X_n$ be a random sample of size n from a normal distribution with mean μ and variance σ^2 . The estimator of θ is $\hat{\theta} = \bar{X}^{-1}$ where $\bar{X} = n^{-1} \sum_{i=1}^n X_i$. When a coefficient of variation $\tau = \sigma/\mu$ is known, the expectation of $\hat{\theta}$ is*

$$E(\hat{\theta}) = \theta \left[1 + \sum_{k=1}^{\infty} \frac{(2k)!}{2^k k!} \left(\frac{\tau^2}{n} \right)^k \right]. \quad (2)$$

Proof of Theorem 1. This theorem was proved in Wongkhao et al. [11]. $\square$

From (2), $\lim_{n \rightarrow \infty} E(\hat{\theta}) = \theta$ and $E(\hat{\theta}/\omega) = \theta$, where $\omega = 1 + \sum_{k=1}^{\infty} ((2k)!/2^k k!) (\tau^2/n)^k$. Thus, the unbiased estimator of θ is $\hat{\theta}/\omega$.

Corollary 2. *From Theorem 1, $\text{var}(\hat{\theta}) \approx \theta^2 \tau^2/n$.*

TABLE 1

Alternative hypothesis	Rejection criterion
$H_1: \theta \neq \theta_0$	$Z_{\text{asympt.}} > z_{\alpha/2}$ or $Z_{\text{asympt.}} < -z_{\alpha/2}$
$H_1: \theta > \theta_0$	$Z_{\text{asympt.}} > z_{\alpha}$
$H_1: \theta < \theta_0$	$Z_{\text{asympt.}} < -z_{\alpha}$

Proof of Corollary 2. This corollary was proved in Wongkhao et al. [11]. $\square$

From the central limit theorem, we use the fact that

$$Z = \frac{\hat{\theta} - \theta}{\sqrt{\text{var}(\hat{\theta})}} \sim N(0, 1). \quad (3)$$

Under H_0 which is true, we get

$$Z_{\text{asympt.}} = \frac{\sqrt{n}}{\hat{\theta}\tau} \left(\frac{\hat{\theta}}{\omega} - \theta_0 \right) \sim N(0, 1). \quad (4)$$

Let z_{α} denote the upper α th quantile of the standard normal distribution. On the basis of the above standard normal distribution, the level- α tests conducted are given in Table 1.

3. Approximate Test for the Reciprocal of a Normal Mean with a Known Coefficient of Variation

In this section, we present an approximate test using the simple approximate expression for the expectation and variance of $\hat{\theta}$. To find a simple approximate expression, we use a Taylor series expansion of $1/x$ around μ :

$$\begin{aligned} \frac{1}{x} &\approx \frac{1}{x} \Big|_{\mu} + (y - \mu) \frac{\partial}{\partial x} \left(\frac{1}{x} \right) \Big|_{\mu} \\ &\quad + \frac{1}{2} (x - \mu)^2 \frac{\partial^2}{\partial x^2} \left(\frac{1}{x} \right) \Big|_{\mu} \\ &\quad + O \left(\left((x - \mu) \frac{\partial}{\partial x} \right)^3 \left(\frac{1}{x} \right) \right). \end{aligned} \quad (5)$$

Theorem 3. *Let $X_1, \dots, X_n$ be a random sample of size n from a normal distribution with mean μ and variance σ^2 . The estimator of θ is $\hat{\theta} = \bar{X}^{-1}$ where $\bar{X} = n^{-1} \sum_{i=1}^n X_i$. The approximate expectation and variance of $\hat{\theta}$ when a coefficient of variation $\tau = \sigma/\mu$ is known are, respectively,*

$$\begin{aligned} E(\hat{\theta}) &\approx \frac{1}{\mu} \left(1 + \frac{\tau^2}{n} \right), \\ \text{var}(\hat{\theta}) &\approx \frac{\theta^2 \tau^2}{n}. \end{aligned} \quad (6)$$

Proof of Theorem 3. Consider random variable $\bar{X}$ where $\bar{X}$ has support $(0, \infty)$. Let $\hat{\theta} = \bar{X}^{-1}$ find approximations for $E(\hat{\theta})$

and $\text{var}(\hat{\theta})$ using Taylor series expansion of $\hat{\theta}$ around μ as in (5). The mean of $\hat{\theta}$ can be found by applying the expectation operator to the individual terms (ignoring all terms higher than two),

$$\begin{aligned}
 E(\hat{\theta}) &= E\left(\frac{1}{\bar{X}}\right) \\
 &\approx E\left(\frac{1}{\bar{X}}\right)\bigg|_{\mu} + E\left[\frac{\partial}{\partial \bar{X}}\left(\frac{1}{\bar{X}}\right)(\bar{X} - E(\bar{X}))\right]\bigg|_{\mu} \\
 &\quad + \frac{1}{2}E\left[\frac{\partial^2}{\partial \bar{X}^2}\left(\frac{1}{\bar{X}}\right)(\bar{X} - E(\bar{X}))^2\right]\bigg|_{\mu} \\
 &\quad + O(n^{-1}) \\
 &\approx \frac{1}{\mu} + 0 + \frac{1}{2}\left(\frac{2}{(E(\bar{X}))^3} \text{var}(\bar{X})\right) \\
 &= \frac{1}{\mu} + \frac{\text{var}(\bar{X})}{\mu^3} = \frac{1}{\mu}\left(1 + \frac{\sigma^2}{n\mu^2}\right) \\
 &= \frac{1}{\mu}\left(1 + \frac{\tau^2}{n}\right).
 \end{aligned} \tag{7}$$

An approximation of the variance of $\hat{\theta}$ is obtained using the first-order terms of the Taylor series expansion:

$$\begin{aligned}
 \text{var}(\hat{\theta}) &= \text{var}\left(\frac{1}{\bar{X}}\right) = E\left[\left(\frac{1}{\bar{X}} - E\left(\frac{1}{\bar{X}}\right)\right)^2\right] \\
 &\approx E\left[\left(\frac{1}{\bar{X}} - \frac{1}{\mu}\right)^2\right] \\
 &\approx E\left[\left(\frac{1}{\mu} + \frac{\partial}{\partial \bar{X}}\left(\frac{1}{\bar{X}}\right)(\bar{X} - E(\bar{X})) - \frac{1}{\mu}\right)^2\right]\bigg|_{\mu} \tag{8} \\
 &= \left(\frac{\partial}{\partial \bar{X}}\left(\frac{1}{\bar{X}}\right)\right)^2 \text{var}(\bar{X})\bigg|_{\mu} \approx \frac{\text{var}(\bar{X})}{\mu^4} = \frac{\sigma^2}{n\mu^4} \\
 &= \frac{\theta^2 \tau^2}{n}.
 \end{aligned}$$

□

It is clear from (7) that $\hat{\theta}$ is asymptotically unbiased ($\lim_{n \rightarrow \infty} E(\hat{\theta}) = \theta$) and $E(\hat{\theta}/\vartheta) = \theta$, where $\vartheta = 1 + \tau^2/n$. Thus, the unbiased estimator of θ is $\hat{\theta}/\vartheta$. From (8), $\hat{\theta}$ is consistent ($\lim_{n \rightarrow \infty} \text{var}(\hat{\theta}) = 0$). Under H_0 , we apply the central limit theorem and Theorem 3,

$$Z_{\text{approx.}} = \frac{\sqrt{n}}{\hat{\theta}\tau} \left(\frac{\hat{\theta}}{\vartheta} - \theta_0 \right) \sim N(0, 1). \tag{9}$$

Based on this we can now conduct the level- α tests (see Table 2).

TABLE 2

Alternative hypothesis	Rejection criterion
$H_1: \theta \neq \theta_0$	$Z_{\text{approx.}} > z_{\alpha/2}$ or $Z_{\text{approx.}} < -z_{\alpha/2}$
$H_1: \theta > \theta_0$	$Z_{\text{approx.}} > z_{\alpha}$
$H_1: \theta < \theta_0$	$Z_{\text{approx.}} < -z_{\alpha}$

4. Simulation Results

In this section, we performed simulation experiments to compare the behavior of the two statistical tests in a variety of situations. The first study compared the type I errors of the two statistical tests and checked how well they behave under the nominal level α . The second study compared their corresponding powers. We take $\tau = 0.1, 0.2, 0.5$ and $\theta_0 = 0.5, 1, 5$. We take $\theta = \theta_0 + \delta$ and estimate the type I errors ($\delta = 0$) and power ($\delta = 0.03, 0.05, 0.10$). The sample sizes are set at $n = 10, 20, 30$, and 50 . To test the following hypothesis, we set the significance level of α at 0.05 :

$$\begin{aligned}
 H_0: \theta &= \theta_0 \\
 \text{versus } H_1: \theta &\neq \theta_0.
 \end{aligned} \tag{10}$$

We repeated the above procedure 20,000 times for each setting using the R statistical software [13] and report the empirical type I errors and powers of the tests in Table 3.

As can be seen from Table 3, the empirical type I errors of both statistical tests were close to the given nominal level and were able to control the probability of type I errors for all situations. In addition, the empirical type I errors of the approximate test were not significantly different from those of the asymptotic test for all scenarios. Regarding the power comparisons, we observed that there was no difference in the empirical powers of the two statistical tests. The powers of both the asymptotic test and the approximate test decreased as τ increased due to the increased variability in the data. Additionally, the empirical powers increased as the sample sizes got larger. However, the empirical powers did not increase or decrease according to the values of δ when $n = 10$ and $\tau = 0.5$. However, the approximate test was much easier to calculate compared to the asymptotic test because the latter was based on an infinite summation.

5. Conclusion

In this paper, we presented two statistical tests for the reciprocal of a normal population mean with a known coefficient of variation. This situation usually arises when the coefficient of variation of the control group is known. The asymptotic test was based on the expectation and variance of the estimator of the reciprocal of a normal mean. The approximate expectation and variance of the estimator by Taylor series expansion were used to develop the approximate test. The simulation study indicated that the approximate test performs as efficiently as the asymptotic test in terms of empirical type I errors and empirical power. However, the computation of the approximate test was less complicated than the asymptotic test.

TABLE 3: The empirical type I errors and powers of the asymptotic test and the approximate test.

θ_0	τ	δ	$n = 10$		$n = 20$		$n = 30$		$n = 50$	
			Asympt.	Approx.	Asympt.	Approx.	Asympt.	Approx.	Asympt.	Approx.
0.5	0.1	0.00	0.0499	0.0499	0.0505	0.0505	0.0497	0.0497	0.0503	0.0503
		0.03	0.1348	0.1348	0.2456	0.2456	0.3402	0.3402	0.5378	0.5378
		0.05	0.3057	0.3058	0.5537	0.5538	0.7417	0.7417	0.9282	0.9282
		0.10	0.8326	0.8327	0.9891	0.9891	0.9992	0.9992	1.0000	1.0000
	0.2	0.00	0.0492	0.0492	0.0499	0.0500	0.0486	0.0487	0.0514	0.0514
		0.03	0.0613	0.0614	0.0833	0.0833	0.1106	0.1106	0.1702	0.1702
		0.05	0.0925	0.0927	0.1684	0.1684	0.2329	0.2329	0.3751	0.3751
		0.10	0.2654	0.2658	0.5151	0.5153	0.7081	0.7081	0.9090	0.9090
	0.5	0.00	0.0538	0.0531	0.0504	0.0502	0.0525	0.0527	0.0504	0.0504
		0.03	0.0450	0.0449	0.0484	0.0489	0.0490	0.0493	0.0562	0.0562
		0.05	0.0418	0.0423	0.0507	0.0509	0.0595	0.0598	0.0821	0.0823
		0.10	0.0493	0.0503	0.0855	0.0866	0.1177	0.1182	0.2038	0.2041
1	0.1	0.00	0.0503	0.0503	0.0489	0.0489	0.0501	0.0501	0.0472	0.0472
		0.03	0.1336	0.1336	0.2418	0.2418	0.3466	0.3466	0.5402	0.5402
		0.05	0.3052	0.3052	0.5708	0.5708	0.7502	0.7502	0.9279	0.9279
		0.10	0.8319	0.8319	0.9889	0.9889	0.9995	0.9995	1.0000	1.0000
	0.2	0.00	0.0520	0.0521	0.0505	0.0505	0.0514	0.0514	0.0512	0.0512
		0.03	0.0634	0.0634	0.0872	0.0873	0.1091	0.1091	0.1585	0.1585
		0.05	0.0976	0.0978	0.1588	0.1588	0.2332	0.2332	0.3698	0.3698
		0.10	0.2521	0.2523	0.5120	0.5121	0.7083	0.7084	0.9067	0.9067
	0.5	0.00	0.0530	0.0526	0.0520	0.0516	0.0492	0.0491	0.0507	0.0505
		0.03	0.0421	0.0421	0.0454	0.0456	0.0491	0.0493	0.0565	0.0567
		0.05	0.0414	0.0422	0.0508	0.0509	0.0609	0.0611	0.0833	0.0835
		0.10	0.0488	0.0498	0.0814	0.0819	0.1233	0.1237	0.2084	0.2087
5	0.1	0.00	0.0491	0.0491	0.0474	0.0474	0.0505	0.0505	0.0494	0.0494
		0.03	0.1398	0.1398	0.2505	0.2505	0.3450	0.3450	0.5327	0.5327
		0.05	0.3032	0.3032	0.5600	0.5600	0.7446	0.7446	0.9272	0.9272
		0.10	0.8314	0.8315	0.9880	0.9880	0.9994	0.9994	1.0000	1.0000
	0.2	0.00	0.0472	0.0472	0.0508	0.0508	0.0502	0.0503	0.0519	0.0519
		0.03	0.0620	0.0621	0.0834	0.0834	0.1089	0.1089	0.1601	0.1601
		0.05	0.0905	0.0904	0.1655	0.1656	0.2335	0.2336	0.3731	0.3731
		0.10	0.2612	0.2613	0.5055	0.5057	0.7035	0.7035	0.9106	0.9106
	0.5	0.00	0.0512	0.0504	0.0538	0.0535	0.0512	0.0512	0.0532	0.0531
		0.03	0.0470	0.0467	0.0455	0.0458	0.0509	0.0511	0.0564	0.0565
		0.05	0.0414	0.0419	0.0504	0.0503	0.0623	0.0626	0.0808	0.0808
		0.10	0.0454	0.0466	0.0846	0.0851	0.1250	0.1255	0.2048	0.2051

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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