Research Article

Properties of Matrix Variate Confluent Hypergeometric Function Distribution

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We study matrix variate confluent hypergeometric function kind 1 distribution which is a generalization of the matrix variate gamma distribution. We give several properties of this distribution. We also derive density functions of $X_2^{-1/2} X_1 X_2^{-1/2}$, $(X_1 + X_2)^{-1/2} X_1 (X_1 + X_2)^{-1/2}$, and $X_1 + X_2$, where $m \times m$ independent random matrices X_1 and X_2 follow confluent hypergeometric function kind 1 and gamma distributions, respectively.

1. Introduction

The matrix variate gamma distribution has many applications in multivariate statistical analysis. The Wishart distribution, which is the distribution of the sample variance covariance matrix when sampling from a multivariate normal distribution, is a special case of the matrix variate gamma distribution.

The purpose of this paper is to give a generalization of the matrix variate gamma distribution and study its properties.

We begin with a brief review of some definitions and notations. We adhere to standard notations (cf. Gupta and Nagar [1]). Let $A = (a_{ij})$ be an $m \times m$ matrix. Then, A' denotes the transpose of A ; $\text{tr}(A) = a_{11} + \dots + a_{mm}$; $\text{etr}(A) = \exp(\text{tr}(A))$; $\det(A)$ = determinant of A ; norm of $A = \|A\|$ = maximum of absolute values of eigenvalues of the matrix A ; $A > 0$ means that A is symmetric positive definite; and $A^{1/2}$ denotes the unique symmetric positive definite square root of $A > 0$. The multivariate gamma function $\Gamma_m(a)$ is defined by

$$\begin{aligned} \Gamma_m(a) &= \int_{X>0} \text{etr}(-X) \det(X)^{a-(m+1)/2} dX \\ &= \pi^{m(m-1)/4} \prod_{i=1}^m \Gamma\left(a - \frac{i-1}{2}\right), \quad \text{Re}(a) > \frac{m-1}{2}. \end{aligned} \quad (1)$$

The $m \times m$ symmetric positive definite random matrix X is said to have a matrix variate gamma distribution, denoted by

$X \sim \text{Ga}(m, \nu, \theta, \Omega)$, if its probability density function (p.d.f.) is given by

$$\frac{\det(X)^{\nu-(m+1)/2} \text{etr}(-\Omega^{-1}X/\theta)}{\Gamma_m(\nu) \theta^{m\nu} \det(\Omega)^\nu}, \quad X > 0, \quad (2)$$

where Ω is a symmetric positive definite matrix of order m , $\theta > 0$, and $\nu > (m-1)/2$. For $\Omega = I_m$, the above density reduces to a standard matrix variate gamma density and in this case we write $X \sim \text{Ga}(m, \nu, \theta)$. Further, if $X_1 \sim \text{Ga}(m, \nu_1, \theta)$ and $X_2 \sim \text{Ga}(m, \nu_2, \theta)$ are independent gamma matrices, then the random matrix $(X_1 + X_2)^{-1/2} X_1 (X_1 + X_2)^{-1/2}$ follows a matrix variate beta type 1 distribution with parameters ν_1 and ν_2 .

By replacing $\text{etr}(-\Omega^{-1}X/\theta)$ by the confluent hypergeometric function of matrix argument ${}_1F_1(\alpha; \beta; -\Omega^{-1}X/\theta)$, a generalization of the matrix variate gamma distribution can be defined by the p.d.f.:

$$C(\nu, \alpha, \beta, \theta, \Omega) \det(X)^{\nu-(m+1)/2} {}_1F_1\left(\alpha; \beta; -\frac{1}{\theta} \Omega^{-1}X\right), \quad (3)$$

where $X > 0$ and $C(\nu, \alpha, \beta, \theta, \Omega)$ is the normalizing constant. In Section 2, it has been shown that, for $\beta - \nu > (m-1)/2$,

$\alpha - \nu > (m-1)/2$, $\nu > (m-1)/2$, $\theta > 0$, and $\Omega > 0$, the normalizing constant can be evaluated as

$$C(\nu, \alpha, \beta, \theta, \Omega) = \frac{\Gamma_m(\alpha) \Gamma_m(\beta - \nu) \det(\theta\Omega)^{-\nu}}{\Gamma_m(\nu) \Gamma_m(\beta) \Gamma_m(\alpha - \nu)}. \quad (4)$$

Therefore, the p.d.f. in (3) can be written explicitly as

$$\frac{\Gamma_m(\alpha) \Gamma_m(\beta - \nu) \det(\theta\Omega)^{-\nu}}{\Gamma_m(\nu) \Gamma_m(\beta) \Gamma_m(\alpha - \nu)} \det(X)^{\nu-(m+1)/2} \cdot {}_1F_1\left(\alpha; \beta; -\frac{1}{\theta}\Omega^{-1}X\right), \quad X > 0, \quad (5)$$

where $\beta - \nu > (m-1)/2$, $\alpha - \nu > (m-1)/2$, $\nu > (m-1)/2$, $\theta > 0$, $\Omega > 0$, and ${}_1F_1$ is the confluent hypergeometric function of the first kind of matrix argument (Gupta and Nagar [1]). Since the density given above involves the confluent hypergeometric function, we will call the corresponding distribution a *confluent hypergeometric function distribution*. We will write $X \sim \text{CH}_m(\nu, \alpha, \beta, \theta, \Omega, \text{kind } 1)$ to say that the random matrix X has a confluent hypergeometric function distribution defined by the density (5). It has been shown by van der Merwe and Roux [2] that the above density can be obtained as a limiting case of a density involving the Gauss hypergeometric function of matrix argument. For $\alpha = \beta$, the density (5) reduces to a matrix variate gamma density and for $\Omega = I_m$ it slides to

$$\frac{\Gamma_m(\alpha) \Gamma_m(\beta - \nu)}{\theta^{\nu} \Gamma_m(\nu) \Gamma_m(\beta) \Gamma_m(\alpha - \nu)} \det(X)^{\nu-(m+1)/2} \cdot {}_1F_1\left(\alpha; \beta; -\frac{1}{\theta}X\right), \quad X > 0, \quad (6)$$

where $\beta - \nu > (m-1)/2$, $\alpha - \nu > (m-1)/2$, $\nu > (m-1)/2$, and $\theta > 0$. In this case we will write $X \sim \text{CH}_m(\nu, \alpha, \beta, \theta, \text{kind } 1)$. The matrix variate confluent hypergeometric function kind 1 distribution occurs as the distribution of the matrix ratio of independent gamma and beta matrices. For $m = 1$, (6) reduces to a univariate confluent hypergeometric function kind 1 density given by (Orozco-Castañeda et al. [3])

$$\frac{\Gamma(\alpha) \Gamma(\beta - \nu)}{\theta^{\nu} \Gamma(\nu) \Gamma(\beta) \Gamma(\alpha - \nu)} x^{\nu-1} {}_1F_1\left(\alpha; \beta; -\frac{x}{\theta}\right), \quad x > 0, \quad (7)$$

where $\beta - \nu > 0$, $\alpha - \nu > 0$, $\nu > 0$, $\theta > 0$, and ${}_1F_1$ is the confluent hypergeometric function of the first kind (Luke [4]). The random variable x having the above density will be designated by $x \sim \text{CH}(\nu, \alpha, \beta, \theta, \text{kind } 1)$. Since the matrix variate confluent hypergeometric function kind 1 distribution is a generalization of the matrix variate gamma distribution, it is reasonable to say that the matrix variate confluent hypergeometric function kind 1 distribution can be used as an alternative to the gamma distribution quite effectively.

Although ample information about matrix variate gamma distribution is available, little appears to have been done in the literature to study the matrix variate confluent hypergeometric function kind 1 distribution.

In this paper, we study several properties including stochastic representations of the matrix variate confluent hypergeometric function kind 1 distribution. We also derive the density function of the matrix quotient of two independent random matrices having confluent hypergeometric function kind 1 and gamma distributions. Further, densities of several other matrix quotients and matrix products involving confluent hypergeometric function kind 1, beta type 1, beta type 2, and gamma matrices are derived.

2. Some Definitions and Preliminary Results

In this section we give some definitions and preliminary results which are used in subsequent sections.

A more general integral representation of the multivariate gamma function can be obtained as

$$\Gamma_m(a) = \det(Y)^a \int_{R>0} \text{etr}(-YR) \det(R)^{a-(m+1)/2} dR, \quad (8)$$

where $\text{Re}(a) > (m-1)/2$ and $\text{Re}(Y) > (m-1)/2$. The above result can be established for real $Y > 0$ by substituting $X = Y^{1/2}RY^{1/2}$ with the Jacobian $J(X \rightarrow R) = \det(Y)^{(m+1)/2}$ in (1) and it follows for complex Y by analytic continuation.

The multivariate generalization of the beta function is given by

$$\begin{aligned} B_m(a, b) &= \int_0^{I_m} \det(R)^{a-(m+1)/2} \det(I_m - R)^{b-(m+1)/2} dR \\ &= \frac{\Gamma_m(a) \Gamma_m(b)}{\Gamma_m(a+b)} = B_m(b, a), \end{aligned} \quad (9)$$

where $\text{Re}(a) > (m-1)/2$ and $\text{Re}(b) > (m-1)/2$.

The generalized hypergeometric function of one matrix, defined in Constantine [5], is given by

$$\begin{aligned} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; X) \\ = \sum_{k=0}^{\infty} \sum_{\kappa \vdash k} \frac{(a_1)_{\kappa} \cdots (a_p)_{\kappa}}{(b_1)_{\kappa} \cdots (b_q)_{\kappa}} \frac{C_{\kappa}(X)}{k!}, \end{aligned} \quad (10)$$

where $a_i, i = 1, \dots, p, b_j, j = 1, \dots, q$ are arbitrary complex numbers, X is an $m \times m$ complex symmetric matrix, $C_{\kappa}(X)$ is the zonal polynomial of $m \times m$ complex symmetric matrix X corresponding to the ordered partition $\kappa = (k_1, \dots, k_m)$, $k_1 \geq \dots \geq k_m \geq 0$, $k_1 + \dots + k_m = k$, and $\sum_{\kappa \vdash k}$ denotes the summation over all partitions κ . The generalized hypergeometric coefficient $(a)_{\kappa}$ used above is defined by

$$(a)_{\kappa} = \prod_{i=1}^m \left(a - \frac{i-1}{2}\right)_{k_i}, \quad (11)$$

where $(a)_r = a(a+1)\cdots(a+r-1)$, $r = 1, 2, \dots$, with $(a)_0 = 1$. Conditions for convergence of the series in (10) are available in the literature. From (10), it follows that

$$\begin{aligned} {}_0F_0(X) &= \sum_{k=0}^{\infty} \sum_{\kappa \vdash k} \frac{C_{\kappa}(X)}{k!} = \text{etr}(X), \\ {}_1F_0(a; X) &= \sum_{k=0}^{\infty} \sum_{\kappa \vdash k} \frac{(a)_{\kappa} C_{\kappa}(X)}{k!} \\ &= \det(I_m - X)^{-a}, \quad \|X\| < 1, \end{aligned} \quad (12)$$

$${}_1F_1(a; c; X) = \sum_{k=0}^{\infty} \sum_{\kappa \vdash k} \frac{(a)_{\kappa} C_{\kappa}(X)}{(c)_{\kappa} k!}, \quad (13)$$

$${}_2F_1(a, b; c; X) = \sum_{k=0}^{\infty} \sum_{\kappa \vdash k} \frac{(a)_{\kappa} (b)_{\kappa} C_{\kappa}(X)}{(c)_{\kappa} k!}, \quad \|X\| < 1. \quad (14)$$

By taking $a_p = b_q = c$ in (10), it can be observed that

$$\begin{aligned} {}_pF_q(a_1, \dots, a_{p-1}, c; b_1, \dots, b_{q-1}, c; X) \\ = {}_{p-1}F_{q-1}(a_1, \dots, a_{p-1}; b_1, \dots, b_{q-1}; X). \end{aligned} \quad (15)$$

Substituting $p = 2$, $q = 1$ in (15) and using (12), the Gauss hypergeometric function ${}_2F_1(a, c; c; X)$ is reduced as

$${}_2F_1(a, c; c; X) = {}_1F_0(a; X) = \det(I_m - X)^{-a}, \quad \|X\| < 1. \quad (16)$$

The integral representations of the confluent hypergeometric function ${}_1F_1$ and the Gauss hypergeometric function ${}_2F_1$ are given by

$$\begin{aligned} {}_1F_1(a; c; X) &= \frac{1}{B_m(a, c-a)} \int_0^{I_m} \det(R)^{a-(m+1)/2} \\ &\cdot \det(I_m - R)^{c-a-(m+1)/2} \text{etr}(XR) dR, \end{aligned} \quad (17)$$

$$\begin{aligned} {}_2F_1(a, b; c; X) &= \frac{1}{B_m(a, c-a)} \\ &\cdot \int_0^{I_m} \frac{\det(R)^{a-(m+1)/2} \det(I_m - R)^{c-a-(m+1)/2}}{\det(I_m - XR)^b} dR, \end{aligned} \quad (18)$$

where $\text{Re}(a) > (m-1)/2$ and $\text{Re}(c-a) > (m-1)/2$.

Further generalizations of (8) and (9) in terms of zonal polynomials, due to Constantine [5], are given as

$$\int_{R>0} \text{etr}(-YR) \det(R)^{a-(m+1)/2} C_{\kappa}(XR) dR \quad (19)$$

$$= \Gamma_m(a) (a)_{\kappa} \det(Y)^{-a} C_{\kappa}(Y^{-1}X),$$

$$\begin{aligned} \int_0^{I_m} \det(R)^{a-(m+1)/2} \det(I_m - R)^{b-(m+1)/2} C_{\kappa}(XR) dR \\ = \frac{B_m(a, b) (a)_{\kappa}}{(a+b)_{\kappa}} C_{\kappa}(X), \end{aligned} \quad (20)$$

respectively.

For $\text{Re}(\alpha) > (m-1)/2$ and $\text{Re}(\beta) > (m-1)/2$, we have

$$\begin{aligned} \int_{R>0} \det(R)^{\alpha-(m+1)/2} \text{etr}(-YR) {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; \\ XR) dR = \Gamma_m(\alpha) \det(Y)^{-\alpha} {}_{p+1}F_q(\alpha, a_1, \dots, a_p; \end{aligned} \quad (21)$$

$$b_1, \dots, b_q; XY^{-1}),$$

$$\begin{aligned} \int_0^{I_m} \det(R)^{\alpha-(m+1)/2} \det(I_m \\ - R)^{\beta-(m+1)/2} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; XR) dR \\ = B_m(\alpha, \beta) {}_{p+1}F_{q+1}(\alpha, a_1, \dots, a_p; \alpha + \beta, b_1, \dots, b_q; X). \end{aligned} \quad (22)$$

We can establish (21) and (22) by expanding ${}_pF_q$ in series form by using (10) and integrating term by term by applying (19) and (20) and finally summing the resulting series.

Note that the series expansions for ${}_1F_1$ and ${}_2F_1$ given in (13) and (14) can be obtained by expanding $\text{etr}(XR)$ and $\det(I_m - XR)^{-b}$, $\|XR\| < 1$, in (17) and (18) and integrating R using (20). Substituting $X = I_m$ in (18) and integrating, we obtain

$${}_2F_1(a, b; c; I_m) = \frac{\Gamma_m(c) \Gamma_m(c-a-b)}{\Gamma_m(c-a) \Gamma_m(c-b)}, \quad (23)$$

where $\text{Re}(c-a-b) > (m-1)/2$, $\text{Re}(c-a) > (m-1)/2$, $\text{Re}(c-b) > (m-1)/2$, and $\text{Re}(c) > (m-1)/2$. The hypergeometric function ${}_1F_1(a; c; X)$ satisfies Kummer's relation

$${}_1F_1(a; c; -X) = \text{etr}(-X) {}_1F_1(c-a; c; X). \quad (24)$$

For properties and further results on these functions the reader is referred to Constantine [5], James [6], Muirhead [7], and Gupta and Nagar [1]. The numerical computation of a hypergeometric function of matrix arguments is very difficult. However, some numerical methods are proposed in recent years; see, Hashiguchi et al. [8] and Koev and Edelman [9].

The generalized hypergeometric function with $m \times m$ complex symmetric matrices X and Y is defined by

$$\begin{aligned} {}_pF_q^{(m)}(a_1, \dots, a_p; b_1, \dots, b_q; X, Y) \\ = \sum_{k=0}^{\infty} \sum_{\kappa \vdash k} \frac{(a_1)_{\kappa} \cdots (a_p)_{\kappa} C_{\kappa}(X) C_{\kappa}(Y)}{(b_1)_{\kappa} \cdots (b_q)_{\kappa} C_{\kappa}(I_m) k!}. \end{aligned} \quad (25)$$

It is clear from the above definition that the order of X and Y is unimportant; that is,

$$\begin{aligned} {}_pF_q^{(m)}(a_1, \dots, a_p; b_1, \dots, b_q; X, Y) \\ = {}_pF_q^{(m)}(a_1, \dots, a_p; b_1, \dots, b_q; Y, X). \end{aligned} \quad (26)$$

Also, if one of the argument matrices is the identity, this function reduces to the one argument function. Further, the two-matrix argument function ${}_pF_q^{(m)}$ can be obtained from the one-matrix function ${}_pF_q$ by averaging over the orthogonal

group $O(m)$ using a result given in James [6, Equation 23]; namely,

$$\int_{O(m)} C_{\kappa}(XHYH') (dH) = \frac{C_{\kappa}(X)C_{\kappa}(Y)}{C_{\kappa}(I_m)}, \quad (27)$$

where (dH) denotes the normalized invariant measure on $O(m)$. That is,

$$\begin{aligned} & {}_pF_q^{(m)}(a_1, \dots, a_p; b_1, \dots, b_q; X, Y) \\ &= \int_{O(m)} {}_pF_q^{(m)}(a_1, \dots, a_p; b_1, \dots, b_q; XHYH') (dH), \end{aligned} \quad (28)$$

given in James [6, Equation 30].

Finally, we define the inverted matrix variate gamma, matrix variate beta type 1, and matrix variate beta type 2 distributions. These definitions can be found in Gupta and Nagar [1] and Iranmanesh et al. [10].

Definition 1. An $m \times m$ random symmetric positive definite matrix X is said to have an inverted matrix variate gamma distribution with parameters μ , θ , and Ψ , denoted by $X \sim \text{InvGa}(m, \mu, \theta, \Psi)$, if its p.d.f. is given by

$$\frac{\det(X)^{-\mu-(m+1)/2} \text{etr}(-\Psi^{-1}X^{-1}/\theta)}{\det(\theta\Psi)^{\mu} \Gamma_m(\mu)}, \quad X > 0, \quad (29)$$

where $\mu > (m-1)/2$, $\theta > 0$, and Ψ is a symmetric positive definite matrix of order m .

Definition 2. An $m \times m$ random symmetric positive definite matrix U is said to have a matrix variate beta type 1 distribution with parameters $a(> (m-1)/2)$ and $b(> (m-1)/2)$, denoted as $U \sim \text{B1}(m, a, b)$, if its p.d.f. is given by

$$\frac{\det(U)^{a-(m+1)/2} \det(I_m - U)^{b-(m+1)/2}}{B_m(a, b)}, \quad 0 < U < I_m. \quad (30)$$

Definition 3. An $m \times m$ random symmetric positive definite matrix V is said to have a matrix variate beta type 2 distribution with parameters $a(> (m-1)/2)$ and $b(> (m-1)/2)$, denoted as $V \sim \text{B2}(m, a, b)$, if its p.d.f. is given by

$$\frac{\det(V)^{a-(m+1)/2} \det(I_m + V)^{-(a+b)}}{B_m(a, b)}, \quad V > 0. \quad (31)$$

Note that if $U \sim \text{B1}(m, a, b)$, then $(I_m - U)^{-1}U \sim \text{B2}(m, a, b)$. Further, if X_1 and X_2 are independent, $X_1 \sim \text{Ga}(m, \nu_1, \theta)$ and $X_2 \sim \text{Ga}(m, \nu_2, \theta)$, then $(X_1 + X_2)^{-1/2}X_1(X_1 + X_2)^{-1/2} \sim \text{B1}(m, \nu_1, \nu_2)$ and $X_2^{-1/2}X_1X_2^{-1/2} \sim \text{B2}(m, \nu_1, \nu_2)$.

We conclude this section by evaluating the normalizing constant $C(\nu, \alpha, \beta, \theta, \Omega)$ in (3). Since the density over its support set integrates to one, we have

$$\begin{aligned} & [C(\nu, \alpha, \beta, \theta, \Omega)]^{-1} \\ &= \int_{X>0} \det(X)^{\nu-(m+1)/2} {}_1F_1\left(\alpha; \beta; -\frac{1}{\theta}\Omega^{-1}X\right) dX. \end{aligned} \quad (32)$$

By rewriting ${}_1F_1$ using Kummer's relation (24) and integrating X by applying (21), we get

$$\begin{aligned} & [C(\nu, \alpha, \beta, \theta, \Omega)]^{-1} \\ &= \Gamma_m(\nu) \det(\theta\Omega)^{\nu} {}_2F_1(\nu, \beta - \alpha; \beta; I_m), \end{aligned} \quad (33)$$

where $\text{Re}(\nu) > (m-1)/2$. Finally, writing ${}_2F_1(\nu, \beta - \alpha; \beta; I_m)$ in terms of multivariate gamma functions by using (23), we obtain

$$\begin{aligned} & [C(\nu, \alpha, \beta, \theta, \Omega)]^{-1} \\ &= \int_{X>0} \det(X)^{\nu-(m+1)/2} {}_1F_1\left(\alpha; \beta; -\frac{1}{\theta}\Omega^{-1}X\right) dX \\ &= \frac{\det(\theta\Omega)^{\nu} \Gamma_m(\nu) \Gamma_m(\beta) \Gamma_m(\alpha - \nu)}{\Gamma_m(\alpha) \Gamma_m(\beta - \nu)}, \end{aligned} \quad (34)$$

where $\text{Re}(\beta - \nu) > (m-1)/2$, $\text{Re}(\alpha - \nu) > (m-1)/2$, $\theta > 0$, $\text{Re}(\nu) > (m-1)/2$, and $\Omega > 0$.

3. Properties

In this section we study several properties of the confluent hypergeometric function kind 1 distribution defined in Section 1. For the sake of completeness, we first state the following results established in Gupta and Nagar [1].

- (1) Let $X \sim \text{CH}_m(\nu, \alpha, \beta, \theta, \Omega, \text{kind } 1)$ and let A be an $m \times m$ constant nonsingular matrix. Then, $AXA' \sim \text{CH}_m(\nu, \alpha, \beta, \theta, A\Omega A', \text{kind } 1)$.
- (2) Let $X \sim \text{CH}_m(\nu, \alpha, \beta, \theta, \Omega, \text{kind } 1)$ and let H be an $m \times m$ orthogonal matrix whose elements are either constants or random variables distributed independent of X . Then, the distribution of X is invariant under the transformation $X \rightarrow HXH'$ if H is a matrix of constants. Further, if H is a random matrix, then H and HXH' are independent.
- (3) Let $X \sim \text{CH}_m(\nu, \alpha, \beta, \theta, \Omega, \text{kind } 1)$. Then, the cumulative distribution function (cdf) of X is derived as

$$\begin{aligned} P(X \leq \Lambda) &= \frac{\Gamma_m(\alpha) \Gamma_m(\beta - \nu) \Gamma_m[(m+1)/2]}{\Gamma_m[\nu + (m+1)/2] \Gamma_m(\beta) \Gamma_m(\alpha - \nu)} \\ &\quad \cdot \det(\theta\Omega)^{-\nu} \det(\Lambda)^{\nu} \\ &\quad \cdot {}_2F_2\left(\alpha, \nu; \beta, \nu + \frac{m+1}{2}; -\frac{1}{\theta}\Omega^{-1}\Lambda\right), \end{aligned} \quad (35)$$

where $\Lambda > 0$.

- (4) Let $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$, where X_{11} is a $q \times q$ matrix. Define $X_{11.2} = X_{11} - X_{12}X_{22}^{-1}X_{21}$ and $X_{22.1} = X_{22} - X_{21}X_{11}^{-1}X_{12}$. If $X \sim \text{CH}_m(\nu, \alpha, \beta, \theta, \text{kind } 1)$, then (i) X_{11} and $X_{22.1}$ are independent, $X_{11} \sim \text{CH}_q(\nu, \alpha - (m-q)/2, \beta - (m-q)/2, \theta, \text{kind } 1)$ and $X_{22.1} \sim \text{CH}_{m-q}(\nu - q/2, \alpha - q/2, \beta - q/2, \theta, \text{kind } 1)$, and (ii) X_{22} and $X_{11.2}$ are independent, $X_{22} \sim \text{CH}_{m-q}(\nu, \alpha - q/2, \beta - q/2, \theta, \text{kind } 1)$ and $X_{11.2} \sim \text{CH}_q(\nu - (m-q)/2, \alpha - (m-q)/2, \beta - (m-q)/2, \theta, \text{kind } 1)$.

- (5) Let A be a $q \times m$ constant matrix of rank $q (\leq m)$. If $X \sim \text{CH}_m(\nu, \alpha, \beta, \theta, \text{kind } 1)$, then $AXA' \sim \text{CH}_q(\nu, \alpha - (m-q)/2, \beta - (m-q)/2, \theta, AA', \text{kind } 1)$ and $(AX^{-1}A')^{-1} \sim \text{CH}_q(\nu - (m-q)/2, \alpha - (m-q)/2, \beta - (m-q)/2, \theta, (AA')^{-1}, \text{kind } 1)$.
- (6) Let $X \sim \text{CH}_m(\nu, \alpha, \beta, \theta, \text{kind } 1)$ and let $\mathbf{a}$ be a nonzero m -dimensional column vector of constants, then $(\mathbf{a}'\mathbf{a})^{-1}(\mathbf{a}'X\mathbf{a}) \sim \text{CH}(\nu, \alpha - (m-1)/2, \beta - (m-1)/2, \theta, \text{kind } 1)$ and $\mathbf{a}'\mathbf{a}(\mathbf{a}'X^{-1}\mathbf{a})^{-1} \sim \text{CH}(\nu - (m-1)/2, \alpha - (m-1)/2, \beta - (m-1)/2, \theta, \text{kind } 1)$. Further, if $\mathbf{y}$ is an m -dimensional random vector, independent of X , and $P(\mathbf{y} \neq \mathbf{0}) = 1$, then it follows that $(\mathbf{y}'\mathbf{y})^{-1}(\mathbf{y}'X\mathbf{y}) \sim \text{CH}(\nu, \alpha - (m-1)/2, \beta - (m-1)/2, \theta, \text{kind } 1)$ and $\mathbf{y}'\mathbf{y}(\mathbf{y}'X^{-1}\mathbf{y})^{-1} \sim \text{CH}(\nu - (m-1)/2, \alpha - (m-1)/2, \beta - (m-1)/2, \theta, \text{kind } 1)$.

It may also be mentioned here that properties (1)–(6) given above are modified forms of results given in Section 8.10 of Gupta and Nagar [1].

If the $m \times m$ random matrices X_1 and X_2 are independent, $X_1 \sim \text{CH}_m(\nu, a_1 + \nu, 2\nu, \theta, \text{kind } 1)$ and $X_2 \sim \text{CH}_m(\nu, a_2 + \nu, 2\nu, \theta, \text{kind } 1)$, $\nu = \gamma/2 + (m+1)/4$, then Roux and van der Merwe [11] have shown that $X_2^{-1/2}X_1X_2^{-1/2}$ has matrix variate beta type 2 distributions with parameters a_2 and a_1 .

The matrix variate confluent hypergeometric function kind 1 distribution can be derived as the distribution of the matrix ratio of independent gamma and beta matrices. It has been shown in Gupta and Nagar [1] that if $Y \sim \text{Ga}(m, \nu, \theta)$ and $U \sim \text{B1}(m, a, b)$, then $U^{-1/2}YU^{-1/2} \sim \text{CH}_m(\nu, a + \nu, a + b + \nu, \theta, \text{kind } 1)$.

The expected values of X and X^{-1} , for $X \sim \text{CH}_m(\nu, \alpha, \beta, \theta, \text{kind } 1)$, can easily be obtained from the above results. For any fixed $\mathbf{a} \in \mathbb{R}^m$, $\mathbf{a} \neq \mathbf{0}$,

$$E \left[\frac{\mathbf{a}'X\mathbf{a}}{\mathbf{a}'\mathbf{a}} \right] = E(\nu_1), \quad (36)$$

where $\nu_1 \sim \text{CH}(\nu, \alpha - (m-1)/2, \beta - (m-1)/2, \theta, \text{kind } 1)$, and

$$E \left[\frac{\mathbf{a}'X^{-1}\mathbf{a}}{\mathbf{a}'\mathbf{a}} \right] = E \left(\frac{1}{\nu_2} \right), \quad (37)$$

where $\nu_2 \sim \text{CH}(\nu - (m-1)/2, \alpha - (m-1)/2, \beta - (m-1)/2, \theta, \text{kind } 1)$. Hence, for all $\mathbf{a} \in \mathbb{R}^m$,

$$\begin{aligned} \mathbf{a}'E(X)\mathbf{a} &= \mathbf{a}'\mathbf{a}E(\nu_1) \\ &= \frac{\nu\theta[\beta - \nu - (m+1)/2]}{[\alpha - \nu - (m+1)/2]} \mathbf{a}'\mathbf{a}, \\ &\quad \alpha - \nu > \frac{m+1}{2}, \quad \beta - \nu > \frac{m+1}{2}, \\ \mathbf{a}'E(X^{-1})\mathbf{a} &= \mathbf{a}'\mathbf{a}E \left(\frac{1}{\nu_2} \right) \\ &= \frac{\alpha - \nu}{\theta[\nu - (m+1)/2](\beta - \nu)} \mathbf{a}'\mathbf{a}, \\ &\quad \nu > \frac{m+1}{2}, \end{aligned} \quad (38)$$

which implies that

$$\begin{aligned} E(X) &= \frac{\nu\theta[\beta - \nu - (m+1)/2]}{[\alpha - \nu - (m+1)/2]} I_m, \\ &\quad \alpha - \nu > \frac{m+1}{2}, \quad \beta - \nu > \frac{m+1}{2}, \\ E(X^{-1}) &= \frac{\alpha - \nu}{\theta[\nu - (m+1)/2](\beta - \nu)} I_m, \\ &\quad \nu > \frac{m+1}{2}. \end{aligned} \quad (39)$$

The Laplace transform of the density of X , where $X \sim \text{CH}_m(\nu, \alpha, \beta, \theta, \text{kind } 1)$, is given by

$$\begin{aligned} &\frac{\theta^{-m\nu} \Gamma_m(\alpha) \Gamma_m(\beta - \nu)}{\Gamma_m(\nu) \Gamma_m(\beta) \Gamma_m(\alpha - \nu)} \int_{X>0} \text{etr} \left(-ZX - \frac{X}{\theta} \right) \\ &\quad \cdot \det(X)^{\nu-(m+1)/2} {}_1F_1 \left(\beta - \alpha; \beta; \frac{X}{\theta} \right) dX \\ &= \frac{\Gamma_m(\alpha) \Gamma_m(\beta - \nu)}{\Gamma_m(\beta) \Gamma_m(\alpha - \nu)} \det(I_m + \theta Z)^{-\nu} {}_2F_1 \left(\nu, \beta \right. \\ &\quad \left. - \alpha; \beta; (I_m + \theta Z)^{-1} \right), \quad \text{Re}(Z) > 0, \end{aligned} \quad (40)$$

where we have used (24) and (21). From the above expression, the Laplace transform of the density of X , where $X \sim \text{CH}_m(\nu, \alpha, \beta, \theta, \Omega, \text{kind } 1)$, is derived as

$$\begin{aligned} &\frac{\Gamma_m(\alpha) \Gamma_m(\beta - \nu)}{\Gamma_m(\beta) \Gamma_m(\alpha - \nu)} \det(I_m + \theta \Omega Z)^{-\nu} \\ &\quad \cdot {}_2F_1 \left(\nu, \beta - \alpha; \beta; (I_m + \theta \Omega Z)^{-1} \right), \quad \text{Re}(Z) > 0. \end{aligned} \quad (41)$$

Theorem 4. Let $X \sim \text{CH}_m(\nu, \alpha, \beta, \theta, \Omega, \text{kind } 1)$; then

$$\begin{aligned} &E \left[\det(X)^h \right] \\ &= \frac{\det(\theta \Omega)^h \Gamma_m(\beta - \nu) \Gamma_m(\nu + h) \Gamma_m(\alpha - \nu - h)}{\Gamma_m(\nu) \Gamma_m(\alpha - \nu) \Gamma_m(\beta - \nu - h)}, \end{aligned} \quad (42)$$

where $\text{Re}(h + \nu) > (m-1)/2$, $\text{Re}(h) < \alpha - \nu - (m-1)/2$, and $\text{Re}(h) < \beta - \nu - (m-1)/2$.

Proof. From the density of X , we have

$$\begin{aligned} E \left[\det(X)^h \right] &= \frac{\det(\theta \Omega)^{-\nu} \Gamma_m(\alpha) \Gamma_m(\beta - \nu)}{\Gamma_m(\nu) \Gamma_m(\beta) \Gamma_m(\alpha - \nu)} \\ &\quad \cdot \int_{X>0} \det(X)^{\nu+h-(m+1)/2} {}_1F_1 \left(\alpha; \beta; -\frac{1}{\theta} \Omega^{-1} X \right) dX. \end{aligned} \quad (43)$$

Now, evaluating the above integral by using (34), we get

$$\begin{aligned} E \left[\det(X)^h \right] &= \frac{\det(\theta \Omega)^{-\nu} \Gamma_m(\alpha) \Gamma_m(\beta - \nu)}{\Gamma_m(\nu) \Gamma_m(\beta) \Gamma_m(\alpha - \nu)} \\ &\quad \cdot \frac{\det(\theta \Omega)^{\nu+h} \Gamma_m(\nu + h) \Gamma_m(\beta) \Gamma_m(\alpha - \nu - h)}{\Gamma_m(\alpha) \Gamma_m(\beta - \nu - h)}, \end{aligned} \quad (44)$$

where $\text{Re}(h + \nu) > (m - 1)/2$, $\text{Re}(h) < \alpha - \nu - (m - 1)/2$, and $\text{Re}(h) < \beta - \nu - (m - 1)/2$. Finally, simplifying the above expression, we get the desired result. $\square$

Corollary 5. Let $x \sim \text{CH}(\nu, \alpha, \beta, \theta, \text{kind } 1)$; then

$$E(x^h) = \frac{\theta^h \Gamma(\beta - \nu) \Gamma(\nu + h) \Gamma(\alpha - \nu - h)}{\Gamma(\nu) \Gamma(\alpha - \nu) \Gamma(\beta - \nu - h)}, \quad (45)$$

where $\text{Re}(h + \nu) > 0$, $\text{Re}(h) < \alpha - \nu$, and $\text{Re}(h) < \beta - \nu$.

$$\begin{aligned} \text{Var}(\det(X)) &= \theta^{2m} \det(\Omega)^2 \prod_{j=1}^m \left[\frac{[\nu - (j - 1)/2] [\beta - \nu - (j + 1)/2]}{[\alpha - \nu - (j + 1)/2]} \left\{ \frac{[\nu - (j - 3)/2] [\beta - \nu - (j + 3)/2]}{[\alpha - \nu - (j + 3)/2]} \right. \right. \\ &\quad \left. \left. - \frac{[\nu - (j - 1)/2] [\beta - \nu - (j + 1)/2]}{[\alpha - \nu - (j + 1)/2]} \right\} \right], \end{aligned} \quad (47)$$

where $\beta > \nu + (m + 3)/2$ and $\alpha > \nu + (m + 3)/2$. For $m \times m$ symmetric matrix A , $E[C_\kappa(AX)]$ is derived as

$$\begin{aligned} E[C_\kappa(AX)] &= \frac{\det(\theta\Omega)^{-\nu} \Gamma_m(\alpha) \Gamma_m(\beta - \nu)}{\Gamma_m(\nu) \Gamma_m(\beta) \Gamma_m(\alpha - \nu)} \\ &\cdot \int_{X>0} C_\kappa(AX) \det(X)^{\nu-(m+1)/2} \\ &\cdot {}_1F_1\left(\alpha; \beta; -\frac{1}{\theta} \Omega^{-1} X\right) dX. \end{aligned} \quad (48)$$

Replacing ${}_1F_1(\alpha; \beta; -\Omega^{-1} X/\theta)$ by its integral representation, namely,

$$\begin{aligned} {}_1F_1\left(\alpha; \beta; -\frac{1}{\theta} \Omega^{-1} X\right) &= \frac{\Gamma_m(\beta)}{\Gamma_m(\alpha) \Gamma_m(\beta - \alpha)} \\ &\cdot \int_0^{I_m} \text{etr}\left(-\frac{1}{\theta} \Omega^{-1/2} X \Omega^{-1/2} Y\right) \det(Y)^{\alpha-(m+1)/2} \\ &\cdot \det(I_m - Y)^{\beta-\alpha-(m+1)/2} dY, \end{aligned} \quad (49)$$

where $\text{Re}(\beta - \alpha) > (m - 1)/2$ and $\text{Re}(\alpha) > (m - 1)/2$, one obtains

$$\begin{aligned} E[C_\kappa(AX)] &= \frac{\det(\theta\Omega)^{-\nu} \Gamma_m(\beta - \nu)}{\Gamma_m(\nu) \Gamma_m(\beta - \alpha) \Gamma_m(\alpha - \nu)} \\ &\cdot \int_0^{I_m} \det(Y)^{\alpha-(m+1)/2} \det(I_m - Y)^{\beta-\alpha-(m+1)/2} \\ &\cdot \int_{X>0} C_\kappa(AX) \det(X)^{\nu-(m+1)/2} \\ &\cdot \text{etr}\left(-\frac{1}{\theta} \Omega^{-1/2} X \Omega^{-1/2} Y\right) dX dY. \end{aligned} \quad (50)$$

Using (42) the mean and the variance of $\det(X)$ are derived as

$$\begin{aligned} E[\det(X)] &= \theta^m \det(\Omega) \\ &\cdot \prod_{j=1}^m \frac{[\nu - (j - 1)/2] [\beta - \nu - (j + 1)/2]}{[\alpha - \nu - (j + 1)/2]}, \end{aligned} \quad (46)$$

where $\beta > \nu + (m + 1)/2$, $\alpha > \nu + (m + 1)/2$, and

Now, evaluating the above integral by using (19), we obtain

$$\begin{aligned} E[C_\kappa(AX)] &= \frac{\theta^k \Gamma_m(\beta - \nu) (\nu)_\kappa}{\Gamma_m(\beta - \alpha) \Gamma_m(\alpha - \nu)} \\ &\cdot \int_0^{I_m} \det(Y)^{\alpha-\nu-(m+1)/2} \det(I_m - Y)^{\beta-\alpha-(m+1)/2} \\ &\cdot C_\kappa(\Omega^{1/2} A \Omega^{1/2} Y^{-1}) dY. \end{aligned} \quad (51)$$

Finally, evaluating the integral involving Y by using (Khatrzi [12])

$$\begin{aligned} &\int_0^{I_m} \det(R)^{a-(m+1)/2} \det(I_m - R)^{b-(m+1)/2} \\ &\cdot C_\kappa(XR^{-1}) dR \\ &= \frac{(-a - b + (m + 1)/2)_\kappa \Gamma_m(a) \Gamma_m(b)}{(-a + (m + 1)/2)_\kappa \Gamma_m(a + b)} C_\kappa(X), \end{aligned} \quad (52)$$

$\text{Re}(a) > \frac{m - 1}{2} + k_1, \text{Re}(b) > \frac{m - 1}{2},$

we get

$$\begin{aligned} E[C_\kappa(AX)] &= \frac{\theta^k (\nu - \beta + (m + 1)/2)_\kappa (\nu)_\kappa}{(\nu - \alpha + (m + 1)/2)_\kappa} C_\kappa(\Omega A), \end{aligned} \quad (53)$$

where $\text{Re}(\alpha - \nu) > (m - 1)/2 + k_1$ and $\text{Re}(\beta - \nu) > (m - 1)/2 + k_1$. Proceeding similarly and using the result (Khatrzi [12])

$$\begin{aligned} &\int_{R>0} \text{etr}(-XR) \det(R)^{a-(m+1)/2} C_\kappa(R^{-1}S) dR \\ &= \frac{(-1)^k \Gamma_m(a)}{(-a + (m + 1)/2)_\kappa} \det(X)^{-a} C_\kappa(XS), \end{aligned} \quad (54)$$

$\text{Re}(a) > \frac{m - 1}{2} + k_1,$

the expected value of $C_\kappa(AX^{-1})$ is derived as

$$\begin{aligned}
 E[C_\kappa(AX^{-1})] &= \frac{\det(\theta\Omega)^{-\nu} \Gamma_m(\beta - \nu)}{\Gamma_m(\nu) \Gamma_m(\beta - \alpha) \Gamma_m(\alpha - \nu)} \\
 &\cdot \int_0^{I_m} \det(Y)^{\alpha-(m+1)/2} \det(I_m - Y)^{\beta-\alpha-(m+1)/2} \\
 &\cdot \int_{X>0} C_\kappa(AX^{-1}) \det(X)^{\nu-(m+1)/2} \\
 &\cdot \text{etr}\left(-\frac{1}{\theta} \Omega^{-1/2} X \Omega^{-1/2} Y\right) dX dY \\
 &= \frac{(-1)^k \theta^{-k} \Gamma_m(\beta - \nu)}{\Gamma_m(\beta - \alpha) \Gamma_m(\alpha - \nu) (-\nu + (m+1)/2)_\kappa} \\
 &\cdot \int_0^{I_m} \det(Y)^{\alpha-\nu-(m+1)/2} \det(I_m - Y)^{\beta-\alpha-(m+1)/2} \\
 &\cdot C_\kappa(\Omega^{-1/2} A \Omega^{-1/2} Y) dY,
 \end{aligned} \tag{55}$$

where $\text{Re}(\nu) > (m-1)/2 + k_1$. Finally, evaluating the above integral using (20) we obtain

$$\begin{aligned}
 E[C_\kappa(AX^{-1})] &= \frac{(-1)^k \theta^{-k} (\alpha - \nu)_\kappa}{(\beta - \nu)_\kappa (-\nu + (m+1)/2)_\kappa} C_\kappa(\Omega^{-1} A), \\
 \text{Re}(\nu) &> \frac{m-1}{2} + k_1.
 \end{aligned} \tag{56}$$

In the next theorem, we derive the confluent hypergeometric function kind 1 distribution using independent beta and gamma matrices.

Theorem 6. Let X_1 and X_2 be independent, $X_1 \sim \text{Ga}(m, \nu, \theta)$ and $X_2 \sim \text{B1}(m, a, b)$. Then, $X_2^{-1/2} X_1 X_2^{1/2} \sim \text{CH}_m(\nu, a + \nu, a + b + \nu, \theta, \text{kind } 1)$.

Proof. See Gupta and Nagar [1]. $\square$

Theorem 7. Let X_1 and X_2 be independent, $X_1 \sim \text{Ga}(m, \nu, \theta)$ and $X_2 \sim \text{B1}(m, a, b)$. Then, $X_1^{1/2} X_2^{-1} X_1^{1/2} \sim \text{CH}_m(\nu, a + \nu, a + b + \nu, \theta, \text{kind } 1)$.

Proof. The result follows from Theorem 6 and the fact that $X_2^{-1/2} X_1 X_2^{1/2}$ and $X_1^{1/2} X_2^{-1} X_1^{1/2}$ have same eigenvalues, and the matrix variate confluent hypergeometric function kind 1 distribution is orthogonally invariant. $\square$

Theorem 8. Let X_1 and X_2 be independent, $X_1 \sim \text{Ga}(m, \nu, \theta)$ and $X_2 \sim \text{B1}(m, a, b)$. Then, $(I_m - X_2)^{-1/2} X_1 (I_m - X_2)^{-1/2} \sim \text{CH}_m(\nu, b + \nu, a + b + \nu, \theta, \text{kind } 1)$.

Proof. Noting that $I_m - X_2 \sim \text{B1}(m, b, a)$ and using Theorem 6 we get the result. $\square$

Theorem 9. Let X_1 and X_2 be independent, $X_1 \sim \text{Ga}(m, \nu, \theta)$ and $X_2 \sim \text{B2}(m, a, b)$. Then, $(I_m + X_2)^{1/2} X_1 (I_m + X_2)^{1/2} \sim \text{CH}_m(\nu, b + \nu, a + b + \nu, \theta, \text{kind } 1)$.

Proof. The desired result is obtained by observing that $(I_m + X_2)^{-1} \sim \text{B1}(m, b, a)$ and using Theorem 6. $\square$

Theorem 10. Let X_1 and X_2 be independent, $X_1 \sim \text{Ga}(m, \nu, \theta)$ and $X_2 \sim \text{B2}(m, a, b)$. Then, $(I_m + X_2^{-1})^{1/2} X_1 (I_m + X_2^{-1})^{1/2} \sim \text{CH}_m(\nu, a + \nu, a + b + \nu, \theta, \text{kind } 1)$.

Proof. Noting that $(I_m + X_2^{-1})^{-1} \sim \text{B1}(m, a, b)$ and using Theorem 6 we get the result. $\square$

Theorem 11. Let X_1 and X_2 be independent, $X_1 \sim \text{Ga}(m, \nu_1, \theta)$ and $X_2 \sim \text{Ga}(m, \nu_2, \theta)$. Then, $(X_1 + X_2) X_1^{-1} (X_1 + X_2) \sim \text{CH}_m(\nu_1 + \nu_2, 2\nu_1 + \nu_2, 2\nu_1 + 2\nu_2, \theta, \text{kind } 1)$.

Proof. It is well known that $X_1 + X_2$ and $(X_1 + X_2)^{-1/2} X_1 (X_1 + X_2)^{-1/2}$ are independent, $X_1 + X_2 \sim \text{Ga}(m, \nu_1 + \nu_2, \theta)$ and $(X_1 + X_2)^{-1/2} X_1 (X_1 + X_2)^{-1/2} \sim \text{B1}(m, \nu_1, \nu_2)$. Therefore, using Theorem 7, $(X_1 + X_2) X_1^{-1} (X_1 + X_2) \sim \text{CH}_m(\nu_1 + \nu_2, 2\nu_1 + \nu_2, 2\nu_1 + 2\nu_2, \theta, \text{kind } 1)$. $\square$

Theorem 12. Let X_1 and X_2 be independent, $X_1 \sim \text{Ga}(m, \nu_1, \theta)$ and $X_2 \sim \text{Ga}(m, \nu_2, \theta)$. Then, $(X_1 + X_2) X_2^{-1} (X_1 + X_2) \sim \text{CH}_m(\nu_1 + \nu_2, \nu_1 + 2\nu_2, 2\nu_1 + 2\nu_2, \theta, \text{kind } 1)$.

Proof. The proof is similar to the proof of Theorem 11. $\square$

4. Distributions of Sum and Quotients

In statistical distribution theory it is well known that if X_1 and X_2 are independent, $X_1 \sim \text{Ga}(m, \nu_1, \theta)$ and $X_2 \sim \text{Ga}(m, \nu_2, \theta)$, then $X_2^{-1/2} X_1 X_2^{-1/2} \sim \text{B2}(m, \nu_1, \nu_2)$, $(X_1 + X_2)^{-1/2} X_1 (X_1 + X_2)^{-1/2} \sim \text{B1}(m, \nu_1, \nu_2)$, and $X_1 + X_2 \sim \text{Ga}(m, \nu_1 + \nu_2)$. In this section we derive similar results when X_1 and X_2 are independent confluent hypergeometric function kind 1 and gamma matrices, respectively.

Theorem 13. Let X_1 and X_2 be independent, $X_1 \sim \text{CH}_m(\nu_1, \alpha_1, \beta_1, \theta, \text{kind } 1)$ and $X_2 \sim \text{Ga}(m, \nu_2, \theta)$. Then, the p.d.f. of $Z = X_2^{-1/2} X_1 X_2^{-1/2}$ is given by

$$\begin{aligned}
 &\frac{\Gamma_m(\alpha_1) \Gamma_m(\beta_1 - \nu_1) \Gamma_m(\nu_1 + \nu_2)}{\Gamma_m(\beta_1) \Gamma_m(\alpha_1 - \nu_1) \Gamma_m(\nu_1) \Gamma_m(\nu_2)} \frac{\det(Z)^{\nu_1-(m+1)/2}}{\det(I_m + Z)^{\nu_1+\nu_2}} \\
 &\cdot {}_2F_1\left(\beta_1 - \alpha_1, \nu_1 + \nu_2; \beta_1; (I_m + Z)^{-1} Z\right), \\
 &Z > 0.
 \end{aligned} \tag{57}$$

Proof. Using the independence, the joint p.d.f. of X_1 and X_2 is given by

$$\begin{aligned}
 &K \det(X_1)^{\nu_1-(m+1)/2} \det(X_2)^{\nu_2-(m+1)/2} \\
 &\cdot \text{etr}\left[-\frac{1}{\theta} (X_1 + X_2)\right] {}_1F_1\left(\beta_1 - \alpha_1; \beta_1; \frac{1}{\theta} X_1\right), \\
 &X_1 > 0, X_2 > 0,
 \end{aligned} \tag{58}$$

where

$$K = \frac{\theta^{-m(\nu_1+\nu_2)} \Gamma_m(\alpha_1) \Gamma_m(\beta_1 - \nu_1)}{\Gamma_m(\nu_1) \Gamma_m(\nu_2) \Gamma_m(\beta_1) \Gamma_m(\alpha_1 - \nu_1)}. \quad (59)$$

Making the transformation $Z = X_2^{-1/2} X_1 X_2^{-1/2}$, $X_2 = X_2$, with the Jacobian $J(X_1, X_2 \rightarrow Z, X_2) = \det(X_2)^{(m+1)/2}$ we obtain the joint p.d.f. of Z and X_2 as

$$K \det(Z)^{\nu_1-(m+1)/2} \det(X_2)^{\nu_1+\nu_2-(m+1)/2} \cdot \exp\left[-\frac{1}{\theta}(I_m + Z)X_2\right] {}_1F_1\left(\beta_1 - \alpha_1; \beta_1; \frac{1}{\theta}ZX_2\right), \quad (60)$$

$$Z > 0, \quad X_2 > 0.$$

Now, integrating X_2 in (60) by applying (21) and substituting for K , we obtain the desired result. $\square$

Corollary 14. Let X_1 and X_2 be independent, $X_1 \sim \text{CH}_m(\nu_1, \alpha_1, \beta_1, \theta, \text{kind } 1)$ and $X_2 \sim \text{Ga}(m, \nu_2, \theta)$. Then, the p.d.f. of $Z_1 = X_2^{1/2} X_1^{-1} X_2^{1/2}$ is given by

$$\frac{\Gamma_m(\alpha_1) \Gamma_m(\beta_1 - \nu_1) \Gamma_m(\nu_1 + \nu_2)}{\Gamma_m(\beta_1) \Gamma_m(\alpha_1 - \nu_1) \Gamma_m(\nu_1) \Gamma_m(\nu_2)} \cdot \frac{\det(Z_1)^{\nu_2-(m+1)/2}}{\det(I_m + Z_1)^{\nu_1+\nu_2}} \cdot {}_2F_1\left(\beta_1 - \alpha_1, \nu_1 + \nu_2; \beta_1; (I_m + Z_1)^{-1}\right), \quad Z_1 > 0. \quad (61)$$

Corollary 15. Let X_1 and X_2 be independent, $X_1 \sim \text{Ga}(m, \nu_1, \theta)$ and $X_2 \sim \text{CH}_m(\nu_2, \alpha_2, \beta_2, \theta, \text{kind } 1)$. Then, the p.d.f. of $Z_3 = X_2^{-1/2} X_1 X_2^{-1/2}$ is given by

$$\frac{\Gamma_m(\alpha_2) \Gamma_m(\beta_2 - \nu_2) \Gamma_m(\nu_1 + \nu_2)}{\Gamma_m(\beta_2) \Gamma_m(\alpha_2 - \nu_2) \Gamma_m(\nu_1) \Gamma_m(\nu_2)} \cdot \frac{\det(Z_3)^{\nu_1-(m+1)/2}}{\det(I_m + Z_3)^{\nu_1+\nu_2}} \cdot {}_2F_1\left(\beta_2 - \alpha_2, \nu_1 + \nu_2; \beta_2; (I_m + Z_3)^{-1}\right), \quad Z_3 > 0. \quad (62)$$

Proof. Interchanging subscripts 1 and 2 in Corollary 14, the p.d.f. of $Z_2 = X_1^{1/2} X_2^{-1} X_1^{1/2}$ is given by

$$\frac{\Gamma_m(\alpha_2) \Gamma_m(\beta_2 - \nu_2) \Gamma_m(\nu_1 + \nu_2)}{\Gamma_m(\beta_2) \Gamma_m(\alpha_2 - \nu_2) \Gamma_m(\nu_1) \Gamma_m(\nu_2)} \cdot \frac{\det(Z_2)^{\nu_1-(m+1)/2}}{\det(I_m + Z_2)^{\nu_1+\nu_2}} \cdot {}_2F_1\left(\beta_2 - \alpha_2, \nu_1 + \nu_2; \beta_2; (I_m + Z_2)^{-1}\right), \quad Z_2 > 0, \quad (63)$$

where $X_1 \sim \text{Ga}(m, \nu_1, \theta)$ and $X_2 \sim \text{CH}_m(\nu_2, \alpha_2, \beta_2, \theta, \text{kind } 1)$. Now, the result follows from the fact that $Z_2 = X_1^{1/2} X_2^{-1} X_1^{1/2}$ and $Z_3 = X_2^{-1/2} X_1 X_2^{-1/2}$ have the same eigenvalues, and the distribution Z_2 is orthogonally invariant. $\square$

Corollary 16. Let X_1 and X_2 be independent, $X_1 \sim \text{CH}_m(\nu_1, \alpha_1, \nu_1 + \nu_2, \theta, \text{kind } 1)$ and $X_2 \sim \text{Ga}(m, \nu_2, \theta)$. Then, $X_2^{-1/2} X_1 X_2^{-1/2} \sim \text{B2}(m, \nu_1, \alpha_1 - \nu_1)$.

Corollary 17. Let the random matrices X_1 and X_2 be independent, $X_1 \sim \text{Ga}(m, \nu_1, \theta)$ and $X_2 \sim \text{CH}_m(\nu_2, \alpha_2, \nu_2 + \nu_1, \theta, \text{kind } 1)$. Then, $X_2^{-1/2} X_1 X_2^{-1/2} \sim \text{B2}(m, \alpha_2 - \nu_2, \nu_2)$.

Corollary 18. Let X_1 and X_2 be independent, $X_1 \sim \text{Ga}(m, \nu_1, \theta)$ and $X_2 \sim \text{Ga}(m, \nu_2, \theta)$. Then, $X_2^{-1/2} X_1 X_2^{-1/2} \sim \text{B2}(m, \nu_1, \nu_2)$.

Theorem 19. Let X_1 , X_2 , and X_3 be independent, $X_1 \sim \text{Ga}(m, \mu, \theta)$, $X_2 \sim \text{B1}(m, a, b)$, and $X_3 \sim \text{Ga}(m, \nu, \theta)$. Then, the p.d.f. of $Z = X_3^{-1/2} X_2^{-1/2} X_1 X_2^{-1/2} X_3^{-1/2}$ is given by

$$\frac{\Gamma_m(a + \mu) \Gamma_m(a + b) \Gamma_m(\mu + \nu)}{\Gamma_m(a + b + \mu) \Gamma_m(a) \Gamma_m(\mu) \Gamma_m(\nu)} \frac{\det(Z)^{\mu-(m+1)/2}}{\det(I_m + Z)^{\mu+\nu}} \cdot {}_2F_1\left(b, \mu + \nu; a + b + \mu; (I_m + Z)^{-1} Z\right), \quad Z > 0. \quad (64)$$

Proof. Using the independence of X_1 and X_2 and Theorem 6, $X_2^{-1/2} X_1 X_2^{-1/2} \sim \text{CH}_m(\mu, a + \mu, a + b + \mu, \text{kind } 1)$. Further, using independence of $X_2^{-1/2} X_1 X_2^{-1/2}$ and X_3 and Theorem 13, we obtain the desired result. $\square$

Corollary 20. Let X_1 , X_2 , and X_3 be independent, $X_1 \sim \text{Ga}(m, \mu, \theta)$, $X_2 \sim \text{B1}(m, a, b)$, and $X_3 \sim \text{Ga}(m, a + b, \theta)$. Then, $X_3^{-1/2} X_2^{-1/2} X_1 X_2^{-1/2} X_3^{-1/2} \sim \text{B2}(m, \mu, a)$.

Proof. For $\nu = a + b$, the p.d.f. of $Z = X_3^{-1/2} X_2^{-1/2} X_1 X_2^{-1/2} X_3^{-1/2}$ given in the above theorem slides to

$$\frac{\Gamma_m(a + \mu)}{\Gamma_m(a) \Gamma_m(\mu)} \frac{\det(Z)^{\mu-(m+1)/2}}{\det(I_m + Z)^{\mu+a+b}} \cdot {}_2F_1\left(b, \mu + a + b; a + b + \mu; (I_m + Z)^{-1} Z\right), \quad Z > 0. \quad (65)$$

Now, simplifying the Gauss hypergeometric function as

$$\begin{aligned} & {}_2F_1\left(b, \mu + a + b; a + b + \mu; (I_m + Z)^{-1} Z\right) \\ &= {}_1F_0\left(b; (I_m + Z)^{-1} Z\right) \\ &= \det\left(I_m - (I_m + Z)^{-1} Z\right)^{-b} = \det(I_m + Z)^b, \end{aligned} \quad (66)$$

where we have used (16), the desired result is obtained. $\square$

Corollary 21. Let V and U be independent, $V \sim \text{B2}(m, \mu, \nu)$ and $U \sim \text{B1}(m, a, b)$. Then, the p.d.f. of $Z = U^{-1/2} V U^{-1/2}$ is given by

$$\frac{\Gamma_m(a + \mu) \Gamma_m(a + b) \Gamma_m(\mu + \nu)}{\Gamma_m(a + b + \mu) \Gamma_m(a) \Gamma_m(\mu) \Gamma_m(\nu)} \frac{\det(Z)^{\mu-(m+1)/2}}{\det(I_m + Z)^{\mu+\nu}} \cdot {}_2F_1\left(b, \mu + \nu; a + b + \mu; (I_m + Z)^{-1} Z\right), \quad Z > 0. \quad (67)$$

Further, if $\nu = a + b$, then $U^{-1/2} V U^{-1/2} \sim \text{B2}(m, \mu, a)$.

Proof. Observe that $Z = X_3^{-1/2} X_2^{-1/2} X_1 X_2^{-1/2} X_3^{-1/2}$ and $X_2^{-1/2} X_3^{-1/2} X_1 X_3^{-1/2} X_2^{-1/2}$ have same eigenvalues and the distribution of Z is orthogonally invariant. Therefore, the random matrices $X_3^{-1/2} X_2^{-1/2} X_1 X_2^{-1/2} X_3^{-1/2}$ and $X_2^{-1/2} X_3^{-1/2} X_1 X_3^{-1/2} X_2^{-1/2}$ have identical distribution. Now, setting $V = X_3^{-1/2} X_1 X_3^{-1/2}$ and $U = X_2$, where $X_3^{-1/2} X_1 X_3^{-1/2} \sim \text{B2}(m, \mu, \nu)$ and $X_2 \sim \text{B1}(m, a, b)$, we observe that $X_3^{-1/2} X_2^{-1/2} X_1 X_2^{-1/2} X_3^{-1/2}$ and $U^{-1/2} V U^{-1/2}$ have identical distribution. $\square$

Theorem 22. Let X_1 and X_2 be independent, $X_1 \sim \text{CH}_m(\nu_1, \alpha_1, \beta_1, \theta, \text{kind } 1)$ and $X_2 \sim \text{Ga}(m, \nu_2, \theta)$. Then, the p.d.f. of $R = (X_1 + X_2)^{-1/2} X_1 (X_1 + X_2)^{-1/2}$ is given by

$$\begin{aligned} & \frac{\Gamma_m(\alpha_1) \Gamma_m(\beta_1 - \nu_1) \Gamma_m(\nu_1 + \nu_2)}{\Gamma_m(\beta_1) \Gamma_m(\alpha_1 - \nu_1) \Gamma_m(\nu_1) \Gamma_m(\nu_2)} \det(R)^{\nu_1 - (m+1)/2} \\ & \cdot \det(I_m - R)^{\nu_2 - (m+1)/2} \\ & \cdot {}_2F_1(\beta_1 - \alpha_1, \nu_1 + \nu_2; \beta_1; R), \quad 0 < R < I_m \end{aligned} \quad (68)$$

and the p.d.f. of $S = X_1 + X_2$ is given by

$$\begin{aligned} & \frac{\theta^{-m(\nu_1 + \nu_2)} \Gamma_m(\alpha_1) \Gamma_m(\beta_1 - \nu_1)}{\Gamma_m(\beta_1) \Gamma_m(\alpha_1 - \nu_1) \Gamma_m(\nu_1 + \nu_2)} \det(S)^{\nu_1 + \nu_2 - (m+1)/2} \\ & \cdot \text{etr}\left(-\frac{1}{\theta} S\right) {}_2F_2\left(\nu_1, \beta_1 - \alpha_1; \nu_1 + \nu_2, \beta_1; \frac{1}{\theta} S\right), \quad (69) \\ & S > 0. \end{aligned}$$

Proof. Substituting $R = (X_1 + X_2)^{-1/2} X_1 (X_1 + X_2)^{-1/2}$ and $S = X_1 + X_2$ with the Jacobian $J(X_1, X_2 \rightarrow R, S) = \det(S)^{(m+1)/2}$ in (58) we obtain the joint p.d.f. of R and S as

$$\begin{aligned} & K \det(R)^{\nu_1 - (m+1)/2} \det(I_m - R)^{\nu_2 - (m+1)/2} \\ & \cdot \det(S)^{\nu_1 + \nu_2 - (m+1)/2} \text{etr}\left(-\frac{1}{\theta} S\right) \\ & \cdot {}_1F_1\left(\beta_1 - \alpha_1; \beta_1; \frac{1}{\theta} RS\right), \quad 0 < R < I_m, S > 0. \end{aligned} \quad (70)$$

Now, integration of S by using (21) yields the density of R . The marginal density of S is obtained by integrating R by using (22). $\square$

It may be remarked here that the density of R given in the above theorem can also be obtained from the density of $Z = X_2^{-1/2} X_1 X_2^{-1/2}$ derived in Theorem 13 by making the transformation $R = (I_m + Z)^{-1} Z$.

Corollary 23. Let X_1 and X_2 be independent random matrices, $X_1 \sim \text{Ga}(m, \nu_1, \theta)$ and $X_2 \sim \text{CH}_m(\nu_2, \alpha_2, \beta_2, \theta, \text{kind } 1)$.

Then, the p.d.f. of $R_1 = (X_1 + X_2)^{-1/2} X_1 (X_1 + X_2)^{-1/2}$ is given by

$$\begin{aligned} & \frac{\Gamma_m(\alpha_2) \Gamma_m(\beta_2 - \nu_2) \Gamma_m(\nu_1 + \nu_2)}{\Gamma_m(\beta_2) \Gamma_m(\alpha_2 - \nu_2) \Gamma_m(\nu_1) \Gamma_m(\nu_2)} \det(R_1)^{\nu_1 - (m+1)/2} \\ & \cdot \det(I_m - R_1)^{\nu_2 - (m+1)/2} \\ & \cdot {}_2F_1(\beta_2 - \alpha_2, \nu_1 + \nu_2; \beta_2; I_m - R_1), \quad 0 < R_1 < I_m \end{aligned} \quad (71)$$

and the p.d.f. of $S_1 = X_1 + X_2$ is given by

$$\begin{aligned} & \frac{\theta^{-m(\nu_1 + \nu_2)} \Gamma_m(\alpha_2) \Gamma_m(\beta_2 - \nu_2)}{\Gamma_m(\beta_2) \Gamma_m(\alpha_2 - \nu_2) \Gamma_m(\nu_1 + \nu_2)} \det(S_1)^{\nu_1 + \nu_2 - (m+1)/2} \\ & \cdot \text{etr}\left(-\frac{1}{\theta} S_1\right) {}_2F_2\left(\nu_2, \beta_2 - \alpha_2; \nu_1 + \nu_2, \beta_2; \frac{1}{\theta} S_1\right), \quad (72) \\ & S_1 > 0. \end{aligned}$$

Proof. Interchanging subscripts 1 and 2 in Theorem 22, the p.d.f. of $R_2 = (X_1 + X_2)^{-1/2} X_2 (X_1 + X_2)^{-1/2}$ is given by

$$\begin{aligned} & \frac{\Gamma_m(\alpha_2) \Gamma_m(\beta_2 - \nu_2) \Gamma_m(\nu_1 + \nu_2)}{\Gamma_m(\beta_2) \Gamma_m(\alpha_2 - \nu_2) \Gamma_m(\nu_1) \Gamma_m(\nu_2)} \det(R_2)^{\nu_2 - (m+1)/2} \\ & \cdot \det(I_m - R_2)^{\nu_2 - (m+1)/2} \\ & \cdot {}_2F_1(\beta_2 - \alpha_2, \nu_1 + \nu_2; \beta_2; R_2), \quad 0 < R_2 < I_m, \end{aligned} \quad (73)$$

where now $X_1 \sim \text{Ga}(m, \nu_1, \theta)$ and $X_2 \sim \text{CH}_m(\nu_2, \alpha_2, \beta_2, \theta, \text{kind } 1)$. The desired result is now obtained by observing that $R_2 = I_m - R_1$. Similarly, the p.d.f. of S_1 is obtained by interchanging subscripts 1 and 2 in the p.d.f. of S . $\square$

Corollary 24. Let the random matrices X_1 and X_2 be independent, $X_1 \sim \text{CH}_m(\nu_1, \alpha_1, \nu_1 + \nu_2, \theta, \text{kind } 1)$ and $X_2 \sim \text{Ga}(m, \nu_2, \theta)$. Then, $(X_1 + X_2)^{-1/2} X_1 (X_1 + X_2)^{-1/2} \sim \text{B1}(m, \nu_1, \alpha_1 - \nu_1)$.

Proof. The desired result is obtained by substituting $\beta_1 = \nu_1 + \nu_2$ in the p.d.f. of R and simplifying the resulting expression by using (16). $\square$

Corollary 25. Let the random matrices X_1 and X_2 be independent, $X_1 \sim \text{Ga}(m, \nu_1, \theta)$ and $X_2 \sim \text{CH}(m, \nu_2, \alpha_2, \nu_1 + \nu_2, \theta, \text{kind } 1)$. Then, $(X_1 + X_2)^{-1/2} X_1 (X_1 + X_2)^{-1/2} \sim \text{B1}(m, \alpha_2 - \nu_2, \nu_2)$.

Proof. The desired result is obtained by substituting $\beta_2 = \nu_1 + \nu_2$ in the p.d.f. of R_1 and simplifying the resulting expression by using (16). $\square$

Corollary 26. Let the random matrices X_1 and X_2 be independent, $X_1 \sim \text{CH}_m(\nu_1, \alpha_1, \alpha_1 + \nu_1 + \nu_2, \theta, \text{kind } 1)$ and $X_2 \sim \text{Ga}(m, \nu_2, \theta)$. Then $X_1 + X_2 \sim \text{CH}_m(\nu_1 + \nu_2, \alpha_1 + \nu_2, \alpha_1 + \nu_1 + \nu_2, \theta, \text{kind } 1)$.

Proof. The result is obtained by substituting $\beta_1 = \alpha_1 + \nu_1 + \nu_2$ in the p.d.f. of S and simplifying the resulting expression by using (15). $\square$

Corollary 27. Let the random matrices X_1 and X_2 be independent, $X_1 \sim \text{Ga}(m, \nu_1, \theta)$ and $X_2 \sim \text{CH}_m(\nu_2, \alpha_2, \alpha_2 + \nu_1 + \nu_2, \theta, \text{kind } 1)$. Then $X_1 + X_2 \sim \text{CH}_m(\nu_1 + \nu_2, \alpha_2 + \nu_1, \alpha_2 + \nu_1 + \nu_2, \theta, \text{kind } 1)$.

Proof. The result is obtained by substituting $\beta_2 = \alpha_2 + \nu_1 + \nu_2$ in the p.d.f. of S and simplifying the resulting expression by using (15). $\square$

Corollary 28. Let the random matrices X_1 and X_2 be independent, $X_1 \sim \text{Ga}(m, \nu_1, \theta)$ and $X_2 \sim \text{Ga}(m, \nu_2, \theta)$. Then, $(X_1 + X_2)^{-1/2} X_1 (X_1 + X_2)^{-1/2} \sim \text{B1}(m, \nu_1, \nu_2)$ and $X_1 + X_2 \sim \text{Ga}(m, \nu_1 + \nu_2, \theta)$.

Proof. Substitute $\beta_1 = \alpha_1$ in the p.d.f. of R or $\beta_2 = \alpha_2$ in the p.d.f. of R_1 and simplify the resulting expression to get the desired result. $\square$

5. Related Distributions

This section gives distributional results for the determinant of the random matrix distributed as confluent hypergeometric function kind 1.

In an unpublished report, Coelho et al. [13] have shown that if z is a positive random variable and $E(z^h)$ is defined for h in some neighborhood of zero, then the moments $E(z^h)$ uniquely identify the distribution of z . In the next theorem, we will use this result to derive the distribution of the product of two independent confluent hypergeometric function kind 1 variables.

Theorem 29. If $x_1 \sim \text{CH}(\nu, \alpha, \beta, \theta, \text{kind } 1)$ and $x_2 \sim \text{CH}(\nu + 1/2, \alpha + 1, \beta + 1, \theta, \text{kind } 1)$ are independent, then $2\sqrt{x_1 x_2} \sim \text{CH}(2\nu, 2\alpha, 2\beta, \theta, \text{kind } 1)$.

Proof. The h th moment of $2\sqrt{x_1 x_2}$ is derived as

$$\begin{aligned} E \left[(2\sqrt{x_1 x_2})^h \right] &= \frac{(2\theta)^h \Gamma(\beta - \nu) \Gamma(\nu + h/2) \Gamma(\alpha - \nu - h/2)}{\Gamma(\nu) \Gamma(\alpha - \nu) \Gamma(\beta - \nu - h/2)} \\ &\cdot \frac{\Gamma(\beta - \nu + 1/2) \Gamma(\nu + 1/2 + h/2) \Gamma(\alpha - \nu + 1/2 - h/2)}{\Gamma(\nu + 1/2) \Gamma(\alpha - \nu + 1/2) \Gamma(\beta - \nu + 1/2 - h/2)}. \end{aligned} \quad (74)$$

Now, using the duplication formula for gamma function, namely,

$$\Gamma(2z) = \frac{\Gamma(z) \Gamma(z + 1/2)}{2^{1-2z} \sqrt{\pi}}, \quad (75)$$

the h th moment of $2\sqrt{x_1 x_2}$ is rewritten as

$$\begin{aligned} E \left[(2\sqrt{x_1 x_2})^h \right] &= \frac{\theta^h \Gamma(2\beta - 2\nu) \Gamma(2\nu + h) \Gamma(2\alpha - 2\nu - h)}{\Gamma(2\nu) \Gamma(2\alpha - 2\nu) \Gamma(2\beta - 2\nu - h)}, \end{aligned} \quad (76)$$

where $\nu > 0$, $\alpha - \nu > 0$, $\beta - \nu > 0$, $2\nu > h$, $2(\alpha - \nu) > h$, and $2(\beta - \nu) > h$.

Finally, comparison of the above expression with the one given in (45) yields the desired result. $\square$

Theorem 30. If $X \sim \text{CH}_m(\nu, \alpha, \beta, \Omega, \theta, \text{kind } 1)$, then $\det(\Omega^{-1/2} X \Omega^{-1/2})$ is distributed as $\prod_{i=1}^m z_i$, where $z_1, \dots, z_m$ are independent, $z_i \sim \text{CH}(\nu - (i-1)/2, \alpha - i + 1, \beta - i + 1, \theta, \text{kind } 1)$, $i = 1, \dots, m$.

Proof. Writing multivariate gamma functions in terms of ordinary gamma function, (42) is rewritten as

$$\begin{aligned} E \left[\det(\Omega^{-1/2} X \Omega^{-1/2})^h \right] &= \prod_{i=1}^m \left[\frac{\theta^h \Gamma[\beta - \nu - (i-1)/2]}{\Gamma[\nu - (i-1)/2] \Gamma[\alpha - \nu - (i-1)/2]} \right. \\ &\cdot \left. \frac{\Gamma[\nu - (i-1)/2 + h] \Gamma[\alpha - \nu - (i-1)/2 - h]}{\Gamma[\beta - \nu - (i-1)/2 - h]} \right]. \end{aligned} \quad (77)$$

Now, comparing the above expression with (45), we get $E[\det(\Omega^{-1/2} X \Omega^{-1/2})^h] = \prod_{i=1}^m E(z_i^h)$. $\square$

Corollary 31. If $X \sim \text{CH}_2(\nu, \alpha, \beta, \theta, \text{kind } 1)$, then

$$2\det(X)^{1/2} \sim \text{CH}(2\nu - 1, 2\alpha - 2, 2\beta - 2, \theta, \text{kind } 1). \quad (78)$$

Proof. For $m = 2$, $\det(X)$ is distributed as $z_1 z_2$, where z_1 and z_2 are independent, $z_1 \sim \text{CH}(\nu, \alpha, \beta, \theta, \text{kind } 1)$ and $z_2 \sim \text{CH}(\nu - 1/2, \alpha - 1, \beta - 1, \theta, \text{kind } 1)$. From Theorem 29, we have $2\sqrt{z_1 z_2} \sim \text{CH}(2\nu - 1, 2\alpha - 2, 2\beta - 2, \theta, \text{kind } 1)$. $\square$

Corollary 32. If $X \sim \text{Ga}(m, \nu, \theta, \Omega)$, then $\det(\Omega^{-1/2} X \Omega^{-1/2})$ is distributed as $\prod_{i=1}^m z_i$, where $z_1, \dots, z_m$ are independent, $z_i \sim \text{Ga}(\nu - (i-1)/2, \theta)$, $i = 1, \dots, m$.

6. Distribution of Eigenvalues

In this section, we derive density of eigenvalues of random matrix distributed as confluent hypergeometric function kind 1.

Theorem 33. Let A be a positive definite random matrix of order m with the p.d.f. $f(A)$. Then, the joint p.d.f. of the eigenvalues $l_1, l_2, \dots, l_m$ of A is given by

$$\frac{\pi^{m^2/2}}{\Gamma_m(m/2)} \prod_{i < j}^m (l_i - l_j) \int_{O(m)} f(HLH') (dH), \quad (79)$$

where $l_1 > l_2 > \dots > l_m > 0$, $L = \text{diag}(l_1, l_2, \dots, l_m)$, and (dH) is the unit invariant Haar measure on the group of orthogonal matrices $O(m)$.

Proof of Theorem 33 and several other results can be found in Muirhead [7].

Theorem 34. If $X \sim \text{CH}_m(\nu, \alpha, \beta, \theta, \Omega, \text{kind } 1)$, then the joint p.d.f. of the eigenvalues $x_1, x_2, \dots, x_m$ of X is given by

$$\frac{\pi^{m^2/2} \Gamma_m(\alpha) \Gamma_m(\beta - \nu) \det(\theta \Omega)^{-\nu}}{\Gamma_m(m/2) \Gamma_m(\nu) \Gamma_m(\beta) \Gamma_m(\alpha - \nu)} \left[\prod_{i < j}^m (x_i - x_j) \right] \cdot \prod_{i=1}^m \left[x_i^{\nu-(m+1)/2} \right] {}_1F_1^{(m)} \left(\alpha; \beta; -\frac{1}{\theta} \Omega^{-1}, L \right), \quad (80)$$

where $0 < x_m < \dots < x_1 < \infty$, $L = \text{diag}(x_1, \dots, x_m)$, and ${}_1F_1^{(m)}$ is the two-matrix argument confluent hypergeometric function.

Proof. The p.d.f. of X is given by (5). Applying Theorem 33, we obtain the joint p.d.f. of the eigenvalues $x_1, \dots, x_m$ of X as

$$\frac{\pi^{m^2/2} \Gamma_m(\alpha) \Gamma_m(\beta - \nu) \det(\theta \Omega)^{-\nu}}{\Gamma_m(m/2) \Gamma_m(\nu) \Gamma_m(\beta) \Gamma_m(\alpha - \nu)} \left[\prod_{i < j}^m (x_i - x_j) \right] \cdot \prod_{i=1}^m \left[x_i^{\nu-(m+1)/2} \right] \cdot \int_{O(m)} {}_1F_1^{(m)} \left(\alpha; \beta; -\frac{1}{\theta} \Omega^{-1} H L H' \right) (dH). \quad (81)$$

Now, using (28), we obtain the desired result. $\square$

7. A Generalized Form

In this section, we give a more general form of the matrix variate confluent hypergeometric function kind 1 distribution by introducing an additional factor $\text{etr}(-\Psi^{-1} X / \theta)$ in the p.d.f. (5). The p.d.f. of X , in this case, is given by

$$\frac{\theta^{-m\nu} \det(\Psi^{-1} + \Omega^{-1})^\nu}{\Gamma_m(\nu) {}_2F_1(\nu, \beta - \alpha; \beta; (\Psi^{-1} + \Omega^{-1})^{-1} \Omega^{-1})} \text{etr} \left(-\frac{1}{\theta} X \right) \cdot \det(X)^{\nu-(m+1)/2} {}_1F_1 \left(\alpha; \beta; -\frac{1}{\theta} \Omega^{-1} X \right), \quad (82)$$

where $X > 0$. We will write $X \sim \text{CH}_m(\nu, \alpha, \beta, \theta, \Omega, \Psi, \text{kind } 1)$ if the density of X is given by (82). For $\Psi = I_m$ and $\theta = 1$, the above p.d.f. reduces to

$$\frac{\det(I_m + \Omega^{-1})^\nu}{\Gamma_m(\nu) {}_2F_1(\nu, \beta - \alpha; \beta; (I_m + \Omega^{-1})^{-1})} \text{etr}(-X) \cdot \det(X)^{\nu-(m+1)/2} {}_1F_1(\alpha; \beta; -\Omega^{-1} X), \quad X > 0, \quad (83)$$

which is a special case of the generalized hypergeometric function density defined by Roux [14].

Theorem 35. Let $Z \mid \Sigma \sim \text{InvGa}(m, \mu, \theta, \Sigma^{-1})$. Further, let the prior distribution of Σ be a generalized matrix variate confluent hypergeometric kind 1 distribution with parameters $\nu, \alpha, \beta, \theta, \Omega$, and $\Psi, \Sigma \sim \text{CH}_m(\nu, \alpha, \beta, \theta, \Omega, \Psi, \text{kind } 1)$. Then,

the marginal distribution of Z is a generalized inverted matrix variate beta with the density

$$m(Z) = \frac{\Gamma_m(\mu + \nu) \det(\Psi^{-1} + \Omega^{-1})^\nu}{\Gamma_m(\nu) \Gamma_m(\mu) {}_2F_1(\nu, \beta - \alpha; \beta; (\Psi^{-1} + \Omega^{-1})^{-1} \Omega^{-1})} \cdot \frac{\det(Z)^{-\mu-(m+1)/2}}{\det(\Psi^{-1} + \Omega^{-1} + Z^{-1})^{\mu+\nu}} \cdot {}_2F_1(\mu + \nu, \beta - \alpha; \beta; (\Psi^{-1} + \Omega^{-1} + Z^{-1})^{-1} \Omega^{-1}), \quad (84)$$

where $Z > 0$.

Proof. By definition, the marginal density of Z , denoted by $m(Z)$, is obtained as

$$m(Z) = \int_{\Sigma > 0} f(Z \mid \Sigma) \pi(\Sigma) d\Sigma. \quad (85)$$

Now, substituting for $f(Z \mid \Sigma)$ and $\pi(\Sigma)$, we get

$$m(Z) = \frac{\theta^{-m(\nu+\mu)} \det(\Psi^{-1} + \Omega^{-1})^\nu}{\Gamma_m(\nu) \Gamma_m(\mu) {}_2F_1(\nu, \beta - \alpha; \beta; (\Psi^{-1} + \Omega^{-1})^{-1} \Omega^{-1})} \cdot \det(Z)^{-\mu-(m+1)/2} \int_{\Sigma > 0} \det(\Sigma)^{\mu+\nu-(m+1)/2} \cdot \text{etr} \left[-\frac{1}{\theta} (\Psi^{-1} + \Omega^{-1} + Z^{-1}) \Sigma \right] \cdot {}_1F_1 \left(\beta - \alpha; \beta; \frac{1}{\theta} \Omega^{-1} \Sigma \right) d\Sigma. \quad (86)$$

Finally, evaluating the above expression by using (21) and simplifying, we get the desired result. $\square$

Theorem 36. Let $Z \mid \Sigma \sim \text{InvGa}(m, \mu, \Sigma^{-1})$. Further, let the prior distribution of Σ be a generalized matrix variate confluent hypergeometric function kind 1 distribution parameters $\nu, \alpha, \beta, \theta, \Omega$, and $\Psi, \Sigma \sim \text{CH}_m(\nu, \alpha, \beta, \theta, \Omega, \Psi, \text{kind } 1)$. Then, the posterior distribution of Σ is a generalized matrix variate confluent hypergeometric kind 1 distribution with parameters $\mu + \nu, \alpha, \beta, \theta, \Omega$, and $(\Psi^{-1} + Z^{-1})^{-1}$.

Proof. By definition and Theorem 35, we have

$$\pi(\Sigma \mid Z) = \frac{f(Z \mid \Sigma) \pi(\Sigma)}{m(Z)}. \quad (87)$$

Now, substituting appropriately, we get

$$\pi(\Sigma \mid Z) = \frac{\theta^{-m(\nu+\mu)} \det(\Psi^{-1} + \Omega^{-1} + Z^{-1})^{\nu+\mu}}{\Gamma_m(\nu + \mu) {}_2F_1(\nu + \mu, \beta - \alpha; \beta; (\Psi^{-1} + \Omega^{-1} + Z^{-1})^{-1} \Omega^{-1})} \cdot \text{etr} \left[-\frac{1}{\theta} (\Psi^{-1} + Z^{-1}) \Sigma \right] \det(\Sigma)^{\nu+\mu-(m+1)/2} \cdot {}_1F_1 \left(\alpha; \beta; -\frac{1}{\theta} \Omega^{-1} \Sigma \right), \quad \Sigma > 0, \quad (88)$$

which is the desired result. $\square$

From the above results it is quite clear that the generalized matrix variate confluent hypergeometric function kind 1 distribution as the prior distribution is conjugate. Thus, this distribution may be used as an alternative to matrix variate gamma distribution.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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