Research Article

A Conditional Fourier-Feynman Transform and Conditional Convolution Product with Change of Scales on a Function Space II

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Using a simple formula for conditional expectations over continuous paths, we will evaluate conditional expectations which are types of analytic conditional Fourier-Feynman transforms and conditional convolution products of generalized cylinder functions and the functions in a Banach algebra which is the space of generalized Fourier transforms of the measures on the Borel class of $L_2[0, T]$. We will then investigate their relationships. Particularly, we prove that the conditional transform of the conditional convolution product can be expressed by the product of the conditional transforms of each function. Finally we will establish change of scale formulas for the conditional transforms and the conditional convolution products. In these evaluation formulas and change of scale formulas, we use multivariate normal distributions so that the conditioning function does not contain present positions of the paths.

1. Introduction

Let $C[0, T]$ denote an analogue of Wiener space which is the space of real-valued continuous functions on the interval $[0, T]$ [1]. The conditional expectations which are types of analytic conditional Fourier-Feynman transforms and conditional convolution products on $C[0, T]$ are introduced by the author [2] using a conditioning function which contains present positions of the paths. According to the author's paper, he investigated the effects that drift has on the conditional Fourier-Feynman transform, the conditional convolution product, and various relationships that occur among them. In particular, he derived several change of scale formulas for the conditional transforms and the conditional convolution products, which simplify evaluating the conditional expectations, because the probability measure used on $C[0, T]$ cannot be scale-invariant [3, 4].

Let a be absolutely continuous on $[0, T]$ and let h be of bounded variation with $h \neq 0$ a.e. on $[0, T]$. Define a stochastic process $Z : C[0, T] \times [0, T] \rightarrow \mathbb{R}$ by

$$Z(x, t) = (h\chi_{[0, t]}, x) + x(0) + a(t), \quad (1)$$

for $x \in C[0, T]$ and $t \in [0, T]$, where $(\cdot, \cdot)$ denotes the Paley-Wiener-Zygmund stochastic integral [1]. For a partition $0 = t_0 < t_1 < \dots < t_n < t_{n+1} = T$ of $[0, T]$, define random vectors $Z_n : C[0, T] \rightarrow \mathbb{R}^{n+1}$ and $Z_{n+1} : C[0, T] \rightarrow \mathbb{R}^{n+2}$ by

$$\begin{aligned} Z_n(x) &= (Z(x, t_0), Z(x, t_1), \dots, Z(x, t_n)), \\ Z_{n+1}(x) &= (Z(x, t_0), Z(x, t_1), \dots, Z(x, t_n), Z(x, t_{n+1})). \end{aligned} \quad (2)$$

With the conditioning function Z_{n+1} which contains the present position $Z(x, T)$ of the path $Z(x, \cdot)$, the author [5] evaluated the conditional Fourier-Feynman transforms and the conditional convolution products of the functions given by

$$\begin{aligned} &\int_{L_2[0, T]} \exp\{i(v, Z(x, \cdot))\} d\sigma(v), \\ &f((v_1, Z(x, \cdot)), \dots, (v_r, Z(x, \cdot))), \end{aligned} \quad (3)$$

where σ is a complex Borel measure of bounded variation on $L_2[0, T]$, $f \in L_p(\mathbb{R}^r)$ with $1 \leq p \leq \infty$, and

$\{v_1, \dots, v_r\}$ is an orthonormal subset of $L_2[0, T]$. He then investigated relationships between the conditional Fourier-Feynman transforms and the conditional convolution products of the functions given by (3).

In this paper, using a simple formula for conditional expectations over continuous paths [6], we evaluate conditional expectations of generalized cylinder functions and the functions in a Banach algebra which plays significant roles in Feynman integration theories and quantum mechanics. We then investigate their relationships. Particularly, we prove that the conditional transform of the convolution product can be expressed by the product of the conditional transforms of each function. Finally we establish change of scale formulas for the conditional expectations. In these evaluation formulas and change of scale formulas we use multivariate normal distributions so that the conditioning function does not contain the present positions of the paths. In fact, with the conditioning function Z_n which does not contain the present position $Z(x, T)$ of the path $Z(x, \cdot)$, we evaluate conditional expectations, that is, the conditional Fourier-Feynman transforms and the conditional convolution products of the functions given by (3). We then show that the L_p -analytic conditional Fourier-Feynman transform $T_q^{(p)}[(F_Z * G_Z)_q | Z_n](\cdot, \vec{\xi}_n | Z_n)$ of the conditional convolution product for the functions F_Z and G_Z which are described in (3) can be expressed by the formula

$$\begin{aligned} & T_q^{(p)} \left[[(F_Z * G_Z)_q | Z_n](\cdot, \vec{\xi}_n | Z_n)(y, \vec{\eta}_n) \right] \\ &= \left[T_q^{(p)} [F_Z | Z_n] \left(\frac{1}{\sqrt{2}} y \right. \right. \\ & \quad \left. \left. + (\sqrt{2} - 1)a, \frac{1}{\sqrt{2}}(\vec{\eta}_n + \vec{\xi}_n) - (\sqrt{2} - 1)\vec{a}_n \right) \right] \\ & \quad \cdot \left[T_q^{(p)} [G_Z | Z_n] \left(\frac{1}{\sqrt{2}} y - a, \frac{1}{\sqrt{2}}(\vec{\eta}_n - \vec{\xi}_n) + \vec{a}_n \right) \right], \end{aligned} \quad (4)$$

for a nonzero real q , almost surely $y \in C[0, T]$, and P_{Z_n} almost surely $\vec{\xi}_n, \vec{\eta}_n \in \mathbb{R}^{n+1}$, where $\vec{a}_n = (a(t_0), a(t_1), \dots, a(t_n))$ and P_{Z_n} is the probability distribution of Z_n on the Borel class of $\mathbb{R}^{n+1}$. Compared with the results in [5, 7], the conditioning function Z_n in this paper does not contain the present position $Z(x, T)$ of $Z(x, \cdot)$ and the effects of drift a depend on the polygonal function of a so that we can generalize the theorems in [5, 7] and the results of this paper do not depend on a particular choice of the initial distribution of the paths.

2. Preliminaries on an Analogue of Wiener Space

We begin this section with introducing a concrete form of an analogue of Wiener space which is our underlying space.

For a positive real T let $C[0, T]$ denote the space of real-valued continuous functions on the time interval $[0, T]$ with the supremum norm. For $\vec{t} = (t_0, t_1, \dots, t_n)$ with $0 = t_0 < t_1 < \dots < t_n \leq T$ let $J_{\vec{t}} : C[0, T] \rightarrow \mathbb{R}^{n+1}$ be the function given by $J_{\vec{t}}(x) = (x(t_0), x(t_1), \dots, x(t_n))$. For B_j ($j = 0, 1, \dots, n$) in the

Borel class $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$, the subset $J_{\vec{t}}^{-1}(\prod_{j=0}^n B_j)$ of $C[0, T]$ is called an interval and let $\mathcal{I}$ be the set of all such intervals. For a probability measure φ on $\mathcal{B}(\mathbb{R})$, let

$$\begin{aligned} m_\varphi \left[J_{\vec{t}}^{-1} \left(\prod_{j=0}^n B_j \right) \right] &= \left[\prod_{j=1}^n \frac{1}{2\pi(t_j - t_{j-1})} \right]^{1/2} \\ &\cdot \int_{B_0} \int_{\prod_{j=1}^n B_j} W_n(\vec{t}, \vec{u}, u_0) d\vec{u} d\varphi(u_0), \end{aligned} \quad (5)$$

where $W(\vec{t}, \vec{u}, u_0) = \exp\{-(1/2) \sum_{j=1}^n ((u_j - u_{j-1})^2 / (t_j - t_{j-1}))\}$ for $\vec{u} = (u_1, \dots, u_n)$. $\mathcal{B}(C[0, T])$, the Borel σ -algebra of $C[0, T]$, coincides with the smallest σ -algebra generated by $\mathcal{I}$ and there exists a unique probability measure w_φ on $C[0, T]$ such that $w_\varphi(I) = m_\varphi(I)$ for all $I \in \mathcal{I}$. This measure w_φ is called an analogue of Wiener measure associated with the probability measure φ [1].

Let $\mathbb{C}$ and $\mathbb{C}_+$ denote the sets of complex numbers and complex numbers with positive real parts, respectively. Let $F : C[0, T] \rightarrow \mathbb{C}$ be integrable and let X_τ be a random vector on $C[0, T]$ assuming that the value space of X_τ is a normed space with the Borel σ -algebra. Then we have the conditional expectation $E[F | X_\tau]$ of F given X_τ from a well-known probability theory [8]. Furthermore, there exists a P_{X_τ} integrable complex-valued function ϕ on the value space of X_τ such that

$$E[F | X_\tau](x) = (\phi \circ X_\tau)(x) \quad \text{for } w_\varphi \text{ a.e. } x \in C[0, T], \quad (6)$$

where P_{X_τ} is the probability distribution of X_τ . The function ϕ is called the conditional w_φ -integral of F given X_τ and it is also denoted by $E[F | X_\tau]$.

For an extended real number p with $1 < p \leq \infty$, suppose that p and p' are related by $1/p + 1/p' = 1$ (possibly $p' = 1$ if $p = \infty$). Let $q \in \mathbb{R} - \{0\}$. For $\lambda \in \mathbb{C}_+$ let F_λ be a measurable function on $C[0, T]$ such that

$$\lim_{\lambda \rightarrow -iq} E[|F_\lambda - F|^{p'}] = 0. \quad (7)$$

Then we write

$$\lim_{\lambda \rightarrow -iq} (w^{p'}) (F_\lambda) = F. \quad (8)$$

Let $0 = t_0 < t_1 < \dots < t_n < t_{n+1} = T$ be a partition of $[0, T]$, where n is a fixed nonnegative integer. Let h be of bounded variation with $h \neq 0$ a.e. on $[0, T]$. Let a be absolutely continuous on $[0, T]$ and define stochastic processes $X, Z : C[0, T] \times [0, T] \rightarrow \mathbb{R}$ by

$$\begin{aligned} X(x, t) &= (h\chi_{[0, t]}, x), \\ Z(x, t) &= X(x, t) + x(0) + a(t), \end{aligned} \quad (9)$$

for $x \in C[0, T]$ and $t \in [0, T]$, where χ denotes an indicator function. Define a random vector $Z_n : C[0, T] \rightarrow \mathbb{R}^{n+1}$ by

$$Z_n(x) = (Z(x, t_0), Z(x, t_1), \dots, Z(x, t_n)), \quad (10)$$

for $x \in C[0, T]$. For $t \in [0, T]$, let $b(t) = \int_0^t [h(s)]^2 ds$ and for any function f on $[0, T]$, define a polygonal function $P_{b,n+1}(f)$ of f by

$$P_{b,n+1}(f)(t) = \sum_{j=1}^{n+1} \left[\frac{b(t_j) - b(t)}{b(t_j) - b(t_{j-1})} f(t_{j-1}) + \frac{b(t) - b(t_{j-1})}{b(t_j) - b(t_{j-1})} f(t_j) \right] \chi_{(t_{j-1}, t_j]}(t) + f(0) \chi_{\{0\}}(t), \quad (11)$$

for $t \in [0, T]$. For $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \dots, \xi_n, \xi_{n+1}) \in \mathbb{R}^{n+2}$, define a polygonal function $P_{b,n+1}(\vec{\xi}_{n+1})$ of $\vec{\xi}_{n+1}$ by (11), where $f(t_j)$ is replaced by ξ_j . If $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$, $P_{b,n}(\vec{\xi}_n)$ is interpreted as $P_{b,n+1}(\vec{\xi}_{n+1})$ with $\xi_{n+1} = \xi_n$. For $x \in C[0, T]$ and $t \in [0, T]$ let

$$\begin{aligned} A(t) &= a(t) - P_{b,n+1}(a)(t), \\ X_{b,n+1}(x, t) &= X(x, t) - P_{b,n+1}(X(x, \cdot))(t), \\ Z_{b,n+1}(x, t) &= Z(x, t) - P_{b,n+1}(Z(x, \cdot))(t). \end{aligned} \quad (12)$$

For $\vec{a}, \vec{u} \in \mathbb{R}^r$, $\lambda \in \mathbb{C}$, and any nonsingular positive $r \times r$ matrix A_r on $\mathbb{R}$, let

$$\begin{aligned} \Psi_r(\lambda, \vec{a}, A_r, \vec{u}) &= \left[\frac{\lambda^r}{(2\pi)^r |A_r|} \right]^{1/2} \\ &\cdot \exp \left\{ -\frac{\lambda}{2} \langle A_r^{-1}(\vec{u} - \vec{a}), \vec{u} - \vec{a} \rangle_{\mathbb{R}} \right\}, \end{aligned} \quad (13)$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ denotes the dot product on $\mathbb{R}^r$. Let I_r be the $r \times r$ identity matrix on $\mathbb{R}$. For a measurable function $F : C[0, T] \rightarrow \mathbb{C}$, let

$$F_Z(x, y) = F(Z(x, \cdot) + y) \quad \text{for } x, y \in C[0, T], \quad (14)$$

let $F_Z^\lambda(x, y) = F_Z(\lambda^{-1/2}x, y)$, and let $Z_n^\lambda(x) = Z_n(\lambda^{-1/2}x)$ for $\lambda > 0$. Suppose that $E[F_Z^\lambda(\cdot, y)]$ exists, where the expectation

is taken over the first variable. By Theorem 2 in [2] and Theorem 7 in [6]

$$\begin{aligned} E[F_Z^\lambda(\cdot, y) | Z_n^\lambda](\vec{\xi}_n) &= \int_{\mathbb{R}} \Psi_1(\lambda, a(T) - a(t_n), b(T) \\ &\quad - b(t_n), \xi_{n+1} - \xi_n) E[F(y + \lambda^{-1/2}X_{b,n+1}(x, \cdot) + A \\ &\quad + P_{b,n+1}(\vec{\xi}_{n+1}))] d\xi_{n+1}, \end{aligned} \quad (15)$$

for $P_{Z_n}^\lambda$ a.e. $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$, where $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \dots, \xi_n, \xi_{n+1})$, $P_{Z_n}^\lambda$ is the probability distribution of Z_n^λ on $(\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}))$ and the expectation is taken over the variable x . Let $I_{F_Z}^\lambda(y, \vec{\xi}_n)$ be the right-hand side of (15). If, for w_φ a.e. $y \in C[0, T]$ and P_{Z_n} a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, $I_{F_Z}^\lambda(y, \vec{\xi}_n)$ has an analytic extension $J_\lambda^*(F_Z)(y, \vec{\xi}_n)$ on $\mathbb{C}_+$, then it is called a generalized analytic conditional Fourier-Wiener transform of F given Z_n with the parameter λ and is denoted by

$$T_\lambda[F_Z | Z_n](y, \vec{\xi}_n) = J_\lambda^*(F_Z)(y, \vec{\xi}_n). \quad (16)$$

Moreover if $T_\lambda[F_Z | Z_n](y, \vec{\xi}_n)$ has a limit as λ approaches $-iq$ through $\mathbb{C}_+$, then it is called a generalized L_1 -analytic conditional Fourier-Feynman transform of F given Z_n with the parameter q and is denoted by

$$T_q^{(1)}[F_Z | Z_n](y, \vec{\xi}_n) = \lim_{\lambda \rightarrow -iq} T_\lambda[F_Z | Z_n](y, \vec{\xi}_n). \quad (17)$$

For $1 < p \leq \infty$, define a generalized L_p -analytic conditional Fourier-Feynman transform $T_q^{(p)}[F_Z | Z_n]$ of F given Z_n by the formula

$$\begin{aligned} T_q^{(p)}[F_Z | Z_n](\cdot, \vec{\xi}_n) \\ = \lim_{\lambda \rightarrow -iq} (w^{p'}) (T_\lambda[F_Z | Z_n](\cdot, \vec{\xi}_n)) \quad (\text{if exists}). \end{aligned} \quad (18)$$

Let G be defined on $C[0, T]$. For $\lambda > 0$ and $y \in C[0, T]$, redefine $F_Z^\lambda(\cdot/\sqrt{2}, y/\sqrt{2})G_Z^\lambda(-\cdot/\sqrt{2}, y/\sqrt{2}) = F_{Z/\sqrt{2}}^\lambda(\cdot, y/\sqrt{2})G_{-Z/\sqrt{2}}^\lambda(\cdot, y/\sqrt{2})$ and suppose that $E[F_Z^\lambda(\cdot/\sqrt{2}, y/\sqrt{2})G_Z^\lambda(-\cdot/\sqrt{2}, y/\sqrt{2})]$ exists. By Theorem 2 in [2] and Theorem 7 in [6]

$$\begin{aligned} E \left[F_Z^\lambda \left(\frac{\cdot}{\sqrt{2}}, \frac{y}{\sqrt{2}} \right) G_Z^\lambda \left(-\frac{\cdot}{\sqrt{2}}, \frac{y}{\sqrt{2}} \right) | Z_n^\lambda \right] (\vec{\xi}_n) &= \int_{\mathbb{R}} \Psi_1(\lambda, a(T) - a(t_n), b(T) - b(t_n), \xi_{n+1} - \xi_n) \\ &\cdot E \left[F \left(\frac{1}{\sqrt{2}} [y + \lambda^{-1/2}X_{b,n+1}(x, \cdot) + A + P_{b,n+1}(\vec{\xi}_{n+1})] \right) \right. \\ &\cdot G \left(\frac{1}{\sqrt{2}} [y - \lambda^{-1/2}X_{b,n+1}(x, \cdot) - A - P_{b,n+1}(\vec{\xi}_{n+1})] \right) \Big] d\xi_{n+1}. \end{aligned} \quad (19)$$

Let $K_{F_Z, G_Z}^\lambda(y, \vec{\xi}_n)$ be the right-hand side of (19). If $K_{F_Z, G_Z}^\lambda(y, \vec{\xi}_n)$ has an analytic extension $J_\lambda^*(F_Z, G_Z)(y, \vec{\xi}_n)$ on $\mathbb{C}_+$, then it is called a generalized conditional convolution product of F and G given Z_n with the parameter λ and denoted by

$$[(F_Z * G_Z)_\lambda | Z_n](y, \vec{\xi}_n) = J_\lambda^*(F_Z, G_Z)(y, \vec{\xi}_n). \quad (20)$$

Moreover if $[(F_Z * G_Z)_\lambda | Z_n](y, \vec{\xi}_n)$ has a limit as λ approaches $-iq$ through $\mathbb{C}_+$, then it is called a generalized

conditional convolution product of F and G given Z_n with the parameter q and denoted by

$$\begin{aligned} & [(F_Z * G_Z)_q | Z_n] (y, \vec{\xi}_n) \\ &= \lim_{\lambda \rightarrow -iq} [(F_Z * G_Z)_\lambda | Z_n] (y, \vec{\xi}_n). \end{aligned} \quad (21)$$

For $j = 1, \dots, n+1$, let

$$\alpha_j = \frac{h}{\sqrt{b(t_j) - b(t_{j-1})}} \chi_{(t_{j-1}, t_j]}, \quad (22)$$

let V be the subspace of $L_2[0, T]$ generated by $\{\alpha_1, \dots, \alpha_n, \alpha_{n+1}\}$, and let $V^\perp$ be the orthogonal complement of V . Let $\mathcal{P} : L_2[0, T] \rightarrow V$ be the orthogonal projection given by

$$\mathcal{P}v = \sum_{j=1}^{n+1} \langle v, \alpha_j \rangle \alpha_j, \quad (23)$$

and let $\mathcal{P}^\perp : L_2[0, T] \rightarrow V^\perp$ be the orthogonal projection. For $\vec{u} \in \mathbb{R}^r$, $\lambda \in \mathbb{C}$, $\vec{v} = (v_1, \dots, v_r) \in L_2^n[0, T]$ and any nonsingular positive $r \times r$ matrix A_r on $\mathbb{R}$, let

$$\begin{aligned} & \langle \vec{v} \alpha_{n+1}, \alpha_{n+1} \rangle = (\langle v_1 \alpha_{n+1}, \alpha_{n+1} \rangle, \dots, \langle v_r \alpha_{n+1}, \alpha_{n+1} \rangle), \\ & \Lambda(A_r) = 1 + [b(T) - b(t_n)] \left\| A_r^{-1} \langle \vec{v} \alpha_{n+1}, \alpha_{n+1} \rangle \right\|_{\mathbb{R}}^2, \\ & \Phi_r(\lambda, \vec{v}, A_r, \vec{u}) = \left[\frac{1}{\Lambda(A_r)} \right]^{1/2} \\ & \cdot \exp \left\{ \frac{\lambda}{2\Lambda(A_r)} [b(T) - b(t_n)] \right. \\ & \cdot \left. \left\langle \vec{u}, A_r^{-1} \langle \vec{v} \alpha_{n+1}, \alpha_{n+1} \rangle \right\rangle_{\mathbb{R}}^2 \right\}. \end{aligned} \quad (24)$$

The following lemmas are useful to prove the results in the next sections [5].

Lemma 1. Let $v \in L_2[0, T]$. Then for w_φ a.e. $x \in C[0, T]$

$$\begin{aligned} & (v, X(x, \cdot)) = (M_h v, x), \\ & (v, P_{b, n+1}(X(x, \cdot))) = (\mathcal{P} M_h v, x), \end{aligned} \quad (25)$$

where $M_h : L_2[0, T] \rightarrow L_2[0, T]$ is the multiplication operator defined by

$$M_h u = hu \quad \text{for } u \in L_2[0, T]. \quad (26)$$

Lemma 2. Let $v \in L_2[0, T]$ and $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \dots, \xi_n, \xi_{n+1}) \in \mathbb{R}^{n+2}$. Then

$$\begin{aligned} & (v, P_{b, n+1}(\vec{\xi}_{n+1})) = (v, P_{b, n}(\vec{\xi}_n)) \\ & + \langle v \alpha_{n+1}, \alpha_{n+1} \rangle (\xi_{n+1} - \xi_n), \end{aligned} \quad (27)$$

where $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n)$.

Lemma 3. Let $\{v_1, \dots, v_r\}$ be a subset of $L_2[0, T]$ such that $\{M_h v_1, \dots, M_h v_r\}$ is an independent set. Then the random vector $(\vec{v}, Z(x, \cdot)) \equiv ((v_1, Z(x, \cdot)), \dots, (v_r, Z(x, \cdot)))$ has the multivariate normal distribution with mean vector $(\vec{v}, a) \equiv ((v_1, a), \dots, (v_r, a))$ and covariance matrix $\Sigma_{M_h} = [\langle M_h v_i, M_h v_j \rangle]_{r \times r}$. Moreover, for any Borel measurable function $f : \mathbb{R}^{r+1} \rightarrow \mathbb{C}$ one has

$$\begin{aligned} & \int_{C[0, T]} f(x(0), (\vec{v}, Z(x, \cdot))) dw_\varphi(x) \\ & \stackrel{*}{=} \int_{\mathbb{R}} \int_{\mathbb{R}^r} f(u_0, \vec{u}) \Psi_r(1, (\vec{v}, a), \Sigma_{M_h}, \vec{u}) d\vec{u} d\varphi(u_0), \end{aligned} \quad (28)$$

where $\stackrel{*}{=}$ means that if either side exists, then both sides exist and they are equal.

Remark 4. Because the Borel sets of $C[0, T]$ are always scale-invariant measurable and we use the Borel class of $C[0, T]$ on which w_φ is defined [1], the scale-invariant measurability is not essential in (7).

3. Conditional Fourier-Wiener and Fourier-Feynman Transforms

Let $1 \leq p \leq \infty$, let r be any fixed positive integer, and let $\{v_1, v_2, \dots, v_r\}$ be an orthonormal subset of $L_2[0, T]$ such that both $\{M_h v_1, \dots, M_h v_r\}$ and $\{\mathcal{P}^\perp M_h v_1, \dots, \mathcal{P}^\perp M_h v_r\}$ are independent sets. Let $\mathcal{A}^{(p)}$ be the space of cylinder functions F having the form

$$F(x) = f((\vec{v}, x)), \quad (29)$$

for w_φ a.e. $x \in C[0, T]$, where $f \in L_p(\mathbb{R}^r)$ and $(\vec{v}, x) = ((v_1, x), \dots, (v_r, x))$. Without loss of generality we can take f to be Borel measurable.

Theorem 5. Let $1 \leq p \leq \infty$ and let $F \in \mathcal{A}^{(p)}$ be given by (29). Then for $\lambda \in \mathbb{C}_+$

$$\begin{aligned} & T_\lambda [F_Z | Z_n] (y, \vec{\xi}_n) \\ &= \int_{\mathbb{R}^r} f(\Sigma_{\mathcal{P}^\perp}^{1/2} \vec{u} + (\vec{v}, y + a + P_{b, n}(\vec{\xi}_n - \vec{a}_n))) \\ & \cdot \Psi_r(\lambda, \vec{0}, I_r, \vec{u}) \Phi_r(\lambda, \vec{v}, \Sigma_{\mathcal{P}^\perp}^{1/2}, \vec{u}) d\vec{u} \\ &= (f_r(\vec{\xi}_n - \vec{a}_n, a, \cdot)) \\ & * (\Psi_r(\lambda, \vec{0}, I_r, \cdot) \Phi_r(\lambda, \vec{v}, \Sigma_{\mathcal{P}^\perp}^{1/2}, \cdot)) (\Sigma_{\mathcal{P}^\perp}^{-1/2}(\vec{v}, y)), \end{aligned} \quad (30)$$

for w_φ a.e. $y \in C[0, T]$ and P_{Z_n} a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, where $\Sigma_{\mathcal{P}^\perp} = [\langle \mathcal{P}^\perp M_h v_i, \mathcal{P}^\perp M_h v_j \rangle]$, $\vec{a}_n = (a(t_0), a(t_1), \dots, a(t_n))$, and

$$f_r(\vec{\xi}_n, a, \vec{u}) = f(\Sigma_{\mathcal{P}^\perp}^{1/2} \vec{u} + (\vec{v}, a + P_{b, n}(\vec{\xi}_n))). \quad (31)$$

Moreover $T_\lambda [F_Z | Z_n](\cdot, \vec{\xi}_n) \in \mathcal{A}^{(p)}$.

Proof. For $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$ let $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \dots, \xi_n, \xi_{n+1})$, where $\xi_{n+1} \in \mathbb{R}$. For $\lambda > 0$ and w_φ a.e. $x \in C[0, T]$ we have by Lemmas 1 and 2

$$\begin{aligned} I_{F_Z}^\lambda(y, \vec{\xi}_n) &= \int_{\mathbb{R}} \int_{\mathbb{R}^r} \Psi_1(\lambda, a(T) - a(t_n), b(T) \\ &\quad - b(t_n), \xi_{n+1} - \xi_n) f(\Sigma_{\mathcal{D}^\perp}^{1/2} \vec{u} + (\vec{v}, y) \\ &\quad + (\vec{v}, A + P_{b,n+1}(\vec{\xi}_{n+1}))) \\ &\quad \cdot \Psi_r(\lambda, \vec{0}, I_r, \vec{u}) d\vec{u} d\xi_{n+1} \\ &= \int_{\mathbb{R}^r} \int_{\mathbb{R}} \Psi_1(\lambda, 0, b(T) - b(t_n), \xi_{n+1}) f(\Sigma_{\mathcal{D}^\perp}^{1/2} \vec{u} \\ &\quad + (\vec{v}, y) + (\vec{v}, A) + (\vec{v}, P_{b,n}(\vec{\xi}_n)) \\ &\quad + (a(T) - a(t_n)) \langle \vec{v} \alpha_{n+1}, \alpha_{n+1} \rangle \\ &\quad + \langle \vec{v} \alpha_{n+1}, \alpha_{n+1} \rangle \xi_{n+1}) \\ &\quad \cdot \Psi_r(\lambda, \vec{0}, I_r, \vec{u}) d\xi_{n+1} d\vec{u}, \end{aligned} \quad (32)$$

where A and Ψ_r are given by (12) and (13), respectively. Using the same method as used in the proof of Theorem 3.2 in [7]

$$\begin{aligned} I_{F_Z}^\lambda(y, \vec{\xi}_n) &= \int_{\mathbb{R}^r} f(\Sigma_{\mathcal{D}^\perp}^{1/2} \vec{u} + (\vec{v}, y) + (\vec{v}, A) \\ &\quad + (\vec{v}, P_{b,n}(\vec{\xi}_n)) + (a(T) - a(t_n)) \langle \vec{v} \alpha_{n+1}, \alpha_{n+1} \rangle) \\ &\quad \cdot \Psi_r(\lambda, \vec{0}, I_r, \vec{u}) \Phi_r(\lambda, \vec{v}, \Sigma_{\mathcal{D}^\perp}^{1/2}, \vec{u}) d\vec{u} \\ &= \int_{\mathbb{R}^r} f(\Sigma_{\mathcal{D}^\perp}^{1/2} \vec{u} + (\vec{v}, y + a + P_{b,n}(\vec{\xi}_n - \vec{a}_n))) \\ &\quad \cdot \Psi_r(\lambda, \vec{0}, I_r, \vec{u}) \Phi_r(\lambda, \vec{v}, \Sigma_{\mathcal{D}^\perp}^{1/2}, \vec{u}) d\vec{u}. \end{aligned} \quad (33)$$

We note that if $1 \leq p < \infty$, then by the change of variable theorem

$$\|f_r(\vec{\xi}_n - \vec{a}_n, a, \cdot)\|_p^p = |\Sigma_{\mathcal{D}^\perp}^{-1/2}| \|f\|_p^p < \infty, \quad (34)$$

and for $\lambda \in \mathbb{C}_+$ we have by Schwarz's inequality

$$\begin{aligned} |\Psi_r(\lambda, \vec{0}, I_r, \vec{u}) \Phi_r(\lambda, \vec{v}, \Sigma_{\mathcal{D}^\perp}^{1/2}, \vec{u})| \\ \leq \left(\frac{|\lambda|}{2\pi} \right)^{r/2} \exp \left\{ -\frac{\operatorname{Re} \lambda}{2\Lambda(\Sigma_{\mathcal{D}^\perp}^{1/2})} \|\vec{u}\|_{\mathbb{R}}^2 \right\}. \end{aligned} \quad (35)$$

Now, by Morera's theorem with aids of Hölder's inequality and the dominated convergence theorem, we have (30) for $\lambda \in \mathbb{C}_+$. Since $f_r(\vec{\xi}_n, a, \cdot) \in L_p(\mathbb{R}^r)$ and $\Psi_r(\lambda, \vec{0}, I_r, \cdot) \Phi_r(\lambda, \vec{v}, \Sigma_{\mathcal{D}^\perp}^{1/2}, \cdot) \in L_1(\mathbb{R}^r)$ by (35), the final result follows by the change of variable theorem and Young's inequality [9]. $\square$

Corollary 6. Let $1 \leq p \leq \infty$ and let $F(\in \mathcal{A}^{(p)})$ be given by (29).

(1) If $n = 0$, then one has for $\lambda \in \mathbb{C}_+$, $y \in C[0, T]$ and $\xi_0 \in \mathbb{R}$

$$\begin{aligned} T_\lambda[F_Z | Z_0](y, \xi_0) &= \int_{\mathbb{R}^r} f(\Sigma_{\mathcal{D}^\perp}^{1/2} \vec{u} + (\vec{v}, y + a)) \\ &\quad \cdot \Psi_r(\lambda, \vec{0}, I_r, \vec{u}) \Phi_r^0(\lambda, \vec{v}, \Sigma_{\mathcal{D}^\perp}^{1/2}, \vec{u}) d\vec{u}, \end{aligned} \quad (36)$$

where for $\Lambda^0(\Sigma_{\mathcal{D}^\perp}^{1/2}) = 1 + b(T) \|\Sigma_{\mathcal{D}^\perp}^{-1/2} \langle \vec{v} \alpha_1, \alpha_1 \rangle\|_{\mathbb{R}}^2$

$$\begin{aligned} \Phi_r^0(\lambda, \vec{v}, \Sigma_{\mathcal{D}^\perp}^{1/2}, \vec{u}) &= \left[\frac{1}{\Lambda^0(\Sigma_{\mathcal{D}^\perp}^{1/2})} \right]^{1/2} \\ &\quad \cdot \exp \left\{ \frac{\lambda b(T)}{2\Lambda^0(\Sigma_{\mathcal{D}^\perp}^{1/2})} \langle \vec{u}, \Sigma_{\mathcal{D}^\perp}^{-1/2} \langle \vec{v} \alpha_1, \alpha_1 \rangle \rangle_{\mathbb{R}}^2 \right\}. \end{aligned} \quad (37)$$

(2) If $r = 1$, then one has

$$\begin{aligned} T_\lambda[F_Z | Z_n](y, \vec{\xi}_n) &= \int_{\mathbb{R}} f(\|\mathcal{D}^\perp M_h v_1\| u \\ &\quad + (v_1, y + a + P_{b,n}(\vec{\xi}_n - \vec{a}_n))) \Delta_1(\lambda, u) du \\ &= (f_1(\vec{\xi}_n - \vec{a}_n, a, \cdot) * \Delta_1(\lambda, \cdot)) \left(\|\mathcal{D}^\perp M_h v_1\|^{-1} \right. \\ &\quad \cdot (v_1, y) \Big), \end{aligned} \quad (38)$$

where for $u \in \mathbb{R}$

$$\begin{aligned} f_1(\vec{\xi}_n, a, u) \\ = f(\|\mathcal{D}^\perp M_h v_1\| u + (v_1, a + P_{b,n}(\vec{\xi}_n))), \end{aligned} \quad (39)$$

and for $\Lambda_1(\mathcal{D}^\perp M_h) = 1 + [b(T) - b(t_n)] (\|\mathcal{D}^\perp M_h v_1\|^{-1} \langle v_1 \alpha_{n+1}, \alpha_{n+1} \rangle)^2$

$$\begin{aligned} \Delta_1(\lambda, u) \\ = \left[\frac{\lambda}{2\pi \Lambda_1(\mathcal{D}^\perp M_h)} \right]^{1/2} \exp \left\{ -\frac{\lambda u^2}{2\Lambda_1(\mathcal{D}^\perp M_h)} \right\}. \end{aligned} \quad (40)$$

(3) If $n = 0$ and $r = 1$, then one has

$$\begin{aligned} T_\lambda[F_Z | Z_0](y, \xi_0) \\ = \int_{\mathbb{R}} f(\|\mathcal{D}^\perp M_h v_1\| u + (v_1, y + a)) \Delta_1^0(\lambda, u) du \\ = (f_1^0(a, \cdot) * \Delta_1^0(\lambda, \cdot)) \left(\|\mathcal{D}^\perp M_h v_1\|^{-1} (v_1, y) \right), \end{aligned} \quad (41)$$

where for $u \in \mathbb{R}$

$$f_1^0(a, u) = f(\|\mathcal{D}^\perp M_h v_1\| u + (v_1, a)), \quad (42)$$

and for $\Lambda_1^0(\mathcal{P}^\perp M_h) = 1 + b(T)(\|\mathcal{P}^\perp M_h v_1\|^{-1} \langle v_1 \alpha_1, \alpha_1 \rangle)^2$

$$\Delta_1^0(\lambda, u) = \left[\frac{\lambda}{2\pi\Lambda_1^0(\mathcal{P}^\perp M_h)} \right]^{1/2} \exp \left\{ -\frac{\lambda u^2}{2\Lambda_1^0(\mathcal{P}^\perp M_h)} \right\}. \quad (43)$$

By the dominated convergence theorem and Theorem 5 we have the following theorem.

Theorem 7. Let $F(\in \mathcal{A}^{(1)})$ be given by (29). Then for a nonzero real q , w_φ a.e. $y \in C[0, T]$, and P_{Z_n} a.e. $\tilde{\xi}_n \in \mathbb{R}^{n+1}$, $T_q^{(1)}[F_Z | Z_n](y, \tilde{\xi}_n)$ exists and it is given by the right-hand side of (30), where λ is replaced by $-iq$. Furthermore $T_q^{(1)}[F_Z | Z_n](\cdot, \tilde{\xi}_n) \in \mathcal{A}^{(\infty)}$.

Using Lemma 1.2 in [10] we can prove the following theorem.

Theorem 8. Let $F(\in \mathcal{A}^{(p)})$ be given by (29) with $1 < p \leq 2$, let q be a nonzero real number, and let $1/p + 1/p' = 1$.

- (1) Suppose that $\langle v_j \alpha_{n+1}, \alpha_{n+1} \rangle = 0$ for $j = 1, \dots, r$. Then for w_φ a.e. $y \in C[0, T]$ and P_{Z_n} a.e. $\tilde{\xi}_n \in \mathbb{R}^{n+1}$, $T_q^{(p)}[F_Z | Z_n](y, \tilde{\xi}_n)$ is given by

$$\begin{aligned} T_q^{(p)}[F_Z | Z_n](y, \tilde{\xi}_n) &= \int_{\mathbb{R}^r} f\left(\Sigma_{\mathcal{P}^\perp}^{1/2} \tilde{u} + (\tilde{v}, y + a + P_{b,n}(\tilde{\xi}_n - \tilde{a}_n))\right) \\ &\cdot \Psi_r(\lambda, \tilde{0}, I_r, \tilde{u}) d\tilde{u} = \left(f_r(\tilde{\xi}_n - \tilde{a}_n, a, \cdot)\right) \\ &\cdot \Psi_r(-iq, \tilde{0}, I_r, \cdot) \left(\Sigma_{\mathcal{P}^\perp}^{-1/2}(\tilde{v}, y)\right), \end{aligned} \quad (44)$$

where f_r is given by (31). Furthermore $T_q^{(p)}[F_Z | Z_n](\cdot, \tilde{\xi}_n) \in \mathcal{A}^{(p')}$.

- (2) If $r = 1$, then for w_φ a.e. $y \in C[0, T]$ and P_{Z_n} a.e. $\tilde{\xi}_n \in \mathbb{R}^{n+1}$, $T_q^{(p)}[F_Z | Z_n](y, \tilde{\xi}_n)$ exists and it is given by the right-hand sides of (38), where λ is replaced by $-iq$. Furthermore $T_q^{(p)}[F_Z | Z_n](\cdot, \tilde{\xi}_n) \in \mathcal{A}^{(p')}$.
- (3) If $n = 0$ and $r = 1$, then for $\xi_0 \in \mathbb{R}$, $T_q^{(p)}[F_Z | Z_0](y, \xi_0)$ is given by the right-hand sides of (41), where λ is replaced by $-iq$.

Using the same method as used in the proof of Theorem 4.2 in [7], we can prove the following theorem.

Theorem 9. Let $F(\in \mathcal{A}^{(p)})$ be given by (29) with $1 \leq p \leq \infty$, and let $q \in \mathbb{R} - \{0\}$.

- (1) Suppose that $\langle v_j \alpha_{n+1}, \alpha_{n+1} \rangle = 0$ for $j = 1, \dots, r$. For w_φ a.e. $y \in C[0, T]$ and P_{Z_n} a.e. $\tilde{\xi}_n, \tilde{\eta}_n \in \mathbb{R}^{n+1}$, let

$$F_r(y, \tilde{\xi}_n, \tilde{\eta}_n) = f((\tilde{v}, y) + (\tilde{v}, 2a + P_{b,n}(\tilde{\xi}_n + \tilde{\eta}_n - 2\tilde{a}_n))).$$

Then

$$\begin{aligned} \int_{C[0,T]} & \left| T_{\tilde{\lambda}}[T_\lambda[F_Z | Z_n](\cdot, \tilde{\xi}_n) | Z_n](y, \tilde{\eta}_n) \right. \\ & \left. - F_r(y, \tilde{\xi}_n, \tilde{\eta}_n) \right|^p dw_\varphi(y) \longrightarrow 0, \end{aligned} \quad (45)$$

for $1 \leq p < \infty$, and for $1 \leq p \leq \infty$

$$\begin{aligned} T_{\tilde{\lambda}}[T_\lambda[F_Z | Z_n](\cdot, \tilde{\xi}_n) | Z_n](y, \tilde{\eta}_n) \\ \longrightarrow F_r(y, \tilde{\xi}_n, \tilde{\eta}_n), \end{aligned} \quad (46)$$

as λ approaches $-iq$ through $\mathbb{C}_+$.

- (2) Suppose that $r = 1$. Let $F_1(y, \tilde{\xi}_n, \tilde{\eta}_n) = f((v_1, y) + (v_1, 2a + P_{b,n}(\tilde{\xi}_n + \tilde{\eta}_n - 2\tilde{a}_n)))$. Then one has (45) if $1 \leq p < \infty$ and has (46) if $1 \leq p \leq \infty$, where F_r is replaced by F_1 .
- (3) Suppose that $n = 0$ and $r = 1$. Let $F_1^0(y, \xi_0, \eta_0) = f((v_1, y) + (v_1, 2a))$. Then one has (45) if $1 \leq p < \infty$ and has (46) if $1 \leq p \leq \infty$, where F_r is replaced by F_1^0 .

4. Relationships between Conditional Fourier-Feynman Transforms and Convolution Products

In this section we evaluate the conditional convolution products and investigate their relationships.

Now, we have the following two theorems by (35) and Theorems 3.2 and 3.3 in [7].

Theorem 10. Let $F(\in \mathcal{A}^{(p_1)})$, $G(\in \mathcal{A}^{(p_2)})$, and f and g be related by (29), respectively, where $1 \leq p_1, p_2 \leq \infty$. Furthermore let $1/p_1 + 1/p'_1 = 1$ and $1/p_2 + 1/p'_2 = 1$. Then for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0, T]$, and P_{Z_n} a.e. $\tilde{\xi}_n \in \mathbb{R}^{n+1}$, $[(F_Z * G_Z)_\lambda | Z_n](y, \tilde{\xi}_n)$ exists and is given by

$$\begin{aligned} [(F_Z * G_Z)_\lambda | Z_n](y, \tilde{\xi}_n) &= \int_{\mathbb{R}^r} f\left(\frac{1}{\sqrt{2}}[(\tilde{v}, y) \right. \\ &+ \Sigma_{\mathcal{P}^\perp}^{1/2} \tilde{u} + (\tilde{v}, a + P_{b,n}(\tilde{\xi}_n - \tilde{a}_n))]\Big) g\left(\frac{1}{\sqrt{2}}[(\tilde{v}, y) \right. \\ &- \Sigma_{\mathcal{P}^\perp}^{1/2} \tilde{u} - (\tilde{v}, a + P_{b,n}(\tilde{\xi}_n - \tilde{a}_n))]\Big) \Psi_r(\lambda, \tilde{0}, I_r, \tilde{u}) \\ &\cdot \Phi_r(\lambda, \tilde{v}, \Sigma_{\mathcal{P}^\perp}^{1/2} \tilde{u}) d\tilde{u}. \end{aligned} \quad (47)$$

Moreover one has $[(F_Z * G_Z)_\lambda | Z_n](\cdot, \tilde{\xi}_n) \in \mathcal{A}^{(1)}$ if either $p_2 \leq p'_1$ or $p_1 \leq p'_2$, $[(F_Z * G_Z)_\lambda | Z_n](\cdot, \tilde{\xi}_n) \in \mathcal{A}^{(p_2)}$ if $p_2 \geq p'_1$, and $[(F_Z * G_Z)_\lambda | Z_n](\cdot, \tilde{\xi}_n) \in \mathcal{A}^{(p_1)}$ if $p_1 \geq p'_2$.

Theorem 11. Let q be a nonzero real number. Then for $\lambda \in \mathbb{C}_+$ or $\lambda = q$ and P_{Z_n} a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, one has the following:

- (1) if $F, G \in \mathcal{A}^{(1)}$, then $[(F_Z * G_Z)_\lambda | Z_n](\cdot, \vec{\xi}_n) \in \mathcal{A}^{(1)}$,
- (2) if $F, G \in \mathcal{A}^{(2)}$, then $[(F_Z * G_Z)_\lambda | Z_n](\cdot, \vec{\xi}_n) \in \mathcal{A}^{(\infty)}$,
- (3) if $F \in \mathcal{A}^{(1)}$ and $G \in \mathcal{A}^{(2)}$, then $[(F_Z * G_Z)_\lambda | Z_n](\cdot, \vec{\xi}_n) \in \mathcal{A}^{(2)}$,
- (4) if $F \in \mathcal{A}^{(1)}$ and $G \in \mathcal{A}^{(1)} \cap \mathcal{A}^{(2)}$, then $[(F_Z * G_Z)_\lambda | Z_n](\cdot, \vec{\xi}_n) \in \mathcal{A}^{(1)} \cap \mathcal{A}^{(2)}$,
- (5) if $F \in \mathcal{A}^{(1)}$ and $G \in \mathcal{A}^{(\infty)}$, then $[(F_Z * G_Z)_\lambda | Z_n](\cdot, \vec{\xi}_n) \in \mathcal{A}^{(\infty)}$.

Theorem 12. Let $F, G \in \bigcup_{1 \leq p \leq \infty} \mathcal{A}^{(p)}$ and let $\vec{a}_n = (a(t_0), a(t_1), \dots, a(t_n))$. Then for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0, T]$, and P_{Z_n} a.e. $\vec{\xi}_n, \vec{\eta}_n \in \mathbb{R}^{n+1}$, one has

$$\begin{aligned} T_\lambda [[(F_Z * G_Z)_\lambda | Z_n](\cdot, \vec{\xi}_n) | Z_n](y, \vec{\eta}_n) \\ = \left[T_\lambda [F_Z | Z_n] \left(\frac{1}{\sqrt{2}} y \right. \right. \\ \left. \left. + (\sqrt{2} - 1) a, \frac{1}{\sqrt{2}} (\vec{\eta}_n + \vec{\xi}_n) - (\sqrt{2} - 1) \vec{a}_n \right) \right] \\ \cdot \left[T_\lambda [G_Z | Z_n] \left(\frac{1}{\sqrt{2}} y - a, \frac{1}{\sqrt{2}} (\vec{\eta}_n - \vec{\xi}_n) + \vec{a}_n \right) \right]. \end{aligned} \quad (48)$$

Proof. We note that $T_\lambda [[(F_Z * G_Z)_\lambda | Z_n](\cdot, \vec{\xi}_n) | Z_n](y, \vec{\eta}_n)$ is well-defined by Theorems 5 and 10. By those theorems as stated above we have for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0, T]$, and P_{Z_n} a.e. $\vec{\xi}_n, \vec{\eta}_n \in \mathbb{R}^{n+1}$

$$\begin{aligned} T_\lambda [[(F_Z * G_Z)_\lambda | Z_n](\cdot, \vec{\xi}_n) | Z_n](y, \vec{\eta}_n) &= \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} f \left(\frac{1}{\sqrt{2}} [(\vec{v}, y) + \Sigma_{\mathcal{D}^\perp}^{1/2} (\vec{u} + \vec{z}) + (\vec{v}, 2a + P_{b,n} (\vec{\xi}_n + \vec{\eta}_n - 2\vec{a}_n))] \right) \\ &\cdot g \left(\frac{1}{\sqrt{2}} [(\vec{v}, y) + \Sigma_{\mathcal{D}^\perp}^{1/2} (\vec{z} - \vec{u}) + (\vec{v}, P_{b,n} (\vec{\eta}_n - \vec{\xi}_n))] \right) \Psi_r(\lambda, \vec{0}, I_r, \vec{u}) \Phi_r(\lambda, \vec{v}, \Sigma_{\mathcal{D}^\perp}^{1/2}, \vec{u}) \Psi_r(\lambda, \vec{0}, I_r, \vec{z}) \\ &\cdot \Phi_r(\lambda, \vec{v}, \Sigma_{\mathcal{D}^\perp}^{1/2}, \vec{z}) d\vec{u} d\vec{z}. \end{aligned} \quad (49)$$

Let $\vec{\alpha} = (1/\sqrt{2})(\vec{z} + \vec{u})$ and $\vec{\beta} = (1/\sqrt{2})(\vec{z} - \vec{u})$. Then we have by the change of variable theorem and Lemma 2

$$\begin{aligned} T_\lambda [[(F_Z * G_Z)_\lambda | Z_n](\cdot, \vec{\xi}_n) | Z_n](y, \vec{\eta}_n) &= \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} f \left(\Sigma_{\mathcal{D}^\perp}^{1/2} \vec{\alpha} + \frac{1}{\sqrt{2}} [(\vec{v}, y + 2a + P_{b,n} (\vec{\xi}_n + \vec{\eta}_n - 2\vec{a}_n))] \right) g \left(\Sigma_{\mathcal{D}^\perp}^{1/2} \vec{\beta} \right. \\ &+ \frac{1}{\sqrt{2}} [(\vec{v}, y + P_{b,n} (\vec{\eta}_n - \vec{\xi}_n))] \left. \right) \Psi_r \left(\lambda, \vec{0}, I_r, \frac{1}{\sqrt{2}} (\vec{\alpha} + \vec{\beta}) \right) \Phi_r \left(\lambda, \vec{v}, \Sigma_{\mathcal{D}^\perp}^{1/2}, \frac{1}{\sqrt{2}} (\vec{\alpha} + \vec{\beta}) \right) \Psi_r \left(\lambda, \vec{0}, I_r, \frac{1}{\sqrt{2}} (\vec{\alpha} - \vec{\beta}) \right) \\ &\cdot \Phi_r \left(\lambda, \vec{v}, \Sigma_{\mathcal{D}^\perp}^{1/2}, \frac{1}{\sqrt{2}} (\vec{\alpha} - \vec{\beta}) \right) d\vec{\alpha} d\vec{\beta} = \frac{\lambda^r}{(2\pi)^r \Lambda(\Sigma_{\mathcal{D}^\perp}^{1/2})} \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} f \left(\Sigma_{\mathcal{D}^\perp}^{1/2} \vec{\alpha} + \left(\vec{v}, \frac{1}{\sqrt{2}} y + \sqrt{2}a + P_{b,n} \left(\frac{1}{\sqrt{2}} (\vec{\xi}_n + \vec{\eta}_n) - \sqrt{2}\vec{a}_n \right) \right) \right) \\ &\cdot g \left(\Sigma_{\mathcal{D}^\perp}^{1/2} \vec{\beta} + \left(\vec{v}, \frac{1}{\sqrt{2}} y + P_{b,n} \left(\frac{1}{\sqrt{2}} (\vec{\xi}_n - \vec{\eta}_n) \right) \right) \right) \\ &\cdot \exp \left\{ -\frac{\lambda}{4} [\|\vec{\alpha} + \vec{\beta}\|_{\mathbb{R}}^2 + \|\vec{\alpha} - \vec{\beta}\|_{\mathbb{R}}^2] + \frac{\lambda}{4\Lambda(\Sigma_{\mathcal{D}^\perp}^{1/2})} [b(T) - b(t_n)] [\langle \vec{\alpha} + \vec{\beta}, \Sigma_{\mathcal{D}^\perp}^{-1/2} \langle \vec{v}\alpha_{n+1}, \alpha_{n+1} \rangle \rangle_{\mathbb{R}}^2 + \langle \vec{\alpha} - \vec{\beta}, \Sigma_{\mathcal{D}^\perp}^{-1/2} \langle \vec{v}\alpha_{n+1}, \alpha_{n+1} \rangle \rangle_{\mathbb{R}}^2] \right\} d\vec{\alpha} d\vec{\beta} \\ &= \left[T_\lambda [F_Z | Z_n] \left(\frac{1}{\sqrt{2}} y + (\sqrt{2} - 1) a, \frac{1}{\sqrt{2}} (\vec{\eta}_n + \vec{\xi}_n) - (\sqrt{2} - 1) \vec{a}_n \right) \right] \left[T_\lambda [G_Z | Z_n] \left(\frac{1}{\sqrt{2}} y - a, \frac{1}{\sqrt{2}} (\vec{\eta}_n - \vec{\xi}_n) + \vec{a}_n \right) \right], \end{aligned} \quad (50)$$

which completes the proof. $\square$

We now have the following theorem from Theorems 5, 10, 11, and 12.

Theorem 13. Let q be a nonzero real number. Then one has the following:

- (1) if $F, G \in \mathcal{A}^{(1)}$, then one has for w_φ a.e. $y \in C[0, T]$ and P_{Z_n} a.e. $\vec{\xi}_n, \vec{\eta}_n \in \mathbb{R}^{n+1}$

$$\begin{aligned} T_q^{(1)} [[(F_Z * G_Z)_q | Z_n](\cdot, \vec{\xi}_n) | Z_n](y, \vec{\eta}_n) \\ = \left[T_q^{(1)} [F_Z | Z_n] \left(\frac{1}{\sqrt{2}} y \right. \right. \end{aligned}$$

$$\begin{aligned}
& + (\sqrt{2} - 1) a, \frac{1}{\sqrt{2}} (\vec{\eta}_n + \vec{\xi}_n) - (\sqrt{2} - 1) \vec{a}_n \Big) \Big] \\
& \cdot \left[T_q^{(1)} [G_Z | Z_n] \left(\frac{1}{\sqrt{2}} y - a, \frac{1}{\sqrt{2}} (\vec{\eta}_n - \vec{\xi}_n) + \vec{a}_n \right) \right], \quad (51)
\end{aligned}$$

(2) if $F \in \mathcal{A}^{(1)}$ and $G \in \mathcal{A}^{(2)}$, then

$$\begin{aligned}
& T_q^{(2)} \left[[(F_Z * G_Z)_q | Z_n] (\cdot, \vec{\xi}_n) | Z_n \right] (y, \vec{\eta}_n) \\
& = \left[T_q^{(1)} [F_Z | Z_n] \left(\frac{1}{\sqrt{2}} y \right. \right. \\
& \quad \left. \left. + (\sqrt{2} - 1) a, \frac{1}{\sqrt{2}} (\vec{\eta}_n + \vec{\xi}_n) - (\sqrt{2} - 1) \vec{a}_n \right) \right] \\
& \cdot \left[T_q^{(2)} [G_Z | Z_n] \left(\frac{1}{\sqrt{2}} y - a, \frac{1}{\sqrt{2}} (\vec{\eta}_n - \vec{\xi}_n) + \vec{a}_n \right) \right]. \quad (52)
\end{aligned}$$

$$\begin{aligned}
& T_\lambda [\Pi_Z | Z_n] (y, \vec{\xi}_n) \\
& = \int_{\mathbb{R}^r} \exp \left\{ -\frac{1}{2\lambda} \left[[b(T) - b(t_n)] \langle \vec{z}, \langle \vec{v} \alpha_{n+1}, \alpha_{n+1} \rangle \rangle_{\mathbb{R}}^2 + \langle \Sigma_{\mathcal{D}^\perp} \vec{z}, \vec{z} \rangle_{\mathbb{R}} \right] + i \langle \vec{z}, (\vec{v}, y + a + P_{b,n} (\vec{\xi}_n - \vec{a}_n)) \rangle_{\mathbb{R}} \right\} d\rho(\vec{z}). \quad (55)
\end{aligned}$$

For a nonzero real q , $T_q^{(p)} [\Pi_Z | Z_n] (y, \vec{\xi}_n)$ exists and it is given by the right-hand side of (55), where λ is replaced by $-iq$. Furthermore $T_\lambda [\Pi_Z | Z_n] (\cdot, \vec{\xi}_n) \in \mathcal{A}^{(\infty)}$ and $T_q^{(p)} [\Pi_Z | Z_n] (\cdot, \vec{\xi}_n) \in \mathcal{A}^{(\infty)}$.

Theorem 15. Let $1 \leq p \leq \infty$ and let Π be given by (54). Then one has that for P_{Z_n} a.e. $\vec{\xi}_n, \vec{\eta}_n \in \mathbb{R}^{n+1}$

$$\begin{aligned}
& \| T_{\bar{\lambda}} [T_\lambda [\Pi_Z | Z_n] (\cdot, \vec{\xi}_n) | Z_n] (\cdot, \vec{\eta}_n) \\
& \quad - \Pi (\cdot + 2a + P_{b,n} (\vec{\xi}_n + \vec{\eta}_n - 2\vec{a}_n)) \|_p \quad (56)
\end{aligned}$$

converges to 0 and for w_φ a.e. $y \in C[0, T]$

$$\begin{aligned}
& T_{\bar{\lambda}} [T_\lambda [\Pi_Z | Z_n] (\cdot, \vec{\xi}_n) | Z_n] (y, \vec{\eta}_n) \\
& \longrightarrow \Pi (y + 2a + P_{b,n} (\vec{\xi}_n + \vec{\eta}_n - 2\vec{a}_n)), \quad (57)
\end{aligned}$$

as λ approaches $-iq$ through $\mathbb{C}_+$.

Theorem 16. Let ψ_1 and ψ_2 and ρ_1 and ρ_2 be related by (53), respectively. Let $\Pi_1(x) = \psi_1((\vec{v}, x))$ and $\Pi_2(x) = \psi_2((\vec{v}, x))$ for w_φ a.e. $x \in C[0, T]$. Then for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0, T]$, and

5. Evaluation Formulas for the Functions in a Banach Algebra

Let ψ be the function defined by

$$\psi(\vec{u}) = \int_{\mathbb{R}^r} \exp \{ i \langle \vec{u}, \vec{z} \rangle_{\mathbb{R}} \} d\rho(\vec{z}) \quad \text{for } \vec{u} \in \mathbb{R}^r, \quad (53)$$

where ρ is a complex-valued Borel measure of bounded variation over $\mathbb{R}^r$. For w_φ a.e. $x \in C[0, T]$, let Π be given by

$$\Pi(x) = \psi((\vec{v}, x)). \quad (54)$$

By Theorem 4.1 in [11], we can prove the following theorem.

Theorem 14. Let $1 \leq p \leq \infty$ and let Π be given by (54). Then for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0, T]$, and P_{Z_n} a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, $T_\lambda [\Pi_Z | Z_n] (y, \vec{\xi}_n)$ is given by

P_{Z_n} a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, $[((\Pi_1)_Z * (\Pi_2)_Z)_\lambda | Z_n] (y, \vec{\xi}_n)$ exists and it is given by

$$\begin{aligned}
& [((\Pi_1)_Z * (\Pi_2)_Z)_\lambda | Z_n] (y, \vec{\xi}_n) \\
& = \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \exp \left\{ -\frac{1}{4\lambda} \left[[b(T) - b(t_n)] \right. \right. \\
& \quad \cdot \langle \langle \vec{v} \alpha_{n+1}, \alpha_{n+1} \rangle, \vec{u} - \vec{w} \rangle_{\mathbb{R}}^2 \\
& \quad \left. \left. + \|\Sigma_{\mathcal{D}^\perp}^{1/2} (\vec{u} - \vec{w})\|_{\mathbb{R}}^2 \right] + \frac{i}{\sqrt{2}} \left[\langle (\vec{v}, y), \vec{u} + \vec{w} \rangle_{\mathbb{R}} \right. \right. \\
& \quad \left. \left. + \langle (\vec{v}, a + P_{b,n} (\vec{\xi}_n - \vec{a}_n)), \vec{u} - \vec{w} \rangle_{\mathbb{R}} \right] \right\} d\rho_1(\vec{u}) d\rho_2(\vec{w}). \quad (58)
\end{aligned}$$

For a nonzero real q , $[((\Pi_1)_Z * (\Pi_2)_Z)_q | Z_n] (y, \vec{\xi}_n)$ is given by the above equation, where λ is replaced by $-iq$. Furthermore, $[((\Pi_1)_Z * (\Pi_2)_Z)_q | Z_n] (\cdot, \vec{\xi}_n) \in \mathcal{A}^{(\infty)}$.

Theorem 17. Let q be a nonzero real number and let $1 \leq p \leq \infty$. Furthermore let Π_1 and Π_2 be as given in Theorem 16. Then one has for w_φ a.e. $y \in C[0, T]$ and P_{Z_n} a.e. $\vec{\xi}_n, \vec{\eta}_n \in \mathbb{R}^{n+1}$

$$\begin{aligned}
& T_q^{(p)} \left[[((\Pi_1)_Z * (\Pi_2)_Z)_q | Z_n] (\cdot, \vec{\xi}_n) | Z_n \right] (y, \vec{\eta}_n) \\
& = \left[T_q^{(p)} [(\Pi_1)_Z | Z_n] \left(\frac{1}{\sqrt{2}} y \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \left(\sqrt{2} - 1 \right) a, \frac{1}{\sqrt{2}} \left(\vec{\eta}_n + \vec{\xi}_n \right) - \left(\sqrt{2} - 1 \right) \vec{a}_n \Bigg] \\
& \cdot \left[T_q^{(p)} \left[\left(\Pi_2 \right)_Z \mid Z_n \right] \left(\frac{1}{\sqrt{2}} y - a, \frac{1}{\sqrt{2}} \left(\vec{\eta}_n - \vec{\xi}_n \right) \right. \right. \\
& \left. \left. + \vec{a}_n \right) \right].
\end{aligned} \tag{59}$$

Let $\mathcal{M}(L_2[0, T])$ be the class of all complex-valued Borel measures of bounded variation over $L_2[0, T]$ and let $\mathcal{S}_{w_\varphi}$ be

the space of all functions F which for $\sigma \in \mathcal{M}(L_2[0, T])$ have the form

$$F(x) = \int_{L_2[0, T]} \exp \{ i(v, x) \} d\sigma(v), \tag{60}$$

for w_φ a.e. $x \in C[0, T]$. Note that $\mathcal{S}_{w_\varphi}$ is a Banach algebra [1].

Theorem 18. Let $1 \leq p \leq \infty$ and let $F(\in \mathcal{S}_{w_\varphi})$ be given by (60). Then for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0, T]$, and P_{Z_n} a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, $T_\lambda[F_Z \mid Z_n](y, \vec{\xi}_n)$ is given by

$$\begin{aligned}
& T_\lambda[F_Z \mid Z_n](y, \vec{\xi}_n) \\
& = \int_{L_2[0, T]} \exp \left\{ -\frac{1}{2\lambda} \left[[b(T) - b(t_n)] \langle v\alpha_{n+1}, \alpha_{n+1} \rangle^2 + \|\mathcal{S}^\perp M_h v\|^2 \right] + i(v, y + a + P_{b,n}(\vec{\xi}_n - \vec{a}_n)) \right\} d\sigma(v).
\end{aligned} \tag{61}$$

For a nonzero real q , $T_q^{(p)}[F_Z \mid Z_n](y, \vec{\xi}_n)$ is given by the right-hand side of the above equality, where λ is replaced by $-iq$. Furthermore, $T_q^{(p)}[F_Z \mid Z_n](\cdot, \vec{\xi}_n) \in \mathcal{A}^{(\infty)}$.

Proof. For $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$ let $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \dots, \xi_n, \xi_{n+1})$, where $\xi_{n+1} \in \mathbb{R}$. Then for $\lambda > 0$, w_φ a.e. $y \in C[0, T]$, and P_{Z_n} a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, we have by Lemmas 1 and 2

$$\begin{aligned}
& I_{F_Z}^\lambda(y, \vec{\xi}_n) = \int_{\mathbb{R}} \Psi_1(\lambda, a(T) - a(t_n), b(T) - b(t_n), \xi_{n+1} - \xi_n) E \left[F(y + \lambda^{-1/2} X_{b,n+1}(x, \cdot) + A + P_{b,n+1}(\vec{\xi}_{n+1})) \right] d\xi_{n+1} \\
& = \left[\frac{\lambda}{2\pi [b(T) - b(t_n)]} \right]^{1/2} \int_{L_2[0, T]} \int_{\mathbb{R}} \exp \left\{ -\frac{\lambda}{2 [b(T) - b(t_n)]} \xi_{n+1}^2 + i\xi_{n+1} \langle v\alpha_{n+1}, \alpha_{n+1} \rangle - \frac{1}{2\lambda} \|\mathcal{S}^\perp M_h v\|^2 \right. \\
& \left. + i \left[(v, y + A + P_{b,n}(\vec{\xi}_n)) + [a(T) - a(t_n)] \langle v\alpha_{n+1}, \alpha_{n+1} \rangle \right] \right\} d\xi_{n+1} d\sigma(v) \\
& = \int_{L_2[0, T]} \exp \left\{ -\frac{1}{2\lambda} \left[[b(T) - b(t_n)] \langle v\alpha_{n+1}, \alpha_{n+1} \rangle^2 + \|\mathcal{S}^\perp M_h v\|^2 \right] + i(v, y + a + P_{b,n}(\vec{\xi}_n - \vec{a}_n)) \right\} d\sigma(v).
\end{aligned} \tag{62}$$

By Morera's theorem and the dominated convergence theorem we have the theorem. $\square$

Theorem 19. Under the assumptions as given in Theorem 18, one has that for P_{Z_n} a.e. $\vec{\xi}_n, \vec{\eta}_n \in \mathbb{R}^{n+1}$

$$\begin{aligned}
& \|T_{\bar{\lambda}}[T_\lambda[F_Z \mid Z_n](\cdot, \vec{\xi}_n) \mid Z_n](\cdot, \vec{\eta}_n) \\
& - F(\cdot + 2a + P_{b,n}(\vec{\xi}_n + \vec{\eta}_n - 2\vec{a}_n))\|_p
\end{aligned} \tag{63}$$

converges to 0 and for w_φ a.e. $y \in C[0, T]$

$$\begin{aligned}
& T_{\bar{\lambda}}[T_\lambda[F_Z \mid Z_n](\cdot, \vec{\xi}_n) \mid Z_n](y, \vec{\eta}_n) \\
& \longrightarrow F(y + 2a + P_{b,n}(\vec{\xi}_n + \vec{\eta}_n - 2\vec{a}_n)),
\end{aligned} \tag{64}$$

as λ approaches $-iq$ through $\mathbb{C}_+$.

Proof. By Theorem 18, $T_{\bar{\lambda}}[T_\lambda[F_Z \mid Z_n](\cdot, \vec{\xi}_n) \mid Z_n](y, \vec{\eta}_n)$ is well-defined so that we have for $\lambda \in \mathbb{C}_+$

$$\begin{aligned}
& T_{\bar{\lambda}}[T_\lambda[F_Z \mid Z_n](\cdot, \vec{\xi}_n) \mid Z_n](y, \vec{\eta}_n) \\
& = \int_{L_2[0, T]} \exp \left\{ -\frac{\operatorname{Re} \lambda}{|\lambda|^2} \left[[b(T) - b(t_n)] \langle v\alpha_{n+1}, \alpha_{n+1} \rangle^2 \right. \right. \\
& \left. \left. + \|\mathcal{S}^\perp M_h v\|^2 \right] \right. \\
& \left. + i \left[(v, y + 2a + P_{b,n}(\vec{\xi}_n + \vec{\eta}_n - 2\vec{a}_n)) \right] \right\} d\sigma(v).
\end{aligned} \tag{65}$$

Applying a similar method as used in the proof of Theorem 4.2 in [11] with minor modifications we can obtain the remainder part of the proof. $\square$

Theorem 20. Let F_1 and F_2 and σ_1 and σ_2 be related by (60), respectively. Then for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0, T]$, and a.e. $\tilde{\xi}_n \in \mathbb{R}^{n+1}$

$$\begin{aligned} & \left[((F_1)_Z * (F_2)_Z)_\lambda \mid Z_n \right] (y, \tilde{\xi}_n) \\ &= \int_{L_2[0, T]} \int_{L_2[0, T]} \exp \left\{ -\frac{1}{4\lambda} \left[[b(T) - b(t_n)] \langle (v_1 - v_2) \alpha_{n+1}, \alpha_{n+1} \rangle^2 + \|\mathcal{P}^\perp M_h(v_1 - v_2)\|^2 \right] \right. \\ & \quad \left. + \frac{i}{\sqrt{2}} \left[(v_1 + v_2, y) + (v_1 - v_2, a + P_{b,n}(\tilde{\xi}_n - \tilde{a}_n)) \right] \right\} d\sigma_1(v_1) d\sigma_2(v_2). \end{aligned} \quad (66)$$

For a nonzero real q , $[((F_1)_Z * (F_2)_Z)_q \mid Z_n](y, \tilde{\xi}_n)$ is given by the above equation, where λ is replaced by $-iq$. Furthermore, $[((F_1)_Z * (F_2)_Z)_q \mid Z_n](\cdot, \tilde{\xi}_n) \in \mathcal{A}^{(\infty)}$.

Proof. For $\tilde{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$ let $\tilde{\xi}_{n+1} = (\xi_0, \xi_1, \dots, \xi_n, \xi_{n+1})$, where $\xi_{n+1} \in \mathbb{R}$. Then for $\lambda > 0$, w_φ a.e. $y \in C[0, T]$, and P_{Z_n} a.e. $\tilde{\xi}_n \in \mathbb{R}^{n+1}$, we have by Lemmas 1 and 2

$$\begin{aligned} & K_{(F_1)_Z, (F_2)_Z}^\lambda(y, \tilde{\xi}_n) \\ &= \left[\frac{\lambda}{2\pi [b(T) - b(t_n)]} \right]^{1/2} \\ & \cdot \int_{L_2[0, T]} \int_{L_2[0, T]} \int_{\mathbb{R}} \int_{C[0, T]} \exp \left\{ -\frac{\lambda}{2 [b(T) - b(t_n)]} \right. \\ & \cdot (\xi_{n+1} - \xi_n - [a(T) - a(t_n)])^2 \\ & + \frac{i}{\sqrt{2}} \left[(v_1, y + \lambda^{-1/2} X_{b,n+1}(x, \cdot) + A + P_{b,n+1}(\tilde{\xi}_{n+1})) \right. \\ & + (v_2, y - \lambda^{-1/2} X_{b,n+1}(x, \cdot) \\ & \quad \left. \left. - A - P_{b,n+1}(\tilde{\xi}_{n+1})) \right] \right\} dw_\varphi(x) d\xi_{n+1} d\sigma_1(v_1) d\sigma_2(v_2) \\ &= \left[\frac{\lambda}{2\pi [b(T) - b(t_n)]} \right]^{1/2} \\ & \cdot \int_{L_2[0, T]} \int_{L_2[0, T]} \int_{\mathbb{R}} \exp \left\{ -\frac{\lambda \xi_{n+1}^2}{2 [b(T) - b(t_n)]} \right. \\ & + \frac{i}{\sqrt{2}} \left[(v_1 + v_2, y) + (v_1 - v_2, A) + (v_1 - v_2, P_{b,n}(\tilde{\xi}_n)) \right. \\ & + [a(T) - a(t_n) + \xi_{n+1}] \langle (v_1 - v_2) \alpha_{n+1}, \alpha_{n+1} \rangle \\ & \quad \left. \left. - \frac{1}{4\lambda} \|\mathcal{P}^\perp M_h(v_1 - v_2)\|^2 \right] \right\} d\xi_{n+1} d\sigma_1(v_1) d\sigma_2(v_2) \\ &= \int_{L_2[0, T]} \int_{L_2[0, T]} \exp \left\{ -\frac{1}{4\lambda} [[b(T) - b(t_n)] \right. \\ & \cdot \langle (v_1 - v_2) \alpha_{n+1}, \alpha_{n+1} \rangle^2 + \|\mathcal{P}^\perp M_h(v_1 - v_2)\|^2] \\ & + \frac{i}{\sqrt{2}} [(v_1 + v_2, y) \\ & \quad \left. + (v_1 - v_2, a + P_{b,n}(\tilde{\xi}_n - \tilde{a}_n))] \right\} d\sigma_1(v_1) d\sigma_2(v_2). \end{aligned} \quad (67)$$

By Morera's theorem and the dominated convergence theorem we have the theorem. $\square$

Theorem 21. Let q be a nonzero real number and let $1 \leq p \leq \infty$. Furthermore let F_1 and F_2 be as given in Theorem 20. Then one has for w_φ a.e. $y \in C[0, T]$ and P_{Z_n} a.e. $\tilde{\xi}_n, \tilde{\eta}_n \in \mathbb{R}^{n+1}$

$$\begin{aligned} & T_q^{(p)} \left[[((F_1)_Z * (F_2)_Z)_q \mid Z_n](\cdot, \tilde{\xi}_n) \mid Z_n \right] (y, \tilde{\eta}_n) \\ &= \left[T_q^{(p)} [(F_1)_Z \mid Z_n] \left(\frac{1}{\sqrt{2}} y \right. \right. \\ & \quad \left. \left. + (\sqrt{2} - 1) a, \frac{1}{\sqrt{2}} (\tilde{\eta}_n + \tilde{\xi}_n) - (\sqrt{2} - 1) \tilde{a}_n \right) \right] \\ & \cdot \left[T_q^{(p)} [(F_2)_Z \mid Z_n] \left(\frac{1}{\sqrt{2}} y - a, \frac{1}{\sqrt{2}} (\tilde{\eta}_n - \tilde{\xi}_n) \right. \right. \\ & \quad \left. \left. + \tilde{a}_n \right) \right]. \end{aligned} \quad (69)$$

Proof. By Theorems 18 and 20 we have for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0, T]$, and P_{Z_n} a.e. $\tilde{\xi}_n, \tilde{\eta}_n \in \mathbb{R}^{n+1}$

$$\begin{aligned} & T_\lambda \left[[((F_1)_Z * (F_2)_Z)_q \mid Z_n](\cdot, \tilde{\xi}_n) \mid Z_n \right] (y, \tilde{\eta}_n) \\ &= \int_{L_2[0, T]} \int_{L_2[0, T]} \exp \left\{ \frac{1}{4qi} [[b(T) - b(t_n)] \right. \\ & \cdot \langle (v_1 + v_2) \alpha_{n+1}, \alpha_{n+1} \rangle^2 \\ & + \|\mathcal{P}^\perp M_h(v_1 + v_2)\|^2] - \frac{1}{4\lambda} [b(T) - b(t_n)] \\ & \cdot \langle (v_1 + v_2) \alpha_{n+1}, \alpha_{n+1} \rangle^2 \\ & + \|\mathcal{P}^\perp M_h(v_1 + v_2)\|^2] \\ & + \frac{i}{\sqrt{2}} [(v_1 + v_2, y + a + P_{b,n}(\tilde{\eta}_n - \tilde{a}_n)) \\ & \quad \left. + (v_1 - v_2, a + P_{b,n}(\tilde{\xi}_n - \tilde{a}_n))] \right\} d\sigma_1(v_1) d\sigma_2(v_2). \end{aligned} \quad (70)$$

Let $T_q^{(p)} [((F_1)_Z * (F_2)_Z)_q \mid Z_n](\cdot, \tilde{\xi}_n) \mid Z_n](y, \tilde{\eta}_n)$ be given by the right-hand side of the above equality, where λ is replaced

by $-iq$. The existence of $T_q^{(1)}[[(F_1)_Z * (F_2)_Z]_q | Z_n](\cdot, \vec{\xi}_n) | Z_n](y, \vec{\eta}_n)$ follows from the dominated convergence theorem. Now let $1 < p \leq \infty$ and $1/p + 1/p' = 1$. Then we have by the dominated convergence theorem

$$\begin{aligned} & \int_{C[0,T]} \left| T_\lambda \left[[(F_1)_Z * (F_2)_Z]_q | Z_n \right] (\cdot, \vec{\xi}_n) | Z_n \right] (y, \vec{\eta}_n) \\ & - T_q^{(p)} \left[[(F_1)_Z * (F_2)_Z]_q | Z_n \right] (\cdot, \vec{\xi}_n) | Z_n \right] (y, \vec{\eta}_n) \right|^{p'} dw_\varphi(y) \\ & \leq \left[\int_{L_2[0,T]} \int_{L_2[0,T]} \left| \exp \left\{ -\frac{1}{4\lambda} [b(T) - b(t_n)] \right. \right. \right. \\ & \quad \cdot \langle (v_1 + v_2) \alpha_{n+1}, \alpha_{n+1} \rangle^2 + \|\mathcal{S}^\perp M_h(v_1 + v_2)\|^2 \Big\} \\ & \quad \left. \left. - \exp \left\{ \frac{1}{4qi} [b(T) - b(t_n)] \langle (v_1 + v_2) \alpha_{n+1}, \alpha_{n+1} \rangle^2 \right. \right. \right. \\ & \quad \left. \left. + \|\mathcal{S}^\perp M_h(v_1 + v_2)\|^2 \right\} \right| d|\sigma_1|(v_1) d|\sigma_2|(v_2) \Big]^{p'} \rightarrow 0, \end{aligned} \quad (71)$$

as λ approaches $-iq$ through $\mathbb{C}_+$, which shows the existence of $T_q^{(p)}[[(F_1)_Z * (F_2)_Z]_q | Z_n](\cdot, \vec{\xi}_n) | Z_n](y, \vec{\eta}_n)$. The equality in the theorem follows from Theorem 18 with simple calculations. $\square$

Remark 22. (1) Theorems 15 and 17 can be obtained from Theorems 9 and 12, respectively. They also can be proved directly with aids of Theorems 14 and 16.

(2) Comparing Theorems 17 and 21 with Theorem 13, the results in Theorems 17 and 21 hold for $1 \leq p \leq \infty$.

6. Change of Scale Formulas for the Transforms and Convolutions

In this section, we derive two types of change of scale formulas for the conditional expectations as described in the previous sections.

For $\lambda \in \mathbb{C}$, $x \in C[0, T]$ and $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ let

$$\begin{aligned} K_1(\lambda, \vec{\xi}_{n+1}, x) &= \left(\frac{|\Sigma_{M_h}|}{|\Sigma_{\mathcal{S}^\perp}|} \right)^{1/2} \\ & \cdot \exp \left\{ \frac{1}{2} \left\| \Sigma_{M_h}^{-1/2} (\vec{v}, X(x, \cdot)) \right\|_{\mathbb{R}}^2 \right. \\ & \quad \left. - \frac{\lambda}{2} \left\| \Sigma_{\mathcal{S}^\perp}^{-1/2} (\vec{v}, Z(x, \cdot) - A - P_{b,n+1}(\vec{\xi}_{n+1})) \right\|_{\mathbb{R}}^2 \right\}. \end{aligned} \quad (72)$$

For a nonzero real q , let $\{\lambda_m\}$ be any sequence in $\mathbb{C}_+$ converging to $-iq$.

By simple calculations, we have the following two theorems.

Theorem 23. Let $1 \leq p \leq \infty$ and let $F \in \mathcal{A}^{(p)}$. Then for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0, T]$, and P_{Z_n} a.e. $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$

$$\begin{aligned} T_\lambda [F_Z | Z_n] (y, \vec{\xi}_n) &= \lambda^{r/2} \int_{\mathbb{R}} \Psi_1(\lambda, a(T) \\ & - a(t_n), b(T) - b(t_n), \xi_{n+1} - \xi_n) \\ & \cdot E [K_1(\lambda, \vec{\xi}_{n+1}, \cdot) F_Z(\cdot, y)] d\xi_{n+1}, \end{aligned} \quad (73)$$

where $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \dots, \xi_n, \xi_{n+1})$ and K_1 is given by (72). If $p = 1$, then

$$\begin{aligned} T_q^{(1)} [F_Z | Z_n] (y, \vec{\xi}_n) &= \lim_{m \rightarrow \infty} \lambda_m^{r/2} \int_{\mathbb{R}} \Psi_1(\lambda_m, a(T) \\ & - a(t_n), b(T) - b(t_n), \xi_{n+1} - \xi_n) \\ & \cdot E [K_1(\lambda_m, \vec{\xi}_{n+1}, \cdot) F_Z(\cdot, y)] d\xi_{n+1}. \end{aligned} \quad (74)$$

Theorem 24. Let $F \in \mathcal{A}^{(p_1)}$ and $G \in \mathcal{A}^{(p_2)}$ with $1 \leq p_1, p_2 \leq \infty$. Then for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0, T]$, and P_{Z_n} a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, $[(F_Z * G_Z)_\lambda | Z_n](y, \vec{\xi}_n)$ is given by the right-hand side of (73) with replacing $F_Z(\cdot, y)$ by $F_Z(\cdot/\sqrt{2}, y/\sqrt{2})G_Z(-\cdot/\sqrt{2}, y/\sqrt{2})$. Moreover if $p_1 = p_2 = 1$, then $[(F_Z * G_Z)_q | Z_n](y, \vec{\xi}_n)$ is given by the right-hand side of (74) with replacing $F_Z(\cdot, y)$ by $F_Z(\cdot/\sqrt{2}, y/\sqrt{2})G_Z(-\cdot/\sqrt{2}, y/\sqrt{2})$.

By Theorems 14 and 23 and the dominated convergence theorem, we have the following theorem.

Theorem 25. Let $1 \leq p \leq \infty$ and let Π be given by (54). Then, for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0, T]$, and P_{Z_n} a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, $T_\lambda[\Pi_Z | Z_n](y, \vec{\xi}_n)$ and $T_q^{(p)}[\Pi_Z | Z_n](y, \vec{\xi}_n)$ are given by the right-hand sides of (73) and (74), respectively, with replacing F_Z by Π_Z .

By Theorems 16 and 24 and the dominated convergence theorem, we have the following theorem.

Theorem 26. Let the assumptions be as given in Theorem 16. Then, for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0, T]$, and P_{Z_n} a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, $[(\Pi_1)_Z * (\Pi_2)_Z]_\lambda | Z_n](y, \vec{\xi}_n)$ and $[(\Pi_1)_Z * (\Pi_2)_Z]_q | Z_n](y, \vec{\xi}_n)$ are given by the right-hand sides of (73) and (74), respectively, with replacing F_Z by $(\Pi_1)_Z(\cdot/\sqrt{2}, y/\sqrt{2})(\Pi_2)_Z(-\cdot/\sqrt{2}, y/\sqrt{2})$.

For $\lambda \in \mathbb{C}$, $x \in C[0, T]$ and $\vec{\xi}_n \in \mathbb{R}^{n+1}$ let

$$\begin{aligned} K_2(\lambda, \vec{\xi}_n, x) &= \left[\frac{|\Sigma_{M_h}|}{\Lambda(\Sigma_{\mathcal{S}^\perp})|\Sigma_{\mathcal{S}^\perp}|} \right]^{1/2} \\ & \cdot \exp \left\{ \frac{1}{2} \left\| \Sigma_{M_h}^{-1/2} (\vec{v}, X(x, \cdot)) \right\|_{\mathbb{R}}^2 \right. \\ & \quad \left. - \frac{\lambda}{2} \left\| \Sigma_{\mathcal{S}^\perp}^{-1/2} (\vec{v}, Z(x, \cdot) - a - P_{b,n}(\vec{\xi}_n - \vec{a}_n)) \right\|_{\mathbb{R}}^2 \right\} \end{aligned}$$

$$\begin{aligned}
& + \frac{\lambda}{2\Lambda(\Sigma_{\mathcal{D}^\perp})} [b(T) - b(t_n)] \\
& \cdot \left\langle \Sigma_{\mathcal{D}^\perp}^{-1} (\tilde{v}, Z(x, \cdot) - a - P_{b,n}(\tilde{\xi}_n - \tilde{a}_n)) \right\rangle_{\mathbb{R}} \\
& \left\langle \tilde{v}\alpha_{n+1}, \alpha_{n+1} \right\rangle_{\mathbb{R}}^2 \Bigg\}. \tag{75}
\end{aligned}$$

Theorem 27. Let $1 \leq p \leq \infty$ and let $F(\in \mathcal{A}^{(p)})$ be given by (29). Then for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0, T]$, and P_{Z_n} a.e. $\tilde{\xi}_n \in \mathbb{R}^{n+1}$

$$T_\lambda [F_Z | Z_n] (y, \tilde{\xi}_n) = \lambda^{r/2} E [K_2 (\lambda, \tilde{\xi}_n, \cdot) F_Z (\cdot, y)], \tag{76}$$

where K_2 is given by (75). If $p = 1$, then

$$\begin{aligned}
T_q^{(1)} [F_Z | Z_n] (y, \tilde{\xi}_n) \\
= \lim_{m \rightarrow \infty} \lambda_m^{r/2} E [K_2 (\lambda_m, \tilde{\xi}_n, \cdot) F_Z (\cdot, y)]. \tag{77}
\end{aligned}$$

Proof. For $\lambda > 0$, w_φ a.e. $y \in C[0, T]$, and P_{Z_n} a.e. $\tilde{\xi}_n \in \mathbb{R}^{n+1}$, we have by Lemma 3

$$\begin{aligned}
\lambda^{r/2} E [K_2 (\lambda, \tilde{\xi}_n, \cdot) F_Z (\cdot, y)] &= \left[\frac{\lambda^r |\Sigma_{M_h}|}{\Lambda(\Sigma_{\mathcal{D}^\perp}) |\Sigma_{\mathcal{D}^\perp}|} \right]^{1/2} \int_{C[0, T]} F(Z(x, \cdot) + y) \exp \left\{ \frac{1}{2} \|\Sigma_{M_h}^{-1/2} (\tilde{v}, Z(x, \cdot) - a)\|_{\mathbb{R}}^2 \right. \\
&- \frac{\lambda}{2} \|\Sigma_{\mathcal{D}^\perp}^{-1/2} (\tilde{v}, Z(x, \cdot) - a - P_{b,n}(\tilde{\xi}_n - \tilde{a}_n))\|_{\mathbb{R}}^2 \\
&+ \frac{\lambda}{2\Lambda(\Sigma_{\mathcal{D}^\perp})} [b(T) - b(t_n)] \left\langle \Sigma_{\mathcal{D}^\perp}^{-1} (\tilde{v}, Z(x, \cdot) - a - P_{b,n}(\tilde{\xi}_n - \tilde{a}_n)) \right\rangle_{\mathbb{R}} \left. \left\langle \tilde{v}\alpha_{n+1}, \alpha_{n+1} \right\rangle_{\mathbb{R}}^2 \right\} dw_\varphi(x) \\
&= \left[\frac{\lambda^r}{(2\pi)^r \Lambda(\Sigma_{\mathcal{D}^\perp}) |\Sigma_{\mathcal{D}^\perp}|} \right]^{1/2} \int_{\mathbb{R}^r} f(\tilde{u} + (\tilde{v}, y)) \exp \left\{ -\frac{\lambda}{2} \|\Sigma_{\mathcal{D}^\perp}^{-1/2} (\tilde{u} - (\tilde{v}, a + P_{b,n}(\tilde{\xi}_n - \tilde{a}_n)))\|_{\mathbb{R}}^2 \right. \\
&+ \frac{\lambda}{2\Lambda(\Sigma_{\mathcal{D}^\perp})} [b(T) - b(t_n)] \left\langle \Sigma_{\mathcal{D}^\perp}^{-1/2} (\tilde{u} - (\tilde{v}, a + P_{b,n}(\tilde{\xi}_n - \tilde{a}_n))) \right\rangle_{\mathbb{R}} \Sigma_{\mathcal{D}^\perp}^{-1/2} \left. \left\langle \tilde{v}\alpha_{n+1}, \alpha_{n+1} \right\rangle_{\mathbb{R}}^2 \right\} d\tilde{u} \\
&= \left[\frac{\lambda^r}{(2\pi)^r \Lambda(\Sigma_{\mathcal{D}^\perp})} \right]^{1/2} \int_{\mathbb{R}^r} f(\Sigma_{\mathcal{D}^\perp}^{1/2} \tilde{u} + (\tilde{v}, y + a + P_{b,n}(\tilde{\xi}_n - \tilde{a}_n))) \exp \left\{ -\frac{\lambda}{2} \|\tilde{u}\|_{\mathbb{R}}^2 \right. \\
&+ \frac{\lambda}{2\Lambda(\Sigma_{\mathcal{D}^\perp})} [b(T) - b(t_n)] \left\langle \tilde{u}, \Sigma_{\mathcal{D}^\perp}^{-1/2} \left. \left\langle \tilde{v}\alpha_{n+1}, \alpha_{n+1} \right\rangle_{\mathbb{R}}^2 \right\rangle_{\mathbb{R}} \right\} d\tilde{u}. \tag{78}
\end{aligned}$$

By the analytic continuation, the dominated convergence theorem, and Theorem 5, we have the theorem. $\square$

Theorem 28. Let $F \in \mathcal{A}^{(p_1)}$ and $G \in \mathcal{A}^{(p_2)}$ with $1 \leq p_1, p_2 \leq \infty$. Then for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0, T]$, and P_{Z_n} a.e. $\tilde{\xi}_n \in \mathbb{R}^{n+1}$, $[(F_Z * G_Z)_\lambda | Z_n](y, \tilde{\xi}_n)$ is given by the right-hand side of (76) with replacing $F_Z(\cdot, y)$ by $F_Z(\cdot/\sqrt{2}, y/\sqrt{2})G_Z(-\cdot/\sqrt{2}, y/\sqrt{2})$. Moreover if $p_1 = p_2 = 1$, then $[(F_Z * G_Z)_q | Z_n](y, \tilde{\xi}_n)$ is given by the right-hand side of (77) with replacing $F_Z(\cdot, y)$ by $F_Z(\cdot/\sqrt{2}, y/\sqrt{2})G_Z(-\cdot/\sqrt{2}, y/\sqrt{2})$.

By Theorems 14, 16, and 27 and the dominated convergence theorem, we have the following two theorems.

Theorem 29. Let $1 \leq p \leq \infty$ and let Π be given by (54). Then, for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0, T]$, and P_{Z_n} a.e. $\tilde{\xi}_n \in \mathbb{R}^{n+1}$, $T_\lambda[\Pi_Z | Z_n](y, \tilde{\xi}_n)$ and $T_q^{(p)}[\Pi_Z | Z_n](y, \tilde{\xi}_n)$ are given by the right-hand sides of (76) and (77), respectively, with replacing F_Z by Π_Z .

Theorem 30. Let the assumptions be as given in Theorem 16. Then, for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0, T]$, and P_{Z_n} a.e. $\tilde{\xi}_n \in \mathbb{R}^{n+1}$, $[(\Pi_1)_Z * (\Pi_2)_Z]_\lambda | Z_n](y, \tilde{\xi}_n)$ and $[(\Pi_1)_Z * (\Pi_2)_Z]_q | Z_n](y, \tilde{\xi}_n)$ are given by the right-hand sides of (76) and (77), respectively, with replacing F_Z by $(\Pi_1)_Z(\cdot/\sqrt{2}, y/\sqrt{2})(\Pi_2)_Z(-\cdot/\sqrt{2}, y/\sqrt{2})$.

Let $\{e_1, e_2, \dots\}$ be a complete orthonormal basis of $L_2[0, T]$. For $m \in \mathbb{N}$, $\lambda \in \mathbb{C}$ and $x \in C[0, T]$ let

$$H_m(\lambda, x) = \exp \left\{ \frac{1 - \lambda}{2} \sum_{j=1}^m (e_j, x)^2 \right\}. \tag{79}$$

We now have the following relationships among the conditional Fourier-Feynman transform, the conditional convolution products, and the generalized Wiener integrals of the functions in $\mathcal{S}_{w_\varphi}$. Their proofs are similar to the proofs of the results in [12] with additional calculations.

Theorem 31. Let $1 \leq p \leq \infty$ and let F be given by (60). Then for $\lambda \in \mathbb{C}_+$, w_φ a.e. $x \in C[0, T]$, and P_{Z_n} a.e. $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$

$$\begin{aligned} T_\lambda [F_Z | Z_n] (y, \vec{\xi}_n) &= \lim_{m \rightarrow \infty} \lambda^{m/2} \int_{\mathbb{R}} \Psi_1 (\lambda, a(T) \\ &\quad - a(t_n), b(T) - b(t_n), \xi_{n+1} - \xi_n) E [H_m (\lambda, x) \\ &\quad \cdot F (y + Z_{b,n+1} (x, \cdot) + P_{b,n+1} (\vec{\xi}_{n+1}))] d\xi_{n+1}, \end{aligned} \quad (80)$$

where $\vec{\xi}_{n+1} = (\xi_0, \dots, \xi_n, \xi_{n+1})$ for $\xi_{n+1} \in \mathbb{R}$. Moreover, $T_q^{(p)} [F_Z | Z_n] (y, \vec{\xi}_n)$ is given by the right-hand side of the above equality with replacing λ by λ_m .

Theorem 32. Let F_1, F_2 , and σ_1, σ_2 be related by (60), respectively. Then for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0, T]$, and P_{Z_n} a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$

$$\begin{aligned} [((F_1)_Z * (F_2)_Z)_\lambda | Z_n] (y, \vec{\xi}_n) &= \lim_{m \rightarrow \infty} \lambda^{m/2} \int_{\mathbb{R}} \Psi_1 (\lambda, a(T) \\ &\quad - a(t_n), b(T) - b(t_n), \xi_{n+1} - \xi_n) E [H_m (\lambda, x) \\ &\quad \cdot F_1 \left(\frac{1}{\sqrt{2}} [y + Z_{b,n+1} (x, \cdot) + P_{b,n+1} (\vec{\xi}_{n+1})] \right) \\ &\quad \cdot F_2 \left(\frac{1}{\sqrt{2}} [y - Z_{b,n+1} (x, \cdot) - P_{b,n+1} (\vec{\xi}_{n+1})] \right)] d\xi_{n+1}. \end{aligned} \quad (81)$$

Moreover, $[((F_1)_Z * (F_2)_Z)_q | Z_n] (y, \vec{\xi}_n)$ is given by the right-hand side of the above equality with replacing λ by λ_m .

Remark 33. Letting $\lambda = \gamma^{-2}$ in the theorems of this section, where $\gamma > 0$, we can obtain the change of scales for $E[F_Z(\gamma, y) | Z_n(\gamma)]$ and $E[F_Z(\gamma \cdot / \sqrt{2}, y / \sqrt{2}) G_Z(-\gamma \cdot / \sqrt{2}, y / \sqrt{2}) | Z_n(\gamma)]$; that is, the scale γ in the functions $F_Z(\gamma, y)$, $G_Z(-\gamma \cdot / \sqrt{2}, y / \sqrt{2})$, and $Z_n(\gamma)$ is moved to K_1, K_2, Ψ_1 , and H_m in evaluating the conditional expectations as above.

Remark 34. (1) An orthonormal subset $\{v_1, v_2, \dots, v_r\}$ of $L_2[0, T]$ such that both $\{M_h v_1, \dots, M_h v_r\}$ and $\{\mathcal{P}^\perp M_h v_1, \dots, \mathcal{P}^\perp M_h v_r\}$ are independent sets exists [12].

(2) Let $\{e_{11}, \dots, e_{1r}\}$ and $\{e_{21}, \dots, e_{2r}\}$ be the orthonormal sets obtained from $\{M_h v_1, \dots, M_h v_r\}$ and $\{\mathcal{P}^\perp M_h v_1, \dots, \mathcal{P}^\perp M_h v_r\}$, respectively, by the Gram-Schmidt orthonormalization process. For $l = 1, \dots, r$, let

$$\begin{aligned} M_h v_l &= \sum_{j=1}^r \alpha_{lj} e_{1j}, \\ \mathcal{P}^\perp M_h v_l &= \sum_{j=1}^r \beta_{lj} e_{2j} \end{aligned} \quad (82)$$

be the linear combinations. Let $B_1 = [\alpha_{lj}]_{r \times r}$ and $B_2 = [\beta_{lj}]_{r \times r}$ be the coefficient matrices of the combinations. Then $M_h^{1/2}$ and $\Sigma_{\mathcal{P}^\perp}^{1/2}$ can be replaced by B_1 and B_2 , respectively, in each expression of the theorems.

(3) It does not mean that $B_1 = \Sigma_{M_h}^{1/2}$ or $B_2 = \Sigma_{\mathcal{P}^\perp}^{1/2}$ in (2). They only satisfy the following equations:

$$\begin{aligned} B_1 B_1^T &= \Sigma_{M_h} = (\Sigma_{M_h}^{1/2})^2, \\ B_2 B_2^T &= \Sigma_{\mathcal{P}^\perp} = (\Sigma_{\mathcal{P}^\perp}^{1/2})^2. \end{aligned} \quad (83)$$

Remark 35. (1) If $y = 0$, then we can obtain the change of scale formulas in [2] with Z_n .

(2) If $a = 0$ and $y = 0$, then we can obtain the results in [12] with the cylinder functions.

(3) The results of this paper are independent of a particular choice of the initial distribution φ .

Competing Interests

The author declares that they have no competing interests.

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