Research Article

Polyhedral Star-Shaped Distributions

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A new method of probabilistic modelling of polyhedrally contoured sample clouds is presented and applied to statistical reasoning for a real dataset. Various representations of the new class of polyhedral star-shaped distributions are derived and basic properties of the moments as well as characteristic and moment generating functions of these distributions are studied. Along with location-scale transformations, estimating and hypothesis testing are dealt with.

1. Introduction

One of nowadays challenges of statistical modelling is the construction of flexible multivariate probability distributions given a dataset. In [1], disparities in premature mortality between high- and low-income US counties are dealt with on the basis of descriptive statistics. For a subsequent step of statistical reasoning, a method of constructing a suitable probabilistic model is needed. Analyzing the correlation between mortality and income cannot be done in a common linear regression model because of absence of homoscedasticity. The present paper provides therefore a new method of constructing flexible multivariate distribution laws being well adapted to practical problems characterized by polyhedral contours of their sample clouds. Moreover, we will particularly suggest a specific model and further statistical reasoning for the premature mortality-income data mentioned above.

Several basic methods and results from the research area of constructing multivariate distributions are surveyed, for example, in [2] and the numerous papers mentioned there. Multivariate densities with given contours were already introduced in [3] and further studied in [4, 5]. Constructing star-shaped distributions on using Minkowski functionals and generalized uniformly distributed random vectors on generalized spheres, proving geometric measure representations of such distributions and stochastic representations of correspondingly distributed random vectors are to be found in [6]. Constructions and representations in the special cases

of norm and antinorm contoured distributions are recently dealt with in [7]. Numerous applications of geometric measure representations are surveyed in the last mentioned two papers and in [8]. In [9] more recent applications to extreme values are presented. More general order statistics are dealt with elsewhere.

The aim of the present paper is to deal with another important special case of star-shaped distributions where the contours are the topological boundaries of polyhedra.

Polyhedral star-shaped distributions can be considered being a subclass of the class of star-shaped distributions. Before studying specific properties of polyhedral star-shaped distributions, we start therefore with a short introduction to the general theory of star-shaped distributions. To this end, we follow [6].

Let $K \subset \mathbb{R}^n$ denote a star body, that is, a nonempty and compact star-shaped set being equal to the closure of its interior and having the origin 0_n in its interior. The functional $h_K : \mathbb{R}^n \rightarrow [0, \infty)$ defined by

$$h_K(x) = \inf \{r > 0 : x \in rK\}, \quad x \in \mathbb{R}^n \quad (1)$$

is known as the Minkowski functional of K . Here, we assume that h_K is positive homogeneous; that is, $h_K(\lambda x) = \lambda h_K(x)$, $\lambda > 0$, and consider $K(r) = rK = \{x \in \mathbb{R}^n : h_K(x) \leq r\}$ and its boundary $S(r) = rS = \{x \in \mathbb{R}^n : h_K(x) = r\}$ being the star ball and star sphere of Minkowski radius $r > 0$, respectively. Since h_K unambiguously defines the considered

star ball, it is possible to study subclasses of star bodies by specifying h_K . One can choose, for example, h_K to be a norm or an antinorm. For the latter notion we refer to [10]. Here, specific representations of h_K are considered later in this paper, if h_K denotes a star-shaped polyhedron. For simplicity, consideration will be restricted throughout this paper to star bodies having the following property.

A countable collection $\mathfrak{F} = \{C_1, C_2, \dots\}$ of pairwise disjoint cones C_j with vertex being the origin 0_n and $\mathbb{R}^n = \bigcup_j C_j$ is called a fan. Let $S_j = S \cap C_j$, $S_j \cap \mathfrak{B}_n = \mathfrak{B}_{S,j}$, where $\mathfrak{B}_n$ denotes the Borel- σ -field in $\mathbb{R}^n$, and $\mathfrak{B}_S = \sigma\{\mathfrak{B}_{S,1}, \mathfrak{B}_{S,2}, \dots\}$. In what follows, the star body K and a set $A \in \mathfrak{B}_S$ are chosen such that for every j the set

$$G(A \cap S_j) = \{\vartheta \in \mathbb{R}^{n-1} : \exists \eta \text{ with } \theta = (\vartheta^T, \eta)^T \in A \cap S_j\} \quad (2)$$

is well defined and for every $\vartheta = (\vartheta_1, \dots, \vartheta_{n-1})^T \in G(A \cap S_j)$ there is uniquely determined $\eta > 0$ satisfying $h_K((\vartheta_1, \dots, \vartheta_{n-1}, \eta)^T) = 1$.

A function $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying the assumptions $0 < \mathcal{J}(g) < \infty$ where $\mathcal{J}(g) = \int_0^\infty r^{n-1} g(r) dr$ is called a density generating function (dgf),

$$\varphi_{g,K}(x) = C(g, K) g(h_K(x)), \quad x \in \mathbb{R}^n \quad (3)$$

a star-shaped density and K its contour defining star body. Such densities are studied in [3, 6, 11]. A probability measure having the density $\varphi_{g,K}$ will be denoted by $\Phi_{g,K}$. Note that the normalizing constant $C(g, K)$ allows the representation

$$C(g, K) = \frac{1}{n\mu(K) \mathcal{J}(g)}, \quad (4)$$

where μ denotes the Lebesgue measure in $\mathbb{R}^n$. For examples of density generating functions, see [12] or Table 1. Moreover, the definition of specific star-shaped densities appears already in earlier work where h_K is prespecified. If h_K is the Euclidean-norm, one considers the class of spherically symmetric distributions; see [12–14] and many other contributions. The class of convex contoured $l_{n,p}$ -symmetric distributions is considered if h_K is the $l_{n,p}$ -norm, $p \geq 1$. This class of distributions was introduced in [15] and studied, for example, in [16–19]. In [20] stochastic and geometric representations are derived also for the case which can be called according to [10] the antinorm contoured case. General norm contoured distributions in $\mathbb{R}^2$ are studied in [8] by choosing h_K in terms of a norm in $\mathbb{R}^2$. Minkowski functionals of ellipsoids and even p -generalized ellipsoids are used in the (re-)construction of common and p -generalized elliptically contoured distributions in [21] and in [6], respectively.

The introduction of h_K as generalized radius functional is closely connected with the notion of generalized surface content measure which turns out to be a suitably chosen non-Euclidean surface content measure on a star sphere. These notions and their properties are very useful in the consideration of nonspherical distributions; see [6, 7, 20, 21].

Here, we recall the local definition of a generalized surface content measure on star spheres from [22], noting that there exists an equivalent integral approach introduced for various special classes of star bodies in [6, 7, 20, 21]. For $A \in \mathfrak{B}_S$, we introduce the central projection cone $\text{CPC}(A) = \{x \in \mathbb{R}^n : x/h_K(x) \in A\}$ and the star sector of star radius $\rho > 0$, $\text{sector}(A, \rho) = \text{CPC}(A) \cap K(\rho)$. The star-generalized surface measure is then defined by

$$\mathfrak{D}_S(A) = f'(\rho), \quad (5)$$

$$\text{where } f(\rho) = \mu(\text{sector}(A, \rho)), \quad A \in \mathfrak{B}_S.$$

The star-generalized uniform probability distribution on the Borel- σ -field $\mathfrak{B}_S$ is defined as $\omega_S(A) = \mathfrak{D}_S(A)/\mathfrak{D}_S(S)$. With these notations, we can recall the geometric measure representation formula of $\Phi_{g,K}(B)$ for every $B \in \mathfrak{B}_n$,

$$\begin{aligned} \Phi_{g,K}(B) &= C(g, K) \int_0^\infty r^{n-1} g(r) \mathfrak{D}_S\left(\left[\frac{1}{r}B\right] \cap S\right) dr \\ &= C(g, K) \mathfrak{D}_S(S) \int_0^\infty r^{n-1} g(r) \mathfrak{F}_S(B, r) dr, \end{aligned} \quad (6)$$

where $r \rightarrow \mathfrak{F}_S(B, r) = \omega_S([(1/r)B] \cap S)$ denotes the star intersection-proportion function (ipf) of the set B . Furthermore, a star-shaped distributed random vector $Y \sim \Phi_{g,K}$ allows the stochastic representation

$$Y \stackrel{d}{=} R_g U_S, \quad (7)$$

where R_g and U_S are stochastically independent, $U_S \sim \omega_S$, and R_g follows the density $f(r) = r^{n-1} g(r)/\mathcal{J}(g)$, $r > 0$. Furthermore, we recall from [7] integral representations of $\mathfrak{D}_S(A)$, $A \in \mathfrak{B}_S^+$, where $\mathfrak{B}_S^+$ denotes the Borel- σ -field on the upper half-sphere of S , if h_K is a norm and an antinorm, respectively. It holds

$$\begin{aligned} \mathfrak{D}_S(A) &= \int_{G(A)} h_{K^*}(N(\vartheta)) d\vartheta, \\ \mathfrak{D}_S(A) &= \int_{G(A)} h_{K^*}(N^\circ(\vartheta)) d\vartheta, \end{aligned} \quad (8)$$

where $G(A) = \{\vartheta \in \mathbb{R}^{n-1} : \exists \eta = \eta(\vartheta) \text{ with } (\vartheta^T, \eta)^T \in A\}$, h_{K^*} denotes the dual norm of h_K , if h_K is a norm, and h_{K^*} denotes the Minkowski functional of the antipolar set K° of K if h_K is an antinorm with K being an element of the class of antinorm balls AN1 defined in [7]. Here, $N(\vartheta)$ is the outer normal vector to S at $(\vartheta^T, \eta)^T$ and $N^\circ(\vartheta)$ is the inner normal vector to S at $(\vartheta^T, \eta)^T$, respectively. For particular representations of h_{K^*} and h_{K^*} if K is a norm or antinorm, respectively, generated by a star-shaped polyhedron, we refer already here to Theorem 6 (d) and (e) below. In this context, we will say that K belongs to the class of antinorm balls AN2 if h_K is an antinorm and $K = \mathbb{P}$ is a star-shaped polyhedron. We notice that convex geometry and certain types of polytopes play also a basic role in [23] where max-stable distributions and all norms that give rise to strictly max-stable distributions are studied.

TABLE 1: Density generating functions, normalizing constants, and moments of R_g .

Type	$g(r)$	$\mathcal{J}(g)$	$\mathbb{E}(R_g)$	$\mathbb{E}(R_g^2)$	$\mathbb{E}(R_g^k)$
Kotz	$r^{2M-2} e^{-\beta r^{2\gamma}}, \quad M, \beta, \gamma \in (0, \infty)$	$\frac{\Gamma((2M+n-2)/2\gamma)}{2\gamma\beta^{(2M+n-2)/2\gamma}}$	$\frac{\Gamma((2M+n-1)/2\gamma)}{\beta^{1/2\gamma}\Gamma((2M+n-2)/2\gamma)}$	$\frac{\Gamma((2M+n)/2\gamma)}{\beta^{1/2\gamma}\Gamma((2M+n-2)/2\gamma)}$	$\frac{\Gamma((2M+n+k-2)/2\gamma)}{\beta^{k/2\gamma}\Gamma((2M+n-2)/2\gamma)}$
Multinormal	$e^{-r^2/2}$	$\frac{\Gamma(n/2)}{2^{1-n/2}}$	$\frac{\Gamma((n+1)/2)\sqrt{2}}{\Gamma(n/2)}$	n	$\frac{\Gamma((n+k)/2)2^{k/2}}{\Gamma(n/2)}$
Pearson VII	$\left(1 + \frac{r^2}{m}\right)^{-M}, \quad M > \frac{n}{2}, m > 0$	$\frac{\Gamma(n/2)\Gamma(M-n/2)}{2\Gamma(M)m^{-n/2}}$	$\frac{\Gamma((n+1)/2)\Gamma(M-(n+1)/2)\sqrt{m}}{\Gamma(n/2)\Gamma(M-n/2)}$	$\frac{mn}{2M-n-2}$	$\frac{\Gamma((n+k)/2)\Gamma(M-(n+k)/2)}{m^{-k/2}\Gamma(n/2)\Gamma(M-n/2)}$
Multivariate t	$\left(1 + \frac{r^2}{m}\right)^{-m/2-n/2}, \quad m \in \mathbb{N}^{>0}$	$\frac{\Gamma(n/2)\Gamma(m/2)}{2\Gamma((m+n)/2)m^{-n/2}}$	$\frac{\Gamma((n+1)/2)\Gamma((m-1)/2)\sqrt{m}}{\Gamma(n/2)\Gamma(m/2)}$	$\frac{mn}{m-2}$	$\frac{\Gamma((n+k)/2)\Gamma((m-k)/2)}{m^{-k/2}\Gamma(n/2)\Gamma(m/2)}$
Multivariate Cauchy	$(1 + r^2)^{-(n+1)/2}$	$\frac{\sqrt{\pi}\Gamma(n/2)}{2\Gamma((n+1)/2)}$	—	—	—
Pearson II	$I_{[0,1]}(r)(1-r^2)^m, \quad m > -1$	$\frac{\Gamma(n/2)\Gamma(m+1)}{2\Gamma(n/2+m+1)}$	$\frac{\Gamma((n+1)/2)\Gamma(n/2+m+1)}{\Gamma(n/2+m+3/2)\Gamma(n/2)}$	$\frac{n}{2m+n+2}$	$\frac{\Gamma((n+k)/2)\Gamma((n/2)+m+1)}{\Gamma((n+k)/2+m+1)\Gamma(n/2)}$
Uniform	$I_{[0,1]}(r)$	$\frac{1}{n}$	$\frac{n}{n+1}$	$\frac{n}{n+2}$	$\frac{n}{n+k}$

The paper is now organized as follows. We present the Minkowski functionals of star-shaped polyhedra in various kinds in Section 2 and derive corresponding representations of the polyhedral star-shaped surface and uniform measures in Section 3. This allows an extension of the ball number function in Section 3.2. The new classes of general polyhedral star-shaped distributions and specific w-star contoured distributions are considered then in Sections 4.1 and 4.2, respectively. Moments, characteristic, and moment generating functions are studied in Section 5, and simulation with a rejection method is discussed in Section 6. Location-scale transformations considered in Section 7 give rise to considering estimating of parameters and testing of hypothesis in Section 8. In Section 8, moreover, we present the probabilistic modelling and further statistical reasoning for the premature mortality-income data mentioned at the beginning of this introduction. We discuss some open problems in Section 9. Appendix mainly shows how various theoretical results apply in one and the same situation.

2. Minkowski Functionals of Star-Shaped Polyhedra

In this section, we specify the representation of h_K if the star ball K is a star-shaped polyhedron. Let $\mathbb{P} \subset \mathbb{R}^n$ be a star-shaped polyhedron. We call $\mathfrak{F} = \{C_1, \dots, C_m\}$ a convex polyhedral fan for $\mathbb{P}$ if $\mathbb{P} \cap C_j$ is for every $j, j \in \{1, \dots, m\}$ a convex polyhedron. Due to the considerations of convex polyhedra in the broad literature of convex geometry, it is possible to represent a convex polyhedron in two different but equivalent ways; see [24]. A convex polyhedron $\mathbb{P} \subset \mathbb{R}^n$ can be given by the set of its vertices $\{p_1, \dots, p_l\}$,

$$\mathbb{P} = \text{conv}(\{p_1, \dots, p_l\}), \quad (\mathcal{VP})$$

where $\text{conv}(M)$ denotes the convex hull of the point set M . Alternatively, it is possible to consider a convex polyhedron as the intersection of suitably chosen closed half-spaces. In this case, there exist a matrix $A \in \mathbb{R}^{k \times n}$ and a vector $b \in \mathbb{R}^k$ such that

$$\mathbb{P} = \{x \in \mathbb{R}^n : Ax \leq b\}, \quad (\mathcal{HP})$$

where “ $\leq$ ” is declared componentwise. The following proposition concludes that star-shaped polyhedra allow analogous representations.

Proposition 1. *If $\mathbb{P} \subset \mathbb{R}^n$ is a star-shaped polyhedron and $\mathfrak{F} = \{C_1, \dots, C_m\}$ a convex polyhedral fan for $\mathbb{P}$ then $\mathbb{P}$ always allows each of the following two equivalent representations.*

- (a) *There exist integer numbers $l_1, \dots, l_m$ and points $p_{j,1}, \dots, p_{j,l_j}$ from $\mathbb{R}^n$, $j \in \{1, \dots, m\}$ such that $\mathbb{P} \cap C_j = \text{conv}(\{0_n, p_{j,1}, \dots, p_{j,l_j}\})$ and*

$$\mathbb{P} = \bigcup_{j=1}^m \text{conv}(\{0_n, p_{j,1}, \dots, p_{j,l_j}\}). \quad (\mathcal{SV})$$

- (b) *There exist matrices $A_j \in \mathbb{R}^{k_j \times n}$ and vectors $b_j \in \mathbb{R}_+^{k_j}$ with positive real components such that*

$$\mathbb{P} = \bigcup_{j=1}^m \{x \in \mathbb{R}^n : A_j x \leq b_j\}. \quad (\mathcal{SH})$$

The following lemma is basic for proving the main results of this paper. It makes use of both representations in Proposition 1. For any $C \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, let $I_C : \mathbb{R}^n \rightarrow \{0, 1\}$ be the indicator function defined by

$$I_C(x) = \begin{cases} 1, & \text{if } x \in C, \\ 0, & \text{otherwise,} \end{cases} \quad (9)$$

$\text{pos } x = \{\alpha x \mid \alpha \geq 0\}$ and $\text{pos } C = \text{conv}(\bigcup_{x \in C} \text{pos } x) = \text{conv}(\bigcup_{\alpha \geq 0} \alpha C)$.

Lemma 2. *Let $\mathbb{P}$ and $\mathfrak{F}$ be as in Proposition 1.*

- (a) *With notation as in $(\mathcal{SV})$, $h_{\mathbb{P}}$ can be equivalently represented as follows:*

$$(a1) \quad h_{\mathbb{P}}(x) = \sum_{j=1}^m I_{C_j}(x) \inf\{r > 0 : x \in \text{conv}(\{0_n, r p_{j,1}, \dots, r p_{j,l_j}\})\}.$$

$$(a2) \quad h_{\mathbb{P}}(x) = \sum_{j=1}^m I_{C_j}(x) \inf\{r > 0 : x \in \{\sum_{i=1}^{l_j} \lambda_i p_{j,i}, \lambda_i \geq 0, \sum_{i=1}^{l_j} \lambda_i = r\}\}.$$

$$(a3) \quad h_{\mathbb{P}}(x) = \sum_{j=1}^m I_{C_j}(x) h_{\mathbb{P}}^{(j)}(x), \text{ where } h_{\mathbb{P}}^{(j)}(x) \text{ denotes the solution of the following linear optimization problem (LOP): minimize } \lambda_1 + \dots + \lambda_{l_j} \text{ subject to the conditions } A^{(j)}(\lambda_1, \dots, \lambda_{l_j})^T = x \text{ and } \lambda_i \geq 0, i \in \{1, \dots, l_j\}, \text{ where } A^{(j)} = (p_{j,1}, \dots, p_{j,l_j}) \in \mathbb{R}^{n \times l_j}, j \in \{1, \dots, m\}.$$

- (a4) *If additionally for every $j, j \in \{1, \dots, m\}$, $\mathbb{P} \cap C_j$ is an n -dimensional simplex then*

$$h_{\mathbb{P}}(x) = \sum_{j=1}^m \left(I_{C_j}(x) \mathbb{1}_n^T \left((A^{(j)})^{-1} x \right) \right), \quad (10)$$

where $\mathbb{1}_n = (1, \dots, 1)^T$ denotes the n -dimensional vector of ones, $C_j = \{x \in \mathbb{R}^n : (A^{(j)})^{-1} x \geq 0\}$, and $\geq$ reads componentwise.

- (b1) *With notation as in $(\mathcal{SH})$, where $\mathbb{P}$ is assumed to have m facets H_j and $C_j = \text{pos } H_j$ for every $j, j \in \{1, \dots, m\}$, there exist reals $a_{j1}, \dots, a_{jn}$, $j = 1, \dots, m$, such that $h_{\mathbb{P}}$ allows the representation*

$$h_{\mathbb{P}}(x) = \sum_{j=1}^m I_{C_j}(x) (a_{j1}x_1 + a_{j2}x_2 + \dots + a_{jn}x_n), \quad (11)$$

$$x = (x_1, \dots, x_n)^T \in \mathbb{R}^n.$$

- (b2) *If $\mathbb{P}$ is additionally convex then*

$$h_{\mathbb{P}}(x) = \max \{A'_1 x, \dots, A'_m x\}, \quad (12)$$

with $A' = (A'_1, \dots, A'_m)^T = (A_1/b_1, \dots, A_m/b_m)^T$, where the quantities $A = (A_1, \dots, A_m)^T$ and $b = (b_1, \dots, b_m)^T$ are equal to those in $(\mathcal{H}\mathcal{P})$ and A_j/b_j , $j \in \{1, \dots, m\}$ is declared componentwise.

For details on the application of the latter representation, we refer to [25]. This representation is used there and elsewhere to study ball numbers, circle numbers, and generalized uniform distributions on the boundaries of platonic bodies and regular convex polygons, respectively.

Proof. Representation (a1) follows immediately by applying the definition of $h_{\mathbb{P}}$ in (1) and $(\mathcal{SV}\mathcal{P})$. Representation (a2) follows then by applying (a1) and the well-known representation

$$\text{conv}(\{x_1, \dots, x_l\}) = \left\{ \sum_{i=1}^l \lambda_i x_i : \lambda_i \geq 0, \sum_{i=1}^l \lambda_i = 1 \right\} \quad (13)$$

for a point set $\{x_1, \dots, x_l\}$ in $\mathbb{R}^n$. From (a2), it follows that $\sum_{i=1}^l \lambda_i$ has to be minimal where $\lambda_i \geq 0$ and $x \in \mathbb{R}^n$ has to be represented by the linear combination $\sum_{i=1}^l \lambda_i p_{j,i}$. From this, (a3) follows. We derive now representation (a4) from (a3). If all $\mathbb{P} \cap C_j$ are n -dimensional simplizes then $A^{(j)} \in \mathbb{R}^{n \times n}$ and $A^{(j)}(\lambda_1, \dots, \lambda_n)^T = x$ is uniquely determined since $A^{(j)}$ is built up by affinely independent points thus satisfying $\det(A^{(j)}) \neq 0$. In turn, the minimum of $\sum_{i=1}^n \lambda_i$ can be calculated directly by $\mathbb{1}_n^T ((A^{(j)})^{-1} x)$. Furthermore, if a given vector x is an element of the simplex $\mathbb{P} \cap C_j$, all $\lambda_1, \dots, \lambda_n$ have to be nonnegative in this case. Thus the vector $\lambda = (A^{(j)})^{-1} x$ satisfies $x \in (\mathbb{P} \cap C_j)$ iff all components of λ are nonnegative, yielding the definition of I_{C_j} .

Representation (b1) can be proved as follows: if H_j , $j = 1, \dots, m$ are the facets of $\mathbb{P}$, each H_j is a subset of the boundary of an n -dimensional, closed half-space. Thus, there exist reals $a_{j1}, \dots, a_{jn}$, $j = 1, \dots, m$, such that

$$\begin{aligned} H_j &= \{x \in \mathbb{P} : a_{j1}x_1 + a_{j2}x_2 + \dots + a_{jn}x_n = 1\}, \\ C_j \cap \mathbb{P} &= \{x \in C_j : a_{j1}x_1 + a_{j2}x_2 + \dots + a_{jn}x_n \leq 1\} \quad (14) \\ &\text{since } C_j = \text{pos } H_j. \end{aligned}$$

From this, it follows

$$\begin{aligned} h_{\mathbb{P}}(x) &= \inf \left\{ r > 0 : \frac{x}{r} \in \mathbb{P} \right\} = \inf \left\{ r > 0 : \frac{x}{r} \right. \\ &\quad \left. \in \bigcup_{j=1}^m (C_j \cap \mathbb{P}) \right\} = \inf \left\{ r > 0 : \frac{x}{r} \right. \\ &\quad \left. \in (\{x \in \mathbb{R}^n : x \in C_1, a_{11}x_1 + \dots + a_{1n}x_n \leq 1\} \cup \dots \right. \\ &\quad \left. \cup \{x \in \mathbb{R}^n : x \in C_m, a_{m1}x_1 + \dots + a_{mn}x_n \leq 1\}) \right\} \end{aligned}$$

$$\begin{aligned} &= \inf \left\{ r > 0 : \sum_{j=1}^m I_{C_j}(x) (a_{j1}x_1 + a_{j2}x_2 + \dots \right. \\ &\quad \left. + a_{jn}x_n) \leq r \right\} = \sum_{j=1}^m I_{C_j}(x) (a_{j1}x_1 + a_{j2}x_2 + \dots \\ &\quad + a_{jn}x_n), \quad \forall x = (x_1, \dots, x_n)^T \in \mathbb{R}^n. \end{aligned} \quad (15)$$

Representation (b2) follows by applying $(\mathcal{H}\mathcal{P})$ and using that

$$\begin{aligned} \mathbb{P} &= \{x \in \mathbb{R}^n : A'x \leq \mathbb{1}_n\} \\ &= \{x \in \mathbb{R}^n : A'_1x \leq 1, A'_2x \leq 1, \dots, A'_mx \leq 1\} \quad (16) \\ &= \{x \in \mathbb{R}^n : \max \{A'_1x, A'_2x, \dots, A'_mx\} \leq 1\}. \end{aligned}$$

□

3. Polyhedral Generalized Surface Content Measure

3.1. Specific Representations. In this section, specific representations of the star-generalized surface content measure $\mathfrak{D}_{\mathbb{S}}$ are considered in case that $K = \mathbb{P}$ is a star-shaped polyhedron and $S = \mathbb{S}$ its boundary. Prior to this, we consider a generalized polar coordinate transformation allowing one of the specific representations derived later on. For another powerful application of these generalized polar coordinates, we refer already here to Section 5.2.

Definition 3. Let $\mathbb{P} \subset \mathbb{R}^n$ be a star-shaped polyhedron. The polyhedral star-shaped polar coordinate transformation $SPOL_{\mathbb{P}} : M_n \rightarrow \mathbb{R}^n$, $M_n = [0, \infty) \times M_n^*$, $M_n^* = [0, \pi)^{\times(n-2)} \times [0, 2\pi)$ is defined by

$$\begin{aligned} x_1 &= \frac{r}{N(\varphi_1, \dots, \varphi_{n-1})} \cos(\varphi_1) \\ x_2 &= \frac{r}{N(\varphi_1, \dots, \varphi_{n-1})} \sin(\varphi_1) \cos(\varphi_2) \\ &\vdots \\ x_{n-1} &= \frac{r}{N(\varphi_1, \dots, \varphi_{n-1})} \sin(\varphi_1), \dots, \sin(\varphi_{n-2}) \\ &\quad \cdot \cos(\varphi_{n-1}) \\ x_n &= \frac{r}{N(\varphi_1, \dots, \varphi_{n-1})} \sin(\varphi_1), \dots, \sin(\varphi_{n-2}) \\ &\quad \cdot \sin(\varphi_{n-1}), \end{aligned} \quad (17)$$

where $N(\varphi_1, \dots, \varphi_{n-1}) = h_{\mathbb{P}}(\cos(\varphi_1), \sin(\varphi_1) \cos(\varphi_2), \dots, \sin(\varphi_1), \dots, \sin(\varphi_{n-1}))$.

We will later also use the notation $x = x(\varphi, r) = (x_1(r, \varphi), \dots, x_n(r, \varphi))$ if $x = SPOL_{\mathbb{P}}(r, \varphi)$.

Lemma 4. The map $SPOL_{\mathbb{P}}$ is almost one-to-one, for $x \neq 0$, its inverse $SPOL_{\mathbb{P}}^{-1}$ is given by

$$r = h_{\mathbb{P}}(x_1, \dots, x_n),$$

$$\varphi_i = \arccos\left(\frac{x_i}{(\sum_{j=i}^n x_j^2)^{1/2}}\right), \quad i = 1, \dots, n-2,$$

$$\varphi_{n-1} = \begin{cases} \operatorname{arccot}\left(\frac{x_{n-1}}{x_n}\right), & x_n > 0, \\ \operatorname{arccot}\left(\frac{x_{n-1}}{x_n}\right) + \pi, & x_n < 0 \end{cases} \quad (18)$$

and its Jacobian is

$$J(SPOL_{\mathbb{P}})(r, \varphi) = r^{n-1} J^*(SPOL_{\mathbb{P}})(\varphi), \quad (r, \varphi) \in M_n \text{ where } J^*(SPOL_{\mathbb{P}})(\varphi_1, \dots, \varphi_{n-1}) = \frac{1}{N(\varphi)^n} \prod_{i=1}^{n-1} \sin(\varphi_i)^{n-i-1}. \quad (19)$$

Proof. It holds

$$h_{\mathbb{P}}(x_1, \dots, x_n) = \frac{r}{N(\varphi_1, \dots, \varphi_{n-1})} \cdot \underbrace{h_{\mathbb{P}}(\cos(\varphi_1), \dots, \sin(\varphi_1), \dots, \sin(\varphi_{n-1}))}_{=N(\varphi_1, \dots, \varphi_{n-1})} \quad (20)$$

and the relations of φ_i , $i = 1, \dots, n-1$, are the same as those in the case of usual polar coordinates. For calculating the Jacobian, we refer to the proof of Theorem 3.1 in [25] that can be extended to the n -dimensional case. $\square$

Let us denote the cdf of a random vector Z by F_Z .

Theorem 5. Let $(R, \Phi_1, \dots, \Phi_{n-1})^T = SPOL_{\mathbb{P}}^{-1}(X)$ be the random polyhedral star-shaped coordinates of X , where $X \sim \Phi_{g, \mathbb{P}}$; then R and $(\Phi_1, \dots, \Phi_{n-1})^T$ are stochastically independent, $F_R(t) = \int_0^t (r^{n-1} / \mathcal{J}(g)) g(r) dr$, and with $\varphi = (\varphi_1, \dots, \varphi_{n-1})$

$$F_{\Phi_i}(\alpha) = \frac{1}{n\mu(\mathbb{P})}$$

$$\cdot \underbrace{\int_0^\pi \dots \int_0^\pi}_{i-1 \text{ times}} \underbrace{\int_0^\pi \dots \int_0^\pi}_{n-i-2 \text{ times}} \int_0^{2\pi} J^*(SPOL_{\mathbb{P}})(\varphi) d(\varphi), \quad (21)$$

$\forall i = 1, \dots, n-2, \alpha \in [0, \pi)$ and

$$F_{\Phi_{n-1}}(\alpha) = \frac{1}{n\mu(\mathbb{P})} \underbrace{\int_0^\pi \dots \int_0^\pi}_{n-2 \text{ times}} \int_0^\alpha J^*(SPOL_{\mathbb{P}})(\varphi) d(\varphi), \quad (22)$$

$\alpha \in [0, 2\pi)$.

Proof. Applying the described method in Remark 11 in [6] for elliptically contoured distributions analogously here yields the representation of F_R . Furthermore, $P(\Phi_i < \alpha) = \Phi_{g, \mathbb{P}}(A(\alpha))$ for every $i = 1, \dots, n-1$, where $A(\alpha) = \{(r, \varphi_1, \dots, \varphi_{n-1}) \in [0, \infty) \times M_n^* : \varphi_i \in [0, \alpha)\}$. Thus with $\varphi = (\varphi_1, \dots, \varphi_{n-1})$

$$P(\Phi_i < \alpha) = \underbrace{\frac{C(g, \mathbb{P})}{1/n\mathcal{J}(g)\mu(\mathbb{P})}}_{=\mathcal{J}(g)} \underbrace{\int_0^\infty r^{n-1} g(r) dr}_{i-1 \text{ times}} \underbrace{\int_0^\pi \dots \int_0^\pi}_{n-i-2 \text{ times}} \int_0^{2\pi} J^*(SPOL_{\mathbb{P}})(\varphi) d(\varphi). \quad (23)$$

$\square$

Let us denote by $SPOL_{\mathbb{P}}^*(\varphi) = SPOL_{\mathbb{P}}(1, \varphi)$ the restriction of $SPOL_{\mathbb{P}}$ to the case $r = 1$ and, analogously, by $SPOL_{\mathbb{P}}^{*-1}$ the inverse of the map $SPOL_{\mathbb{P}}^* : M_n^* \rightarrow \mathbb{S}$. Now, the following representations of $\mathfrak{D}_{\mathbb{S}}$ can be proved.

Theorem 6. Let $\mathbb{P} \subset \mathbb{R}^n$ be a star-shaped polyhedron and $B \in (\mathbb{S} \cap \mathfrak{B}_n)$.

(a) $\mathfrak{D}_{\mathbb{S}}$ satisfies the polyhedral star-shaped polar coordinate representation

$$\mathfrak{D}_{\mathbb{S}}(B) = \int_{SPOL_{\mathbb{P}}^{*-1}(B)} J^*(SPOL_{\mathbb{P}})(\varphi) d\varphi. \quad (24)$$

(b1) With notation as in $(\mathcal{SHP})$ and Lemma 2 (b1), $\mathfrak{D}_{\mathbb{S}}$ satisfies the star-spherical coordinate representation

$$\mathfrak{D}_{\mathbb{S}}(B) = \sum_{j=1}^m \int_{G(B \cap (C_j \cap \mathbb{P}))} \frac{1}{|a_{jn}|} d\vartheta. \quad (25)$$

(b2) Let additionally vol_{n-1} denote the volume in $\mathbb{R}^{n-1}$, $B = \bigcup_{j=1}^m \mathcal{B}_j$ where $\mathcal{B}_j \subseteq (C_j \cap \mathbb{P})$, $\operatorname{int}(\mathcal{B}_j) \cap \operatorname{int}(\mathcal{B}_i) \neq \emptyset$ if $i \neq j$, $i, j \in \{1, \dots, m\}$; then $\mathfrak{D}_{\mathbb{S}}$ satisfies the facet-content representation

$$\mathfrak{D}_{\mathbb{S}}(B) = \sum_{j=1}^m \frac{\operatorname{vol}_{n-1}(\mathcal{B}_j)}{\sqrt{a_j^T a_j}}, \quad (26)$$

where $a_j = (a_{j1}, \dots, a_{jn})^T$. Note that $\mathfrak{D}_{\mathbb{S}}$ is proportional to the Lebesgue measure in $\mathbb{R}^{n-1}$ in every sector $C_j \cap \mathbb{P}$, $j = 1, \dots, m$.

- (c) Let additionally $\mathbb{P} \cap C_j$, $j \in \{1, \dots, m\}$ be n -dimensional simplices and $\mathbb{P} \cap C_j$ and $B \cap C_j$, $j \in \{1, \dots, m\}$ represented according to $(\mathcal{V}\mathcal{P})$. Then there exist integers k_j such that $B \cap C_j = \bigcup_{i=1}^{k_j} \text{conv}(\{b_{(j),i1}, \dots, b_{(j),in}\})$ and $\text{int}(\text{conv}(\{b_{(j),i1}, \dots, b_{(j),in}\})) \cap \text{int}(\text{conv}(\{b_{(j),i21}, \dots, b_{(j),i2n}\})) = \emptyset$, $i_1 \neq i_2$, where $\text{conv}(\{b_{(j),i1}, \dots, b_{(j),in}\})$ are $(n-1)$ -dimensional simplices. Then $\mathfrak{D}_{\mathbb{S}}$ satisfies the simplicial representation

$$\mathfrak{D}_{\mathbb{S}}(B) = \frac{1}{(n-1)!} \sum_{j=1}^m V(B \cap C_j), \quad (27)$$

where

$$V(B \cap C_j) = \begin{cases} \sum_{i=1}^{k_j} |\det(b_{(j),i1}, \dots, b_{(j),in})|, & \text{if } B \cap C_j \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases} \quad (28)$$

- (d) If additionally $h_{\mathbb{P}}$ denotes a norm and $N(\vartheta) = (\nabla \eta(\vartheta), -1)^T$ is the outer normal vector to the norm sphere $\mathbb{S}$ at the point $(\vartheta^T, \eta(\vartheta))^T$, then, with notations as in $(\mathcal{S}\mathcal{V}\mathcal{P})$, $\mathfrak{D}_{\mathbb{S}}$ satisfies the dual norm representation

$$\mathfrak{D}_{\mathbb{S}}(A) = \int_{G(A)} \max\{p_1^T N(\vartheta), \dots, p_l^T N(\vartheta)\} d\vartheta, \quad A \in \mathfrak{B}_{\mathbb{S}}^+, \quad (29)$$

$$G(A) = \{\vartheta \in \mathbb{R}^{n-1} : \exists \eta = \eta(\vartheta) \text{ with } (\vartheta^T, \eta)^T \in A\}.$$

- (e) If additionally $h_{\mathbb{P}}$ denotes an antinorm and $\mathbb{P} \in \text{AN2}$ and $\overline{N}(\vartheta) = (\nabla \eta(\vartheta), 1)^T$ is the inner normal vector to the antinorm sphere $\mathbb{S}$ at the point $(\vartheta^T, \eta(\vartheta))^T$, then with notations as in $(\mathcal{S}\mathcal{H}\mathcal{P})$ the Minkowski functional of the antipolar set $\mathbb{P}^\circ$ of $\mathbb{P}$ defined by $h_{\mathbb{P}^\circ}(u) = \sum_{i=1}^m I_{C_i}(u) \inf\{u^T y : y \in \mathbb{S} \cap C_i\}$, $u \in \mathbb{R}^n$ and $\mathfrak{D}_{\mathbb{S}}$ satisfies the (antipolar set) representation

$$\mathfrak{D}_{\mathbb{S}}(B) = \int_{G(B)} h_{\mathbb{P}^\circ}(\overline{N}(\vartheta)) d\vartheta. \quad (30)$$

Proof. Since

$$\begin{aligned} \mu(\text{sector}_{\mathbb{P}}(B, \rho)) &= \int_{\text{sector}_{\mathbb{P}}(B, \rho)} dx \\ &= \int_0^\rho \int_{\text{SPOL}_{\mathbb{P}}^{-1}(B)} r^{n-1} J^*(\text{SPOL}_{\mathbb{P}})(\varphi) d(\varphi, r), \end{aligned} \quad (31)$$

(5) yields (a). Analogously,

$$\begin{aligned} \mu(\text{sector}_{\mathbb{P}}(B, \rho)) &= \int_{\text{sector}_{\mathbb{P}}(B, \rho)} dx \\ &= \int_0^\rho r^{n-1} dr \sum_{i=1}^m \int_{G(B \cap C_i \cap \mathbb{P})} J_j^*(\vartheta) d\vartheta, \end{aligned} \quad (32)$$

where, for every $\vartheta \in G(\mathbb{S}_j)$, $J_j^*(\vartheta)$, $j = 1, \dots, m$ is the value of the Jacobian of the star-spherical coordinate transformation; see [6]. Since rH_j , $j = 1, \dots, m$ can be represented by star-spherical coordinates as

$$\begin{aligned} rH_j &= r \cdot \left\{ (\vartheta, \eta_j(\vartheta))^T \in \mathbb{P} : \eta_j(\vartheta) \right. \\ &= \left. \frac{1}{a_{jn}} \left(1 - \sum_{i=1}^{n-1} a_{ji} \vartheta_i \right) \right\}, \quad j = 1, \dots, m, \end{aligned} \quad (33)$$

it follows by applying Lemma 1 in [6] that $J_j^*(\vartheta) = 1/|a_{jn}|$. Thus, (5) yields (b1). Considering part (b2), it holds $\mu(\text{sector}_{\mathbb{P}}(B, 1)) = \sum_{j=1}^m \mu(\text{sector}_{\mathbb{P}}(\mathcal{B}_j, 1))$, where every $\text{sector}_{\mathbb{P}}(\mathcal{B}_j, 1)$, $j = 1, \dots, m$ is a convex cone with base $\mathcal{B}_j$ and cusp in the origin. Thus $\mu(\text{sector}_{\mathbb{P}}(\mathcal{B}_j, 1)) = \text{vol}_{n-1}(\mathcal{B}_j)/(n\sqrt{a_j^T a_j})$, where $1/\sqrt{a_j^T a_j}$ denotes the Euclidean length of the height of $\text{sector}_{\mathbb{P}}(\mathcal{B}_j, 1)$. According to Remark 2 in [6], it holds $\mathfrak{D}_{\mathbb{S}}(B) = \mathfrak{D}_{\mathbb{S}}(\mathbb{S})\mu(\text{sector}_{\mathbb{P}}(B, 1))/\mu(\mathbb{P})$. Thus

$$\mathfrak{D}_{\mathbb{S}}(B) = \frac{\mathfrak{D}_{\mathbb{S}}(\mathbb{S})}{\frac{n\mu(\mathbb{P})}{=1}} \sum_{j=1}^m \frac{\text{vol}_{n-1}(\mathcal{B}_j)}{\sqrt{a_j^T a_j}}, \quad (34)$$

and (b2) is proved. Considering part (c), it holds

$$\begin{aligned} \mu(\text{sector}_{\mathbb{P}}(B \cap C_j, 1)) &= \sum_{i=1}^{k_j} \mu(\text{conv}(\{0_n, b_{(j),i1}, \dots, b_{(j),in}\})), \\ & \quad j = 1, \dots, m, \end{aligned} \quad (35)$$

and since $\mu(\text{conv}(\{0_n, b_{(j),i1}, \dots, b_{(j),in}\})) = (1/n!) |\det(b_{(j),i1}, \dots, b_{(j),in})|$ it follows

$$\begin{aligned} \mathfrak{D}_{\mathbb{S}}(B) &= \frac{\mathfrak{D}_{\mathbb{S}}(\mathbb{S})}{\frac{\mu(\mathbb{P})}{=n}} \mu(\text{sector}_{\mathbb{P}}(B, 1)) \\ &= \frac{1}{(n-1)!} \sum_{j=1}^m V(B \cap C_j). \end{aligned} \quad (36)$$

Considering part (d) and applying (8), it remains to prove that $h_{\mathbb{P}^\circ}(N(\vartheta)) = \max\{p_1^T N(\vartheta), \dots, p_l^T N(\vartheta)\}$. According to [26] it is well known that $h_{\mathbb{P}^\circ}$ equals the support function of $\mathbb{P}$; thus $h_{\mathbb{P}^\circ}(u) = \sup\{u^T y : y \in \mathbb{P}\}$, $u \in \mathbb{R}^n$, and it follows $h_{\mathbb{P}^\circ}(u) = \max\{p_1^T u, \dots, p_l^T u\}$.

Considering part (e) we can use the results from Lemma 1 in [6], according to which it holds

$$\mathfrak{D}_{\mathbb{S}}(B) = \int_{G(B)} \left| (\vartheta^T, \eta(\vartheta)) \overline{N}(\vartheta) \right| d\vartheta. \quad (37)$$

With notations as in $(\mathcal{SHP})$, it follows for every $i = 1, \dots, m$ that $\mathbb{S} \cap C_i = \{x \in \mathbb{R}^n : a_{i1}x_1 + \dots + a_{in}x_n = 1\}$, $\eta(\vartheta) = (1/a_{in})(1 - \sum_{j=1}^{n-1} a_{ij}\vartheta_j)$ and $\overline{N}(\vartheta) = (a_{i1}/a_{in}, \dots, a_{i(n-1)}/a_{in}, 1)^T$. Thus, for every $(\vartheta^T, \eta(\vartheta))^T \in \mathbb{S} \cap C_i$,

$$\begin{aligned} \left| (\vartheta^T, \eta(\vartheta)) \overline{N}(\vartheta) \right| &= \left| \frac{1}{a_{in}} \right| \\ &= \inf \{ \overline{N}(\vartheta) y : y \in \mathbb{S} \cap C_i \}. \end{aligned} \quad (38)$$

Since the antisupport function $\mathfrak{h}_{\mathbb{P}}^{\mathfrak{F}}$ of $\mathbb{P}$ with respect to $\mathfrak{F}$ is defined by

$\mathfrak{h}_{\mathbb{P}}^{\mathfrak{F}}(u) = \sum_{i=1}^m I_{C_i}(u) \inf \{u^T y : y \in \mathbb{S} \cap C_i\}$, $u \in \mathbb{R}^n$ and it is shown in Lemma 10 in [7] that $\mathfrak{h}_{\mathbb{P}}^{\mathfrak{F}} = h_{\mathbb{P}^{\circ}}$, part (e) is proved. $\square$

According to [7] it is possible to represent the antipolar set $\mathbb{P}^{\circ}$ of a $\mathbb{P} \in \text{AN2}$ by

$$\mathbb{P}^{\circ} = \left\{ \lambda(u) u : 0 \leq \lambda(u) h_{\mathbb{P}^{\circ}}(u) \leq 1, u \in S_E^{(n-1)} \right\}, \quad (39)$$

where $S_E^{(n-1)}$ denotes the Euclidean unit sphere in $\mathbb{R}^n$. For an illustration, we refer to Section 4.2, where antipolar sets of w-stars are considered. For an application of certain representations of Theorem 6, we refer to Example A.1 in the Appendix.

3.2. Extension of the Ball Number Function. The circumference and area content properties of Euclidean circles which motivate the generalization of the circle number π have been discussed to a certain extent first in [27] for $l_{2,p}$ -circles. These considerations were later extended to ellipses and general star-circles. Related multivariate studies started by introducing the generalized surface and volume properties of $l_{n,p}$ -balls and were followed up in [6, 21]. General ball numbers being values attained by the ball number function are defined in the literature, and it is also stated there as a challenging problem to extend the ball number function to as many as possible further classes of generalized balls. In this respect, an extension of the ball number function to the class of regular convex polygons can be found in [25], and an extension to platonic bodies is considered elsewhere by the authors. Furthermore, in Section 7 of [25], several other possible extensions of the circle number function one could think about are discussed.

Note that the equation

$$\mu(\mathbb{P}(r)) = \int_0^r \mathfrak{D}_{\mathbb{S}}(\mathbb{S}(\rho)) d\rho \quad (40)$$

reflects a certain generalization of the method of indivisibles of Cavalieri and Torricelli and that the ratios

$$\frac{\mu(\mathbb{P}(r))}{r^n} = \frac{\mathfrak{D}_{\mathbb{S}}(\mathbb{S}(r))}{nr^{n-1}} \quad (41)$$

not only do coincide but are even independent of r . This motivates the following definition.

Definition 7. The polyhedral star ball number $\pi_{\mathbb{P}}$ is defined by $\pi_{\mathbb{P}} = \mu(\mathbb{P})$.

Note that $\pi_{\mathbb{P}} = \mathfrak{D}_{\mathbb{S}}(\mathbb{S})/n$. For concrete values of $\pi_{\mathbb{P}}$ in the special case of regular convex polygons, see [25]. Ball numbers of platonic bodies are dealt with elsewhere. Apart from this extension of the domain of the ball number function, $\pi_{\mathbb{P}}$ will be used in this paper as kind of normalizing constant for polyhedral star-shaped contoured distributions.

3.3. Polyhedral Star-Shaped Generalized Uniform Distributions. The notion of a polyhedral generalized surface content measure makes it possible to define the polyhedral star-shaped generalized uniform distribution on $\mathbb{S}$ as

$$\omega_{\mathbb{S}}(A) = \frac{\mathfrak{D}_{\mathbb{S}}(A)}{\mathfrak{D}_{\mathbb{S}}(\mathbb{S})}, \quad A \in \mathfrak{B}_{\mathbb{S}}. \quad (42)$$

Note that plugging in any of the representations of Theorem 6 into (42) yields the polyhedral star-shaped polar coordinate, star-spherical coordinate, and facet-content, simplicial and dual norm representations of the generalized uniform distribution, respectively. A numerical example of the application of $\omega_{\mathbb{S}}$ can be found in Example A.1 in the Appendix.

Let $(\Omega, \mathfrak{A}, P)$ be a probability space and G a random vector being defined on Ω and taking values in $\mathbb{R}^n$. Assume further that G follows the uniform probability distribution on $\mathbb{P}$; that is, $P(G \in A) = \mu(A)/\mu(\mathbb{P})$, for $A \in (\mathbb{P} \cap \mathfrak{B}_n)$, and put $Y = G/h_{\mathbb{P}}(G)$, where division is defined componentwise. The proof of the following result can be done analogously to that in case of platonically generalized uniform distributions on platonic spheres.

Theorem 8. The random vector Y follows the polyhedral star-shaped generalized uniform distribution on $\mathbb{S}$.

4. Polyhedral Star-Shaped Distributions

4.1. Representations. The results in the previous section can be used for representing general polyhedral star-shaped distributions. Doing this, we follow the considerations in [6–8, 12, 20, 21, 28], where stochastic representations like (7) were repeatedly exploited.

A random vector $X : \Omega \rightarrow \mathbb{R}^n$ is said to follow a polyhedral star-shaped distribution if there exists a random variable $R : \Omega \rightarrow [0, \infty)$ such that X satisfies the stochastic representation

$$X \stackrel{d}{=} R \cdot U_{\mathbb{S}}, \quad (43)$$

where $U_{\mathbb{S}} \sim \omega_{\mathbb{S}}$ is polyhedral star-shaped generalized uniformly distributed on $\mathbb{S}$ and $U_{\mathbb{S}}$ and R are stochastically independent. The set of all polyhedral star-shaped distributions

on $\mathfrak{B}_n$ will be denoted by $\text{PStSh}^{(n)}$ and its subset of continuous distributions by

$$\text{CPStSh}^{(n)} = \left\{ \Phi_{g,\mathbb{P}} : \mathbb{P} \text{ is a star-shaped polyhedron, } g \text{ is a dgf} \right\}. \quad (44)$$

Applying (6) and the representations of Theorem 6, we can state the following specific geometric measure representation formulae of continuous polyhedral star-shaped distributions. For the application of geometric measure representation formulae of other classes of distributions, we refer to [28–30].

Proposition 9. *Let the assumptions from Theorem 6 be fulfilled and $B \in \mathfrak{B}_n$. Then $\Phi_{g,\mathbb{P}}(B)$ allows correspondingly*

(a) *polyhedral star-shaped polar coordinate representation*

$$\Phi_{g,\mathbb{P}}(B) = \frac{1}{n\pi_{\mathbb{P}}\mathcal{J}(g)} \int_0^\infty \int_{\text{SPOLE}^{n-1}([(1/r)B] \cap \mathbb{S})} r^{n-1} g(r) \cdot J^*(\text{SPOLE}_{\mathbb{P}})(\varphi) d(\varphi) dr, \quad (45)$$

(b1) *star-spherical coordinate representation*

$$\Phi_{g,\mathbb{P}}(B) = \frac{1}{n\pi_{\mathbb{P}}\mathcal{J}(g)} \int_0^\infty r^{n-1} g(r) \cdot \sum_{j=1}^m \int_{G([(1/r)B] \cap \mathbb{S}) \cap (C_j \cap \mathbb{P})} \frac{1}{|a_{jn}|} d\vartheta dr, \quad (46)$$

(b2) *facet-content representation*

$$\Phi_{g,\mathbb{P}}(B) = \frac{1}{n\pi_{\mathbb{P}}\mathcal{J}(g)} \int_0^\infty r^{n-1} g(r) \sum_{j=1}^m \frac{\text{vol}_{n-1}(B_j(r))}{\sqrt{a_j^T a_j}} dr, \quad (47)$$

if additionally $[(1/r)B] \cap \mathbb{S} = \bigcup_{j=1}^m \mathcal{B}_j(r)$ where $\mathcal{B}_j(r) \subseteq (C_j \cap \mathbb{P})$ and $\text{int}(\mathcal{B}_i(r)) \cap \text{int}(\mathcal{B}_j(r)) = \emptyset$, if $i \neq j$, $i, j \in \{1, \dots, m\}$,

(c) *simplicial representation*

$$\Phi_{g,\mathbb{P}}(B) = \frac{1}{n\pi_{\mathbb{P}}\mathcal{J}(g)(n-1)!} \cdot \int_0^\infty r^{n-1} g(r) \sum_{j=1}^m V\left(\left(\left[\frac{1}{r}B\right] \cap \mathbb{S}\right) \cap C_j\right) dr, \quad (48)$$

(d) *dual norm representation*

$$\Phi_{g,\mathbb{P}}(B) = \frac{1}{n\pi_{\mathbb{P}}\mathcal{J}(g)} \int_0^\infty \int_{G([(1/r)B] \cap \mathbb{S})} r^{n-1} g(r) \cdot \max\{p_1^T N(\vartheta), \dots, p_l^T N(\vartheta)\} d(\vartheta) dr, \quad (49)$$

(e) *(antipolar set) representation*

$$\Phi_{g,\mathbb{P}}(B) = \frac{1}{n\pi_{\mathbb{P}}\mathcal{J}(g)} \cdot \int_0^\infty \int_{G([(1/r)B] \cap \mathbb{S})} r^{n-1} g(r) h_{\mathbb{P}^\circ}(\overline{N}(\vartheta)) d\vartheta dr, \quad (50)$$

(f) *sector measure representation*

$$\Phi_{g,\mathbb{P}}(B) = \frac{1}{\pi_{\mathbb{P}}\mathcal{J}(g)} \cdot \int_0^\infty r^{n-1} g(r) \mu\left(\text{sector}_{\mathbb{P}}\left(\left[\frac{1}{r}B\right] \cap \mathbb{S}, 1\right)\right) dr. \quad (51)$$

Note that for the sector measure also the notion of cone measure is used; see [17, 31]

Finally note that if $X \sim \Phi_{g,\mathbb{P}}$ then we write (43) as

$$X \stackrel{d}{=} R_g \cdot U_{\mathbb{S}}, \quad (52)$$

where the nonnegative random variable R_g is independent of $U_{\mathbb{S}}$.

4.2. The Class of $\mathbb{w}$ -Star Contoured Distributions in $\mathbb{R}^2$. The polyhedral star-shaped distributions considered in this section may be alternatively norm or antinorm contoured. For simplicity, consideration is restricted here to the two-dimensional case. Let $\mathbb{P}_{\mathbb{w}} \subset \mathbb{R}^2$ be a $\mathbb{w}$ (vertices) driven star-shaped polygon defined for arbitrary $\mathbb{w} \in (0, \infty)$ by the vertices $q_1 = (1, 0)$, $q_2 = (\mathbb{w}, \mathbb{w})$, $q_3 = (0, 1)$, $q_4 = (-\mathbb{w}, \mathbb{w})$, $q_5 = (-1, 0)$, $q_6 = (-\mathbb{w}, -\mathbb{w})$, $q_7 = (0, -1)$, and $q_8 = (\mathbb{w}, -\mathbb{w})$. We call $\mathbb{P}_{\mathbb{w}}$ a $\mathbb{w}$ -star. For an illustration of different $\mathbb{w}$ -stars, see Figure 1. Note that a $\mathbb{w}$ -star appears to be convex if $\mathbb{w} \in [1/2, 1]$ and radially concave if $\mathbb{w} \in (0, 1/2) \cup (1, \infty)$. For representing the Minkowski functional $h_{\mathbb{P}_{\mathbb{w}}}$ of a $\mathbb{w}$ -star, it is convenient to make use of the fans $\mathfrak{C}_1 = \bigcup_{i=1}^4 C_{1,i}$ and $\mathfrak{C}_2 = \bigcup_{i=1}^4 C_{2,i}$ if $\mathbb{w} \in (0, 1)$ or $\mathbb{w} \geq 1$, respectively; see again Figure 1. If $\mathbb{w} \in (0, 1)$ then

$$\begin{aligned} h_{\mathbb{P}_{\mathbb{w}}}(x, y) &= I_{C_{1,1}}(x, y) \left(x + \frac{1-\mathbb{w}}{\mathbb{w}} |y| \right) + I_{C_{1,2}}(x, y) \\ &\quad \cdot \left(\frac{1-\mathbb{w}}{\mathbb{w}} |x| + y \right) + I_{C_{1,3}}(x, y) \\ &\quad \cdot \left(-x + \frac{1-\mathbb{w}}{\mathbb{w}} |y| \right) + I_{C_{1,4}}(x, y) \\ &\quad \cdot \left(\frac{1-\mathbb{w}}{\mathbb{w}} |x| - y \right) \end{aligned} \quad (53)$$

FIGURE 1: w -stars in case $0 < w < 1$ (a) and in case $w \geq 1$ (b).

and if $w \geq 1$ then

$$\begin{aligned}
 h_{\mathbb{P}_w}(x, y) &= I_{C_{2,1}}(x, y) \\
 &\cdot \left(\max\{x, y\} + \frac{1-w}{w} \min\{x, y\} \right) + I_{C_{2,2}}(x, y) \\
 &\cdot \left(\max\{-x, y\} + \frac{1-w}{w} \min\{-x, y\} \right) \\
 &+ I_{C_{2,3}}(x, y) \\
 &\cdot \left(\max\{-x, -y\} + \frac{1-w}{w} \min\{-x, -y\} \right) \\
 &+ I_{C_{2,4}}(x, y) \left(\max\{x, -y\} + \frac{1-w}{w} \min\{x, -y\} \right).
 \end{aligned} \tag{54}$$

Surprisingly enough, it follows that $h_{\mathbb{P}_w}(x, y) = \max\{|x|, |y|\} + ((1-w)/w)\min\{|x|, |y|\}$ for every $w \in (0, \infty)$. This result enables us to introduce the w -star contoured densities

$$\begin{aligned}
 \varphi_{g, \mathbb{P}_w}(x, y) &= \frac{1}{8w\mathcal{F}(g)} \\
 &\cdot g \left(\max\{|x|, |y|\} + \frac{1-w}{w} \min\{|x|, |y|\} \right), \tag{55} \\
 (x, y)^T &\in \mathbb{R}^2.
 \end{aligned}$$

Note that since $\mu(\mathbb{P}_w) = 4w$ the normalizing constant applies by (4). If $w \in [1/2, 1]$ then $h_{\mathbb{P}_w}$ denotes a norm. Thus, applying

Theorem 6 (d), we can represent $\mathfrak{D}_{\mathbb{S}_w}$ of a set $A \in \mathfrak{B}_{\mathbb{S}_w}^+$ by the dual norm representation as

$$\begin{aligned}
 \mathfrak{D}_{\mathbb{S}_w}(A) &= \int_{G(A)} \max\{|N_1(\vartheta)|, |N_2(\vartheta)|\}, \\
 &w(|N_1(\vartheta)| + |N_2(\vartheta)|) d\vartheta,
 \end{aligned} \tag{56}$$

where $N(\vartheta) = (N_1(\vartheta), N_2(\vartheta))^T$. Note that further representations of the star-generalized surface content measure can be found in [7] for a special class of radially concave star bodies and in [25] for regular polygons, especially in the case that these polygons have an odd number of vertices and thus are not symmetric.

If $w \in (0, 1/2) \cup (1, \infty)$ then $h_{\mathbb{P}_w}$ denotes an antinorm. Thus, applying Theorem 6 (e), we can represent $\mathfrak{D}_{\mathbb{S}_w}$ of a set $B \in \mathfrak{B}_{\mathbb{S}_w}$ by the (antipolar set) representation

$$\begin{aligned}
 \mathfrak{D}_{\mathbb{S}_w}(B) &= \int_{G(B)} \sum_{i=1}^8 I_{C_i} \left(\left(\frac{a_{i1}}{a_{i2}}, 1 \right)^T \right) \left(w \left(\frac{a_{i1}}{a_{i2}} + 1 \right) \right) d\vartheta \\
 &= \int_{G(B)} w(|\bar{N}_1(\vartheta)| + |\bar{N}_2(\vartheta)|) d\vartheta,
 \end{aligned} \tag{57}$$

where the sectors C_i , $i = 1, \dots, 8$, are equivalent to those in Example A.2, a_{i1} and a_{i2} , $i = 1, \dots, 8$, are equivalent to the entries of the matrix A in Example A.2, $\bar{N}(\vartheta) = (\bar{N}_1(\vartheta), \bar{N}_2(\vartheta))^T$, and the antipolar set $\mathbb{P}_w^\circ$ of $\mathbb{P}_w$ is the l_1 -ball of Minkowski radius $1/w$, $\mathbb{P}_w^\circ = \{(x_1, x_2)^T \in \mathbb{R}^2 : |x_1| + |x_2| \leq 1/w\}$.

5. Expectations

5.1. Moments of Polyhedral Star-Shaped Distributions. Let $G = (G_1, \dots, G_n)^T$ be uniformly distributed on a star-shaped polyhedron $\mathbb{P}$. Such vector has dgf $g(r) = I_{[0,1]}(r)$. Since $\mathcal{J}(g) = 1/n$ the density of G is

$$\varphi_{g,\mathbb{P}}(x) = \frac{I_{[0,1]}(h_{\mathbb{P}}(x))}{\mu(\mathbb{P})} = \frac{I_{\mathbb{P}}(x)}{\mu(\mathbb{P})}, \quad x \in \mathbb{R}^n. \quad (58)$$

With $\mathbb{E}(G_i^k) = (1/\mu(\mathbb{P})) \int_{\mathbb{P}} x_i^k d\mu$, $i = 1, \dots, n$, the vector of moments of order k is defined as $\mathbb{E}(G^k) = (\mathbb{E}(G_1^k), \dots, \mathbb{E}(G_n^k))^T$. Mixed moments of order $k = k_1 + \dots + k_n$, $k_j \geq 0$, $j \in \{1, \dots, n\}$ are

$$\mathbb{E}(G_1^{k_1} \dots G_n^{k_n}) = \frac{1}{\mu(\mathbb{P})} \int_{\mathbb{P}} x_1^{k_1} \dots x_n^{k_n} d\mu. \quad (59)$$

The numerical computation of these moments of G requires the integration of polynomials over star-shaped polyhedra. To solve this computational problem, especially in higher dimensions, one can use the software package “LattE Integrale” which computes integrals of polynomials over convex polyhedra. The polyhedra can be given by $(\mathcal{V}\mathcal{P})$ or $(\mathcal{H}\mathcal{P})$, and the results are exact if the entries (vertices in case of $(\mathcal{V}\mathcal{P})$ or A and b in case of $(\mathcal{H}\mathcal{P})$) are rational numbers; see [32–34]. Note that star-shaped polyhedra can always be divided into disjoint convex polyhedra $\mathbb{P} \cap C_j$, $j \in \{1, \dots, m\}$, say. Moreover, $\mu(\mathbb{P})$ can also be computed by integrating the function $f(x) = 1$ over $\mathbb{P}$.

Theorem 10. Let $\mathbb{P} \subset \mathbb{R}^n$ be a star-shaped polyhedron, $G = (G_1, \dots, G_n)^T$ uniformly distributed on $\mathbb{P}$ and $X = (X_1, \dots, X_n)^T \sim \Phi_{g,\mathbb{P}}$.

(a) If the moment $\mathbb{E}(R_g^k)$ exists then $\mathbb{E}(X^k) = ((n + k)/n)\mathbb{E}(R_g^k)\mathbb{E}(G^k)$.

(b) If $\mathbb{E}(R_g^k)$ exists then, for all $k_j \geq 0$, $j \in \{1, \dots, n\}$ with $k = k_1 + \dots + k_n$,

$$\mathbb{E}(X_1^{k_1} \dots X_n^{k_n}) = \frac{n+k}{n} \mathbb{E}(R_g^k) \mathbb{E}(G_1^{k_1} \dots G_n^{k_n}). \quad (60)$$

(c) In case of existence, the covariance matrix $\text{cov}(X)$ of X allows the representation

$$\begin{aligned} \text{cov}(X) &= \frac{n+2}{n} \mathbb{E}(R_g^2) \mathbb{E}(GG^T) \\ &\quad - \left(\frac{n+1}{n}\right)^2 (\mathbb{E}(R_g))^2 \mathbb{E}(G) \mathbb{E}(G)^T. \end{aligned} \quad (61)$$

Proof. By the stochastic representation in (52), we have $\mathbb{E}(X^k) = \mathbb{E}(R_g^k \cdot U_{\mathbb{S}}^k) = \mathbb{E}(R_g^k) \mathbb{E}(U_{\mathbb{S}}^k)$. According to Theorem 8, $G \stackrel{d}{=} h_{\mathbb{P}}(G) \cdot U_{\mathbb{S}}$ where $h_{\mathbb{P}}(G)$ and $U_{\mathbb{S}}$ are stochastically

independent, $h_{\mathbb{P}}(G)$ follows the density function $f_{(h_{\mathbb{P}}(G))}(t) = I_{[0,1]}(t)nt^{n-1}$, and

$$\begin{aligned} \mathbb{E}((h_{\mathbb{P}}(G))^k) &= \int_{-\infty}^{\infty} t^k f_{(h_{\mathbb{P}}(G))}(t) dt = n \int_0^1 t^{n+k-1} dt \\ &= \frac{n}{n+k}. \end{aligned} \quad (62)$$

Since $\mathbb{E}(G^k) = \mathbb{E}((h_{\mathbb{P}}(G))^k) \mathbb{E}(U_{\mathbb{S}}^k)$ it follows (a). Parts (b) and (c) can be proved analogously. $\square$

For basic types of dgf g and representations of the moments of R_g , we refer to [12] and to Table 1.

5.2. Characteristic Functions. Integral representations of characteristic functions of arbitrary star-shaped distributed random vectors are derived in [6]. Specific representations are proved for spherically distributed random vectors in [35], for $l_{n,1}$ -symmetrically distributed random vectors in [36] and for $l_{n,p}$ -symmetrically distributed random vectors in [37]. We present a specific integral representation for continuous polyhedral star-shaped distributed random vectors by applying the polyhedral star-shaped polar coordinates to the arbitrary representations from [6]. This change of the coordinate system is very fruitful for the visualization of characteristic functions of continuous polyhedral star-shaped distributions by numerical methods; see Figures 2 and 3.

Theorem 11. If $X \sim \Phi_{g,\mathbb{P}}$ then its characteristic function allows the representation

$$\begin{aligned} \phi_X(t) &= C_{g,\mathbb{P}}(n) \\ &\quad \cdot \left(\int_0^{\infty} \left[\int_{M_n^*} \cos\left(\sum_{i=1}^n t_i x_i(r, \varphi)\right) J^*(\text{SPOL}_{\mathbb{P}})(\varphi) d\varphi \right] \right. \\ &\quad \cdot r^{n-1} g(r) dr \\ &\quad + i \int_0^{\infty} \left[\int_{M_n^*} \sin\left(\sum_{i=1}^n t_i x_i(r, \varphi)\right) J^*(\text{SPOL}_{\mathbb{P}})(\varphi) d\varphi \right] \\ &\quad \cdot r^{n-1} g(r) dr \Big), \quad t \in \mathbb{R}^n. \end{aligned} \quad (63)$$

Proof. Applying the stochastic representation (52) and Theorem 1.1.6 in [38], it is proved in Theorem 9 in [6] that

$$\phi_X(t) = \frac{1}{\mathcal{J}(g)} \int_0^{\infty} \phi_{U_{\mathbb{S}}}(rt) r^{n-1} g(r) dr, \quad t \in \mathbb{R}^n. \quad (64)$$

Applying the representation of $\phi_{U_{\mathbb{S}}}(t)$ in Remark 10 (a) of [6] and the polyhedral star-shaped polar coordinate transformation to (64) yields (63).

Note that if $U_{\mathbb{S}}$ is symmetrically distributed with respect to the origin, $U_{\mathbb{S}} \stackrel{d}{=} -U_{\mathbb{S}}$, the imaginary part in equation (63) vanishes. To be specific, let $(X, Y)^T$ follow the w -star contoured distribution $\Phi_{g,\mathbb{P}_w}$. The characteristic function of $(X, Y)^T$ is

$$\phi_{g,w}(t_1, t_2) = \frac{1}{8w\mathcal{F}(g)} \cdot \int_0^\infty \int_0^{2\pi} \cos\left(\frac{t_1 r \cos(\varphi) + t_2 r \sin(\varphi)}{\max\{|\cos(\varphi)|, |\sin(\varphi)|\} + ((1-w)/w) \min\{|\cos(\varphi)|, |\sin(\varphi)|\}}\right) \cdot \frac{rg(r)}{(\max\{|\cos(\varphi)|, |\sin(\varphi)|\} + ((1-w)/w) \min\{|\cos(\varphi)|, |\sin(\varphi)|\})^2} d\varphi dr. \quad (65)$$

Note that the use of polyhedral star-shaped polar coordinates yields an integral representation of the characteristic function where the range of integration is much easier to describe than if one makes use of other types of coordinates. We exploit this fact for drawing Figures 2 and 3 where this characteristic function is visualized for the particular dgf $g_0(r) = (1 + r^2)^{-2}$. $\square$

5.3. Moment Generating Functions. Considering a random vector $X \sim \Phi_{g,\mathbb{P}}$ the moment generating function $M_X : \mathbb{R}^n \rightarrow \mathbb{R}$ can be formally represented by

$$M_X(t_1, \dots, t_n) = C(g, \mathbb{P}) \cdot \int_{\mathbb{R}^n} e^{(t_1 x_1 + \dots + t_n x_n)} g(h_{\mathbb{P}}(x_1, \dots, x_n)) d(x_1, \dots, x_n). \quad (66)$$

It is possible to specify representation (66) given a certain type of dgf and a certain representation of $\mathbb{P}$. The results are shown in the following theorem.

Theorem 12. Let $X \sim \Phi_{g,\mathbb{P}}$ and the special Kotz-type dgf $g(r) = e^{-\beta r}$, $\beta > 0$ be given. If $h_{\mathbb{P}}$ is represented by Lemma 2 (b1) and it exists an $\epsilon > 0$ such that

$$h_{\mathbb{P}(t)}(x) = \sum_{i=1}^m I_{C_i}(x) \left(\left(a_{i1} - \frac{t_1}{\beta} \right) x_1 + \dots + \left(a_{in} - \frac{t_n}{\beta} \right) x_n \right) \quad (67)$$

for all $t = (t_1, \dots, t_n)^T \in [-\epsilon, \epsilon]^n$ is the Minkowski functional of the star-shaped polyhedron $\mathbb{P}(t) = \{x \in \mathbb{R}^n : h_{\mathbb{P}(t)}(x) \leq 1\}$ then

$$M_X(t) = \frac{\pi_{\mathbb{P}(t)}}{\pi_{\mathbb{P}}}, \quad t = (t_1, \dots, t_n)^T \in [-\epsilon, \epsilon]^n. \quad (68)$$

Proof. Since $C(g, \mathbb{P}) = \beta^n / n\pi_{\mathbb{P}}\Gamma(n)$ it holds

$$\begin{aligned} M_X(t) &= \frac{\beta^n}{n\pi_{\mathbb{P}}\Gamma(n)} \int_{\mathbb{R}^n} e^{\langle t, x \rangle} e^{-\beta h_{\mathbb{P}}(x)} dx = \frac{\beta^n}{n\pi_{\mathbb{P}}\Gamma(n)} \\ &\cdot \sum_{i=1}^m \int_{C_i} e^{-\beta((a_{i1}-t_1/\beta)x_1 + \dots + (a_{in}-t_n/\beta)x_n)} d(x_1, \dots, x_n) \quad (69) \\ &= \frac{\beta^n}{n\pi_{\mathbb{P}}\Gamma(n)} \underbrace{\int_{\mathbb{R}^n} e^{-\beta h_{\mathbb{P}(t)}(x)} dx}_{= n\pi_{\mathbb{P}(t)}\Gamma(n)/\beta^n} = \frac{\pi_{\mathbb{P}(t)}}{\pi_{\mathbb{P}}}. \end{aligned}$$

$\square$

The specific representation (68) of the moment generating function can be applied to compute vectors of moments of polyhedral star-shaped distributions. For an illustration, we refer to Example A.2 in the Appendix.

6. Simulation of Polyhedral Star-Shaped Distributions

The stochastic representations of Theorem 8 and (52) offer the possibility to generate polyhedral star-shaped distributed random points for a given contour defining star-shaped polyhedron $\mathbb{P} \subset \mathbb{R}^n$ and a dgf g . By generating a random number R_g from the density $f_{R_g}(r) = \mathcal{F}(g)^{-1} r^{n-1} g(r)$ and independently of this a polyhedral star-shaped generalized uniformly distributed random point $U_{\mathbb{S}}$ on $\mathbb{S}$, the product $R_g U_{\mathbb{S}}$ gives the desired polyhedral star-shaped distributed random point. $U_{\mathbb{S}}$ can be simulated by generating a uniformly on $\mathbb{P}$ distributed random point G and applying Theorem 8; that is, $U_{\mathbb{S}} = G/h_{\mathbb{P}}(G)$. It remains the simulation of G . To this end, we use a cuboid $Q \subset \mathbb{R}^n$ satisfying $\mathbb{P} \subseteq Q$ and apply the rejection method. For basic details on rejection sampling, we refer to [39]. Applications of the rejection method to the simulation of the p -generalized Gaussian distribution can be found in [29] and to the simulation of the uniform distribution on platonic bodies elsewhere. All single steps are summarized in Algorithm 1.

The sampling method in step (1) has to be specified in accordance with the chosen dgf g . For the dgfs in Table 1, sampling methods can be found in [12]. Proving that G is uniformly distributed on $\mathbb{P}$ and the stopping time of step (2) is finite can be done analogously to Appendix 1 of [29]. A suitable cuboid Q can be found as follows. If $\mathbb{P}$ is represented by $(\mathcal{SV}\mathcal{P})$, one can put

$$\begin{aligned} q_{\min,i} &= \min \{p_{j,k}[i], \quad j = 1, \dots, m, \quad k = 1, \dots, l_j\} \\ &\quad i = 1, \dots, n, \\ q_{\max,i} &= \max \{p_{j,k}[i], \quad j = 1, \dots, m, \quad k = 1, \dots, l_j\} \\ &\quad i = 1, \dots, n, \end{aligned} \quad (70)$$

where $p_{j,k}[i]$ denotes the i th component of the vertex $p_{j,k}$. If $\mathbb{P}$ is represented by $(\mathcal{SH}\mathcal{P})$, one can solve for every $i = 1, \dots, n$, the optimization problems: minimize $q_{\min,i} = x_i$ (maximize $q_{\max,i} = x_i$) subject to $h_{\mathbb{P}}(x_1, \dots, x_n) \leq 1$. Since it appears to be easier to find a suitable Q in case of $(\mathcal{SV}\mathcal{P})$, note that it is possible to transform $\mathbb{P}$ from $(\mathcal{SH}\mathcal{P})$ to $(\mathcal{SV}\mathcal{P})$ according to [40, 41].

FIGURE 2: $\phi_{g_0, w}$ with $w = 0.25$ (a) and its level sets (b). FIGURE 3: $\phi_{g_0, w}$ with $w = 0.8$ (a) and its level sets (b).

Input: $g, h_{\mathbb{P}}, Q = [q_{\min, 1}, q_{\max, 1}] \times \cdots \times [q_{\min, n}, q_{\max, n}]$

Output: X from $\Phi_{g, \mathbb{P}}$

Algorithm

- (1) Sample R_g from $f_{R_g}(r) = \mathcal{J}(g)^{-1} r^{n-1} g(r)$.
- (2) Sample G uniformly from Q until $h_{\mathbb{P}}(G) \leq 1$.
- (3) Build $U_{\mathbb{S}} = G/h_{\mathbb{P}}(G)$ componentwise.
- (4) Build $X = R_g \cdot U_{\mathbb{S}}$ componentwise.

Return X

ALGORITHM 1: Sampling algorithm for polyhedral star-shaped distributions $\Phi_{g, \mathbb{P}}$.

Example 13. Let X be w -star contoured distributed, $X \sim \Phi_{g, \mathbb{P}_w}$ with the Kotz-type dgf $g(r) = g_{K(M, \beta, \gamma)}(r) =$

$r^{2M-2} e^{-\beta r^{2\gamma}}$ where $M, \beta, \gamma \in (0, \infty)$. Figures 4, 5, and 6 show 2000 points simulated independently according to $\Phi_{g, \mathbb{P}_w}$.

7. Location-Scale Transformations of Continuous Polyhedral Star-Shaped Distributions

Throughout this section, we assume that R_g has a finite second-order moment and use the notation $X \sim \Phi_{g, \mathbb{P}, \vartheta, \Psi}$, where ϑ and Ψ denote expectation vector and covariance matrix of X , respectively. We consider location and scale transformations of X and study transformed density level sets.

Theorem 14. Let $X \sim \Phi_{g, \mathbb{P}, \vartheta, \Psi}$ with symmetric and positive definite matrix Ψ . If $\Sigma \in \mathbb{R}^{n \times n}$ is another symmetric and

FIGURE 4: 2000 points (a) simulated according to $\varphi_{g_{K(1,0.5,1)}, \mathbb{P}_{0.3}}$ (b).

positive definite matrix and $\mu \in \mathbb{R}^n$ then there exist regular lower triangular matrices D and V from $\mathbb{R}^{n \times n}$ such that $DD^T = \Psi$, $VV^T = \Sigma$ and the random vector $Z = VD^{-1}(X - \vartheta) + \mu$ follows the distribution $\Phi_{g, \mathbb{P}^{LT}, \mu, \Sigma}$, where $\mathbb{P}^{LT} = VD^{-1}(\mathbb{P} - \vartheta) + \mu$, and the density is represented as

$$\begin{aligned} \varphi_{g, \mathbb{P}^{LT}}(z) \\ = \frac{\det(D) C(g, \mathbb{P})}{\det(V)} g(h_{\mathbb{P}}(DV^{-1}(z - \mu) + \vartheta)), \quad (71) \\ z \in \mathbb{R}^n. \end{aligned}$$

Proof. For the existence of D and V , see [42]. Furthermore, $\mathbb{E}(Z) = VD^{-1}(\mathbb{E}(X) - \vartheta) + \mu = \mu$, and $\text{cov}(Z) = VD^{-1}\Psi(D^{-1})^T V^T = VV^T = \Sigma$. The representation of $\varphi_{g, \mathbb{P}}(z)$ follows by the density transformation formula. $\square$

For an example of the application of the linear transformation in Theorem 14, see Example A.3 in the Appendix. Note that the linear transformation does not only transform location and scale of the given random vector but also transforms the contour defining star-shaped polyhedron $\mathbb{P}$ into a polyhedron being generally not congruent with $\mathbb{P}$. Referring to this, see again Figures 7 and 8. Since noncongruent change of shape is not always a desired property, because $\mathbb{P}$ may be strictly chosen, we introduce a second, orthogonal transformation by using Givens rotations and Givens matrices. For further information about Givens rotations and matrices, see [42].

Definition 15. Let $l \in \{1, \dots, n-1\}$ and $k \in \{l+1, \dots, n\}$. The n -dimensional (l, k) -Givens matrix $S_n(l, k; \alpha) = (S_n(l, k; \alpha))_{(i,j)}$, $i, j \in \{1, \dots, n\}$ is defined by

$$S_n(l, k; \alpha)_{(i,j)} = \begin{cases} \cos(\alpha), & i = j = l \vee i = j = k, \\ \sin(\alpha), & i = l \wedge j = k, \\ -\sin(\alpha), & i = k \wedge j = l, \\ 1, & i = j \wedge i \neq l \wedge i \neq k, \\ 0, & \text{otherwise.} \end{cases} \quad (72)$$

Note that, for every dimension n , there exist $n(n-1)/2$ Givens matrices. A Givens rotation of a vector $x \in \mathbb{R}^n$ defined by $S_n(l, k; \alpha)x$ is an orthogonal transformation and rotates x within the l - k -plane of $\mathbb{R}^n$ by angle α . It follows that the application of all possible matrices $S_n(l, k; \alpha)$ to x allows an orthogonal transformation in all $n(n-1)/2$ two-dimensional planes of $\mathbb{R}^n$. For this reason, the product of all $n(n-1)/2$ n -dimensional Givens-matrices is defined by

$$\begin{aligned} S_n(\alpha_1, \dots, \alpha_{(1/2)n(n-1)}) \\ =: \prod_{i=1}^{n-1} \left(\prod_{j=i+1}^n S_n(i, j; \alpha_{(1/2)(i-1)(2n-i+j-i)}) \right). \quad (73) \end{aligned}$$

By now, it is possible to transform a random vector $X \sim \Phi_{g, \mathbb{P}, \vartheta, \Psi}$ orthogonally with given angles $\alpha_1, \dots, \alpha_{n(n-1)/2}$. The transformed contour defining polyhedron then represents a rotated but congruent representative of given $\mathbb{P}$. The proof of the following theorem can be done analogously to that of Theorem 14.

Theorem 16. Let $X \sim \Phi_{g, \mathbb{P}, \vartheta, \Psi}$. Given the angles $\alpha_1, \dots, \alpha_{n(n-1)/2}$, the random vector $Z = S_n(\alpha_1, \dots, \alpha_{n(n-1)/2})X$ is $\Phi_{g, \mathbb{P}^{OT}, \mu, \Sigma}$ distributed, where $\mathbb{P}^{OT} = S_n(\alpha_1, \dots, \alpha_{n(n-1)/2})\mathbb{P}$, $\Sigma = S_n(\alpha_1, \dots, \alpha_{n(n-1)/2})\Psi S_n(\alpha_1, \dots, \alpha_{n(n-1)/2})^T$, $\mu = S_n(\alpha_1, \dots, \alpha_{n(n-1)/2})\vartheta$, and $\varphi_{g, \mathbb{P}^{OT}}(z) = C(g, \mathbb{P})g(h_{\mathbb{P}}(S_n(\alpha_1, \dots, \alpha_{n(n-1)/2})^T z))$ is the density of Z .

For an illustration of how Theorem 16 applies, see Example A.4 in the Appendix. In case $n = 2$, the Givens rotation reduces to a case considered in [43] for a geometric parametrization of the two-dimensional Gaussian law.

8. Statistics in Location-Scale Families

8.1. Estimating and Testing. Estimating the parameters μ and Σ from a $\Phi_{g, \mathbb{P}, \mu, \Sigma}$ -distributed sample can be done using

FIGURE 5: 2000 points (a) simulated according to $\varphi_{g_{K(3,0.5,1)}, \mathbb{P}_{0.8}}$ (b). FIGURE 6: 2000 points (a) simulated according to $\varphi_{g_{K(1,5,2)}, \mathbb{P}_3}$ (b).

different methods. In a reasonable way, classical empirical moments

$$\hat{\mu} = \frac{1}{k} \sum_{r=1}^k X_r, \quad (74)$$

$$\hat{\Sigma} = \frac{1}{k} \sum_{r=1}^k (X_r - \hat{\mu})(X_r - \hat{\mu})^T$$

turn out to be method of moments estimators. To find maximum-likelihood estimators of the lower triangular matrix $V \in \mathbb{R}^{n \times n}$ with $v_{ii} \neq 0, \forall i$, and the vector $\rho \in \mathbb{R}^n$ in a $\Phi_{g, \mathbb{P}^{LT}, V\theta + \rho, V\Psi V^T}$ -model, one has to numerically solve a nonlinear minimizing problem. Note that there are various algorithms introduced in the literature to solve such problems. To this end, we refer to [44–46] and for a global overview to [47], respectively. A concrete application of this method will be given below.

Similarly, deriving maximum-likelihood estimates of $S_n(\alpha_1, \dots, \alpha_{n/2(n-1)})$ and ρ in a $\Phi_{g, \mathbb{P}^{OT}, S_n(\alpha_1, \dots, \alpha_{n/2(n-1)})\mu, S_n(\alpha_1, \dots, \alpha_{n/2(n-1)})\Psi S_n(\alpha_1, \dots, \alpha_{n/2(n-1)})}$ model needs to solve a

slightly modified nonlinear problem. Moreover, robust parameter estimation following [48, 49] can be seen to successfully work in the present distribution class. An analogous remark holds true for the application in the following section.

Now we discuss how to use one-dimensional Kolmogorov-Smirnov tests to verify if a dataset is distributed according to a given polyhedral star-shaped distribution. To this end, we use the polyhedral star-shaped polar coordinate transformation and Theorem 5 from Section 3.1 to convert the random vector $X \sim \Phi_{g, \mathbb{P}, \vartheta, \Psi}$ into a tuple of random variables and applying the well-known Kolmogorov-Smirnov test to each random variable. Assuming a given realization $(x_1, \dots, x_k)$ of independent and identically distributed random vectors $X_1, \dots, X_k, X_i \in \mathbb{R}^n, i = 1, \dots, k$, we can apply the transformation $SPOL_{\mathbb{P}}^{-1}$ for every $i = 1, \dots, k$ to the realization x_i to generate polyhedral star-shaped polar coordinates $r_i, \varphi_{i,1}, \dots, \varphi_{i,n-1}, i = 1, \dots, k$. For a given polyhedral star-shaped distribution $\Phi_{g, \mathbb{P}, \vartheta, \Psi}$ and $X \sim \Phi_{g, \mathbb{P}, \vartheta, \Psi}$ we can apply Theorem 5 and test the empirical distribution functions of the realizations against the distributions of the

TABLE 2: Model estimates.

	Estimates
Method of moments	$\hat{\mu} = \begin{pmatrix} 39.27992 \\ 44.28225 \end{pmatrix}, \hat{\Sigma} = \begin{pmatrix} 118.70887 & -61.49893 \\ -61.49893 & 66.09405 \end{pmatrix}$
Maximum-likelihood	(a) $\hat{\rho} = \begin{pmatrix} 37.65567 \\ 42.24069 \end{pmatrix}, \hat{V} = \begin{pmatrix} 0.6957135 & 0 \\ -0.1285103 & 0.6720302 \end{pmatrix}$ (b) $\hat{\rho} = \begin{pmatrix} 36.6896 \\ 42.0099 \end{pmatrix}, S_2(\hat{\alpha}) = \begin{pmatrix} \cos(0.1343) & \sin(0.1343) \\ -\sin(0.1343) & \cos(0.1343) \end{pmatrix}$
M-estimation	$\hat{\rho} = \begin{pmatrix} 35.7269 \\ 42.7719 \end{pmatrix}, S_2(\hat{\alpha}) = \begin{pmatrix} \cos(0.1437) & \sin(0.1437) \\ -\sin(0.1437) & \cos(0.1437) \end{pmatrix}$

random variables $R, \Phi_1, \dots, \Phi_{n-1}$. This can be done by the Kolmogorov-Smirnov test statistics

$$T_k(r) = \sup_t |F_k(t, r) - F_R(t)|,$$

$$T_k(\varphi_j) = \sup_{\alpha} |F_k(\alpha, \varphi_j) - F_{\Phi_j}(\alpha)|, \quad (75)$$

$$j = 1, \dots, n-1,$$

where $F_k(t, r)$ and $F_k(\alpha, \varphi_j)$ are the usual empirical distribution functions. Since we consider the marginal distributions we can reject our realization to be $\Phi_{g, \mathbb{P}, \vartheta, \Psi}$ distributed if the tests are rejected for one polar coordinate. Note that the integral representations of F_R and F_{Φ_j} , $j = 1, \dots, n-1$ are difficult to compute, especially in high dimensions. To this end, we can use Algorithm 1 to simulate $\tilde{k}$ independently, $\Phi_{g, \mathbb{P}, \vartheta, \Psi}$ distributed realizations $(\tilde{x}_1, \dots, \tilde{x}_{\tilde{k}})$, apply Theorem 5 to $(\tilde{x}_1, \dots, \tilde{x}_{\tilde{k}})$, replace the exact cdf F_R and F_{Φ_j} , $j = 1, \dots, n-1$ by the empirical cdf $F_{\tilde{k}}(t, \tilde{r})$ and $F_{\tilde{k}}(\alpha, \tilde{\varphi}_j)$, $j = 1, \dots, n-1$, and perform two sample Kolmogorov-Smirnov tests with test statistics

$$T_{k, \tilde{k}}(r, \tilde{r}) = \sup_t |F_k(t, r) - F_{\tilde{k}}(t, \tilde{r})|,$$

$$T_{k, \tilde{k}}(\varphi_j, \tilde{\varphi}_j) = \sup_{\alpha} |F_k(\alpha, \varphi_j) - F_{\tilde{k}}(\alpha, \tilde{\varphi}_j)|, \quad (76)$$

$$j = 1, \dots, n-1.$$

8.2. Application. To conclude our considerations we model a polyhedral star-shaped distribution based on real data using the methods we described in this paper. To be more specific, we consider premature mortality and the median household income in all counties of the state Georgia (USA) and model their joint distribution to describe how premature mortality is influenced by household income in that particular state. Doing this is inspired by the descriptive statistical analysis in [1].

Premature mortality is defined as the all-cause, age-adjusted mortality rate for all individuals based on all deaths occurring before the age of 75 that could have been prevented. The data for the sample of all 159 counties in Georgia are from the CDC Compressed Mortality database

(<https://wonder.cdc.gov>) and from the County Health Ranking of the USA (<http://www.countyhealthrankings.org>) and averaged from 2008 through 2011. For an illustration of the sample, see the scatter plot in Figure 9. Testing this sample for normality via Shapiro-Wilk test is significantly high rejected with a resulting p value $p = 1.516 \cdot 10^{-7}$. Since the density of the bivariate normal distribution is characterized by elliptical level sets (see [43]), the intention to model the data is here to keep the multinormal dgf $g(r) = e^{-r^2/2}$ but to change the contours of the level sets using a specific star-shaped polygon. Inspired by the optical appearance of the data point cloud, we choose $\mathbb{P}$ to be the triangle defined by its vertices $p_1 = (-7, -6)^T$, $p_2 = (-11, 19)^T$, and $p_3 = (21, -6)^T$; see Figure 9. This results in the star-shaped polyhedral distribution $\Phi_{g, \mathbb{P}, \vartheta, \Psi}$, where

$$\vartheta = \begin{pmatrix} 1.87997 \\ 4.3866 \end{pmatrix},$$

$$\Psi = \begin{pmatrix} 203.13237 & -98.91335 \\ -98.91335 & 141.42441 \end{pmatrix}, \quad (77)$$

$$\Phi_{g, \mathbb{P}, \vartheta, \Psi}(x, y)$$

$$= \frac{1}{700} e^{-(1/2)(\max\{-25x/199-4y/199, 25x/333+32y/333, -y/6\})^2},$$

$$x, y \in \mathbb{R}.$$

For the modelling we divide the data randomly into two groups, one for the estimation and one for testing the estimated models. Applying the estimation methods from Section 8.1 for the first group of our data yields estimates that can be found in Table 2.

Applying Theorem 14 with the method of moments estimates and the maximum-likelihood estimates (a) yields $\Phi_{g, \mathbb{P}^{LT}, \hat{\mu}, \hat{\Sigma}}$ and $\Phi_{g, \mathbb{P}^{LT}, \hat{V}, \hat{\rho}, \hat{V}\Psi\hat{V}^T}$, respectively. The application of Theorem 16 with the maximum-likelihood estimates (b) and the m-estimates, respectively, yields $\Phi_{g, \mathbb{P}^{OT}, \hat{\rho}, S_2(\hat{\alpha})\vartheta + \hat{\rho}, S_2(\hat{\alpha})\Psi S_2(\hat{\alpha})^T}$. We tested the second group of the data against these estimated models by using the two sample Kolmogorov-Smirnov tests from Section 8.1 with 10^5 simulated random points of the estimated models. The results are shown in Table 3. It turns out that the estimated

models we generated using orthogonal transformation cannot be accepted as appropriate models of the data in this case. The choice of the contour defining polygon $\mathbb{P}$ was not good enough to model the data properly by using orthogonal transformations of $\Phi_{g,\mathbb{P},\vartheta,\Psi}$. Since the greatest p values of our

tests are received with the maximum-likelihood estimates (a), we prefer $\Phi_{g,\mathbb{P}^{LT},\widehat{V}\vartheta+\widehat{\rho},\widehat{V}\Psi\widehat{V}^T}$ as distribution model of the data, where

$$\begin{aligned}\widehat{V}\vartheta + \widehat{\rho} &= \begin{pmatrix} 38.96359 \\ 44.94702 \end{pmatrix}, \\ \widehat{V}\Psi\widehat{V}^T &= \begin{pmatrix} 98.31958 & -64.40732 \\ -64.40732 & 84.31032 \end{pmatrix}, \\ \varphi_{g,\mathbb{P}^{LT}}(x, y) &= 0.003 \\ &\cdot \exp\left(\frac{1}{2}(\max(-0.1818x - 0.0312y + 8.161, -0.0449x - 0.2585y + 12.611, 0.1313x + 0.149y - 11.2397))^2\right), \\ &x, y \in \mathbb{R}.\end{aligned}\tag{78}$$

For an illustration of the empirical cdf of the marginal distributions used for the two sample Kolmogorov-Smirnov tests of our preferred model, see Figure 10. The density and its level sets of the model are visualized in Figure 11. Since the value of the correlation of our estimated model is -0.7074 we can conclude a strong negative correlation between premature mortality and household income which is a first hint to that lower income results can be accompanied by higher premature mortality rates in Georgia. Furthermore, in a simulation of 10^6 independent $\Phi_{g,\mathbb{P}^{LT},\widehat{V}\vartheta+\widehat{\rho},\widehat{V}\Psi\widehat{V}^T}$ distributed random points, we observed that 36.7% of the points were related to an income higher than 38.96359 and a mortality rate lower than 44.94702, and 36.2% of the points were related to an income lower than 38.96359 and a mortality rate higher than 44.94702. Even when choosing a tolerance level of a half standard deviation $\sigma/2$, the rates are approximately equal, namely, 20.3% (income higher than $38.96359 + \sigma/2$ and a mortality rate lower than $44.94702 - \sigma/2$) and 19.6% (income lower than $38.96359 - \sigma/2$ and a mortality rate higher than $44.94702 + \sigma/2$). This is a second hint to that a higher household income is accompanied by a lower premature mortality rate in Georgia.

9. Discussion and Outlook

In Sections 8.1 and 8.2 we considered m-estimates assuming fully specified dgf g . Referring to Table 1 where density generating functions are given with unknown parameters, one could ask for a simultaneous estimation of these parameters and m-estimates. The simultaneous m-estimation of $\alpha_1, \dots, \alpha_{n(n-1)/2}$, ρ , and parameters of g failed so far by numerical problems. Solving this is postponed to future work.

In Section 8.1 we presented univariate Kolmogorov-Smirnov tests to test data against a given polyhedral star-shaped distribution $\Phi_{g,\mathbb{P},\vartheta,\Psi}$. Since with this method we test with univariate marginal distributions of $\Phi_{g,\mathbb{P},\vartheta,\Psi}$, it

is an open problem to develop a multivariate test for the class of polyhedral star-shaped distributions to validate also the dependencies in the model. Doing this, however, is postponed to future work.

Appendix

Example A.1. Let $\mathbb{P}$ be a pentagon given by the vertices $p_1 = (3, 0)$, $p_2 = (1, 1)$, $p_3 = (-1, 3)$, $p_4 = (-3, 0)$, and $p_5 = (2, -3)$ and $\mathfrak{C} = \{C_1, C_2, C_3, C_4, C_5\}$ a convex polyhedral fan for $\mathbb{P}$, where $C_1 = \{(x, y)^T \in \mathbb{R}^2 : y > 0, y \leq x\}$, $C_2 = \{(x, y)^T \in \mathbb{R}^2 : y > x, -3x \leq y\}$, $C_3 = \{(x, y)^T \in \mathbb{R}^2 : y < -3x, y \geq 0\}$, $C_4 = \{(x, y)^T \in \mathbb{R}^2 : y < 0, y \leq -3/2x\}$, and $C_5 = \{(x, y)^T \in \mathbb{R}^2 : y > -3/2x, y \leq 0\}$; see Figure 12. Applying Lemma 2 (b1), where $H_1 = \overline{p_1 p_2}$, $H_2 = \overline{p_2 p_3}$, $H_3 = \overline{p_3 p_4}$, $H_4 = \overline{p_4 p_5}$, and $H_5 = \overline{p_5 p_1}$ the Minkowski functional of $\mathbb{P}$ can be represented as

$$\begin{aligned}h_{\mathbb{P}}(x, y) &= I_{C_1}(x, y) \left(\frac{1}{3}x + \frac{2}{3}y \right) \\ &\quad + I_{C_2}(x, y) \left(\frac{1}{2}x + \frac{1}{2}y \right) \\ &\quad + I_{C_3}(x, y) \left(-\frac{1}{3}x + \frac{2}{9}y \right) \\ &\quad + I_{C_4}(x, y) \left(-\frac{1}{3}x - \frac{5}{9}y \right) \\ &\quad + I_{C_5}(x, y) \left(\frac{1}{3}x - \frac{1}{9}y \right), \\ &(x, y)^T \in \mathbb{R}^2.\end{aligned}\tag{A.1}$$

FIGURE 7: Tetragon $\mathbb{P}$ (a) and level sets of the density of X (b) in Example A.3. FIGURE 8: Linear transformed contour defining tetragon $\mathbb{P}^{LT}$ (a) and density level sets of the density of Y (b) in Example A.3.

Let $q = (-2, 1.5)$ and $B = \{x \in (\text{conv}(p_1, p_2) \cup \text{conv}(p_2, p_3) \cup \text{conv}(p_3, q))\}$.

Note that $\mathfrak{D}_{\mathbb{S}}(B)$ can be calculated applying

(i) the polyhedral star-shaped polar coordinate representation as

$$\begin{aligned} \mathfrak{D}_{\mathbb{S}}(B) &= \int_{\text{SPOL}_P^{*(-1)}(B)} J^*(\text{SPOL}_P(\varphi)) d\varphi \\ &= \int_0^{\pi + \arctan(-3/4)} \frac{1}{h_P(\cos(\varphi), \sin(\varphi))^2} d\varphi \\ &= \int_0^{\pi/4} \frac{1}{((1/3)\cos(\varphi) + (2/3)\sin(\varphi))^2} d\varphi \end{aligned}$$

$$\begin{aligned} &+ \int_{\pi/4}^{\pi + \arctan(-3)} \frac{1}{((1/2)\cos(\varphi) + (1/2)\sin(\varphi))^2} d\varphi \\ &+ \int_{\pi + \arctan(-3)}^{\pi + \arctan(-3/4)} \frac{1}{(-(1/3)\cos(\varphi) + (2/9)\sin(\varphi))^2} d\varphi \\ &= 3 + 4 + 4.5 = 11.5, \end{aligned} \tag{A.2}$$

(ii) the star-spherical coordinate representation as

$$\begin{aligned} \mathfrak{D}_{\mathbb{S}}(B) &= \int_1^3 \frac{3}{2} d\vartheta + \int_{-1}^1 2 d\vartheta + \int_{-2}^{-1} \frac{9}{2} d\vartheta \\ &= 3 + 4 + 4.5 = 11.5, \end{aligned} \tag{A.3}$$

FIGURE 9: Median household income and premature mortality in the counties of Georgia, averaged from 2008 through 2011 (a) and contour defining triangle $\mathbb{P}$ (b).

FIGURE 10: Simulated empirical cdf of the marginal distributions of R (a) and Φ (b) of the second group of data and $\Phi_{g, \mathbb{P}^{LT}, \nabla \vartheta + \hat{\rho}, \nabla \Psi \nabla^T}$.

(iii) the facet-content representation by calculating $\sqrt{a_1 a_1^T} = 3/\sqrt{5}$, $\sqrt{a_2 a_2^T} = 2/\sqrt{2}$, $\sqrt{a_3 a_3^T} = 9/\sqrt{13}$, $\text{vol}_1(\mathcal{B}_1) = \sqrt{2^2 + 1^2} = \sqrt{5}$, $\text{vol}_1(\mathcal{B}_2) = \sqrt{2^2 + 2^2} = 2\sqrt{2}$, $\text{vol}_1(\mathcal{B}_3) = \sqrt{1^2 + 1.5^2} = \sqrt{13}/2$, and

$$\mathfrak{D}_{\mathbb{S}}(B) = \frac{3}{\sqrt{5}}\sqrt{5} + \frac{2}{\sqrt{2}}2\sqrt{2} + \frac{9}{\sqrt{13}}\frac{1}{2}\sqrt{13} = 11.5 \text{ or } (A.4)$$

(iv) the simplicial representation as

$$\mathfrak{D}_{\mathbb{S}}(B) = \frac{1}{(2-1)!} \left(\left| \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix} \right| + \left| \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \right| + \left| \begin{pmatrix} -1 & -2 \\ 3 & \frac{3}{2} \end{pmatrix} \right| \right) = 11.5. \quad (A.5)$$

Applying the definition of the polyhedral star-shaped generalized uniform distribution (42) we can calculate $\omega_{\mathbb{S}}(B)$ by $\omega_{\mathbb{S}}(B) = \mathfrak{D}_{\mathbb{S}}(B)/\mathfrak{D}_{\mathbb{S}}(\mathbb{S}) = (23/2)/34 = 23/68$. Note that it is also possible to calculate this ratio using the sector measure and cone measure, respectively, since $\mu(\text{sector}_{\mathbb{P}}(B, 1))/\mu(\mathbb{P}) = (23/4)/17 = 23/68$.

TABLE 3: Kolmogorov-Smirnov tests of the data against the estimated distribution models.

	p value of the K. S. test for R	p value of the K. S. test for Φ
Method of moments	0.7782463	0.4186046
Maximum-likelihood	(a) 0.9205786 (b) $6.992589 \cdot 10^{-8}$	(a) 0.6398323 (b) 0.1791245
M-estimation	$3.928482 \cdot 10^{-8}$	0.2240325

Example A.2. In this example we consider again the class of w -stars from Section 4.2. Apart from the representation of $h_{\mathbb{P}_w}$ in the mentioned section it is also possible to apply Lemma 2 (b2) and represent $h_{\mathbb{P}_w}$ by

$$h_{\mathbb{P}_w}(x) = \sum_{i=1}^8 I_{C_i}(x, y) (a_{i1}x + a_{i2}y), \quad (A.6)$$

where a_{i1} and a_{i2} are for all $i = 1, \dots, 8$ the elements of the matrix

$$A = \begin{pmatrix} 1 & \frac{1-w}{w} & -\frac{1-w}{w} & -1 & -1 & -\frac{1-w}{w} & \frac{1-w}{w} & 1 \\ \frac{1-w}{w} & 1 & 1 & \frac{1-w}{w} & -\frac{1-w}{w} & -1 & -1 & -\frac{1-w}{w} \end{pmatrix}^T \quad (\text{A.7})$$

and $C_i = C_{1, \lfloor i/2 \rfloor + 1 \bmod 4} \cap C_{2, \lfloor i/2 \rfloor}$, $i = 1, \dots, 8$, where $C_{1,j}$ and $C_{2,j}$, $j = 1, \dots, 4$ are equivalent to those in Section 4.2. Since $\mathbb{P}(t_1, t_2) = \{(x, y) \in \mathbb{R}^2 : \sum_{i=1}^8 I_{C_i}(x, y)((a_{i1} - t_1)x + (a_{i2} - t_2)y)\}$, $(t_1, t_2)^T \in [-\epsilon, \epsilon]^{\times 2}$, $\epsilon > 0$ it follows

$$M_X(t_1, t_2) = \frac{1}{8} \sum_{i=0}^3 \frac{1/(1 + (-1)^i t_1) + 1/(1 + (-1)^{i+1} t_2)}{1 + (-1)^i w t_1 + (-1)^{i+1} w t_2},$$

$$(t_1, t_2)^T \in [-\epsilon, \epsilon]^{\times 2},$$

$$\frac{\partial}{\partial t_j} M_X(t_1, t_2) = \frac{1}{8}$$

$$\cdot \sum_{i=0}^3 \left[\frac{(-1)^{i+1} w (1/(1 + (-1)^i t_1) + 1/(1 + (-1)^{i+1} t_2))}{(1 + (-1)^i w t_1 + (-1)^{i+1} w t_2)^2} \right. \\ \left. + \frac{(-1)^{i+1}}{(1 + (-1)^i w t_1 + (-1)^{i+1} w t_2)(1 + (-1)^i t_j)} \right],$$

$$j = 1, 2,$$

$$\frac{\partial^2}{\partial t_1} M_X(t_1, t_2) = \frac{1}{4}$$

$$\cdot \sum_{i=0}^3 \left[\frac{w^2 (1/(1 + (-1)^i t_1) + 1/(1 + (-1)^{i+1} t_2))}{(1 + (-1)^i w t_1 + (-1)^{i+1} w t_2)^3} \right. \quad (\text{A.8})$$

$$+ \frac{w}{(1 + (-1)^i w t_1 + (-1)^{i+1} w t_2)^2 (1 + (-1)^i t_1)^2}$$

$$+ \left. \frac{1}{(1 + (-1)^i w t_1 + (-1)^{i+1} w t_2)(1 + (-1)^i t_1)^3} \right],$$

$$\frac{\partial^2}{\partial t_2} M_X(t_1, t_2) = \frac{1}{4}$$

$$\cdot \sum_{i=0}^3 \left[\frac{w^2 (1/(1 + (-1)^i t_1) + 1/(1 + (-1)^{i+1} t_2))}{(1 + (-1)^i w t_1 + (-1)^{i+1} w t_2)^3} \right. \\ + \frac{w}{(1 + (-1)^i w t_1 + (-1)^{i+1} w t_2)^2 (1 + (-1)^{i+1} t_2)^2} \\ \left. + \frac{1}{(1 + (-1)^i w t_1 + (-1)^{i+1} w t_2)(1 + (-1)^{i+1} t_2)^3} \right].$$

Thus we can calculate the first and second vector of moments by

$$\mathbb{E}(X) = \begin{pmatrix} \frac{\partial}{\partial t_1} M_X(t_1, t_2) \\ \frac{\partial}{\partial t_2} M_X(t_1, t_2) \end{pmatrix} \Big|_{(t_1, t_2) = (0,0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\mathbb{E}(X^2) = \begin{pmatrix} \frac{\partial^2}{\partial t_1^2} M_X(t_1, t_2) \\ \frac{\partial^2}{\partial t_1 \partial t_2} M_X(t_1, t_2) \\ \frac{\partial^2}{\partial t_2^2} M_X(t_1, t_2) \end{pmatrix} \Big|_{(t_1, t_2) = (0,0)} \quad (\text{A.9})$$

$$= \begin{pmatrix} 2w^2 + w + 1 \\ 0 \\ 2w^2 + w + 1 \end{pmatrix}.$$

Example A.3. Let $\mathbb{P} \subset \mathbb{R}^2$ be the tetragon defined by the vertices $p_1 = (4, -1)$, $p_2 = (1, 1)$, $p_3 = (-1, 3)$, and $p_4 = (-1, -1)$. Considering the convex polyhedral fan $\mathfrak{C} = \{C_1, C_2, C_3, C_4\}$ for $\mathbb{P}$, where

$$C_1 = \{(x, y)^T \in \mathbb{R}^2 : x > y, -y \leq \frac{1}{4}x\},$$

$$C_2 = \{(x, y)^T \in \mathbb{R}^2 : x \leq y, -3x < y\}, \quad (\text{A.10})$$

$$C_3 = \{(x, y)^T \in \mathbb{R}^2 : -3x \geq y, y > x\},$$

$$C_4 = \{(x, y)^T \in \mathbb{R}^2 : y \leq x, -y > \frac{1}{4}x\},$$

as shown in Figure 7(a), we can represent $h_{\mathbb{P}}$ applying Lemma 2 (b1) as $h_{\mathbb{P}}(x, y) = I_{C_1}(x, y)((2/5)x + (3/5)y) + I_{C_2}(x, y)(x/2 + y/2) - I_{C_3}(x, y)x - I_{C_4}(x, y)y$, $(x, y)^T \in \mathbb{R}^2$.

Assuming the dgf $g(r) = (1 + r^2/2)^{-4}$, it follows $\mathcal{J}(g) = 1/3$ and $C(g, \mathbb{P}) = 1/6$, since $\pi_{\mathbb{P}} = 9$. The random vector $X \sim \Phi_{g, \mathbb{P}, \vartheta, \Psi}$ follows the density $f_{g, \mathbb{P}}(x, y) = 1/6 \cdot (1 + h_{\mathbb{P}}(x, y)^2/2)^{-4}$ and first- and second-order moments are

$$\vartheta = \frac{\pi\sqrt{2}}{96} \begin{pmatrix} 16 \\ 7 \end{pmatrix}, \quad (\text{A.11})$$

$$\Psi = \begin{pmatrix} 2.896 & -1.0917 \\ -1.0917 & 1.7469 \end{pmatrix}.$$

If we wish to transform X by a linear transformation into a random vector $Y \sim \Phi_{g, \mathbb{P}^{LT}, 0_2, E_2}$ where E_2 denotes the two-dimensional identity matrix, we can apply Theorem 14. It

FIGURE 11: Density levels sets (a) and density (b) of $\Phi_{g, \mathbb{P}^{LT}, \tilde{V}\vartheta + \hat{\rho}, \tilde{V}\Psi\tilde{V}^T}$. FIGURE 12: Pentagon $\mathbb{P}$ and the convex polyhedral fan $\mathfrak{C}$ for $\mathbb{P}$ in Example A.1.

follows then $Y = D^{-1}(X - \vartheta)$, where D can be calculated by Cholesky decomposition of Ψ ; thus

$$D = \begin{pmatrix} 1.7018 & 0 \\ -0.6415 & 1.1556 \end{pmatrix}, \quad (A.12)$$

$$D^{-1} = \begin{pmatrix} 0.5876 & 0 \\ 0.3262 & 0.86537 \end{pmatrix}.$$

Y then follows the density

$$f_{g, \mathbb{P}^{LT}}(x, y) = 1.96656 \cdot C(g, \mathbb{P}) \cdot \left(1 + \frac{1}{2} h_{\mathbb{P}}(D(x, y)^T + \vartheta)^2\right)^{-4}. \quad (A.13)$$

For an illustration of the transformed contour defining tetragon $\mathbb{P}^{LT}$ and the density level sets of the transformed density, see Figure 8.

Example A.4. Assume that the polyhedron $\mathbb{P} \subset \mathbb{R}^2$, the dof g , and the random vector $X \sim \Phi_{g, \mathbb{P}, \vartheta, \Psi}$ are the same as in Example A.3. Applying Theorem 16 with

$$S_2(5.74) = \begin{pmatrix} 0.856 & -0.5168 \\ 0.5168 & 0.856 \end{pmatrix}, \quad (A.14)$$

it follows that $Y = S_2(5.74)X$ is $\Phi_{g, \mathbb{P}^{OT}, S_2(5.74)\vartheta, \text{diag}(3.555, 1.0878)}$ distributed, and the resulting density function allows the representation

$$f_{g, \mathbb{P}^{OT}}(x, y) = C(g, \mathbb{P}) \left(1 + \frac{1}{2} h_{\mathbb{P}}(S_2(\alpha)^T(x, y)^T)\right)^{-4}. \quad (A.15)$$

For an illustration of the transformed contour defining tetragon $\mathbb{P}^{OT}$ and the density level sets of the transformed density, see Figure 13.

FIGURE 13: Orthogonal transformed contour defining tetragon $\mathbb{P}^{OT}$ (a) and density level sets of the density of Y (b) in Example A.4.

Competing Interests

The authors declare further that there is no conflict of interests regarding the publication of this paper.

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