Hindawi
Journal of Probability and Statistics
Volume 2018, Article ID 5902839, 9 pages
https://doi.org/10.1155/2018/5902839

Research Article

Hybrid Clayton-Frank Convolution-Based Bivariate Archimedean Copula

Maxwell Akwasi Boateng,[1] **Akoto Yaw Omari-Sasu**,[2] **Richard Kodzo Avuglah**,[2] **and Nana Kena Frempong**[2]

[1] *Faculty of Engineering, Ghana Technology University College, Accra, Ghana*
[2] *Department of Mathematics, Kwame Nkrumah University of Science and Technology, Kumasi, Ghana*

Correspondence should be addressed to Maxwell Akwasi Boateng; mboateng@gtuc.edu.gh

Received 5 October 2017; Accepted 8 February 2018; Published 14 March 2018

Academic Editor: Ramón M. Rodríguez-Dagnino

This study exploits the closure property of the converse convolution operator to come up with a hybrid Clayton-Frank Archimedean copula for two random variables. Pairs of random variables were generated and the upper tail observation of the cumulative distribution function (CDF) was used to assess the right skew behavior of the proposed model. Various values of the converse convolution operator were used to see their effect on the proposed model. The simulation covered lengths $n = 10^i$, $i = 2, 3, 4, 5$, and 6. The proposed model was compared with about 40 other bivariate copulas (both Archimedean and elliptical). The proposed model had parameters that spanned the entire real line, thus removing restrictions on the parameters. The parameters theta and omega were varied for a selected interval and the hybrid Clayton-Frank model was, in most cases, found to outperform the other copulas under consideration.

1. Introduction

It is quite apparent that identifying and quantifying dependencies is the core of econometric modeling, especially when it comes to risk management. Pearson's correlation factor for dependence has over time been used in this regard although it has a deficiency when it comes to issues of nonlinear dependence. Several arguments have been put forward in this regard [1] and Embrechts et al. [2]. In order to get a measure that gives more information about the dependence structure, copulas were preferred to Pearson's correlation coefficient.

Copulas give a way of isolating the marginal behavior from the dependence structure McNeil et al. [3]. Alcock and Hatherley [4] suggested that, through copulas, the nonnormal dependence structure could be modeled by using only uniforms of the marginal distributions which allowed violation of the assumptions of normality and linear dependence. This meant that the marginals could be modeled using each type of distributions without influencing the dependence structure between them.

The use of copulas has increased greatly in all fields of study since several phenomena tend to have certain dependencies amongst or between them, no matter how small the dependencies are. Boateng et al. [5] focused on the likelihood of a pair of random variables having either an Archimedean copula or an elliptical copula. Their work involved simulating several pairs of random variables and selecting an appropriate bivariate copula family. The corresponding parameter estimates were obtained by maximum likelihood estimation. They compared AICs of the various bivariate copulas under consideration, using about forty (40) bivariate copulas, for sample sizes 30, 300, 1000, 10000, 100000, and 1000000. Their result showed that, between the Archimedean and elliptical copulas, the Archimedean copulas were the most likely to fit the simulated pairs of random variables.

There have been several applications of copulas. Some of the applications being simulation of multivariate sea storms [6]; dependence structure between the stock and foreign exchange markets [7]; operational risk management [8]; portfolio optimization in the presence of dependent financial returns with long memory [9]; risk evaluation of droughts across the Pearl River basin, China [10]; probabilistic assessment of flood risks [11]; estimation of distribution algorithms for coverage problem of wireless sensor network [12]; risk assessment of hydroclimatic variability on groundwater levels in the Manjara basin aquifer in India [13]; models of tourists' time use and expenditure behavior with self-selection [14]; modeling wind speed dependence in system reliability assessment using copulas [15]; dependence between crude oil spot and futures markets [16]; stochastic modeling of power demand [17].

This study explores the possibility of a combination of existing bivariate Archimedean copulas performing better or just as well as the individual copulas.

2. Method

2.1. Copula. A copula is a function $C: [0,1]^2 \rightarrow [0,1]$ satisfying the following requirements:

(1) Grounded: $C(0,v) = C(u,0) = 0$

(2) Uniform marginals: $C(u,1) = u$, $C(1,v) = v$

(3) 2-increasing: $C(u_1,v_1) - C(u_1,v_2) - C(u_2,v_1) + C(u_2,v_2) \geq 0$, for $u_1 > u_2$ and $v_1 > v_2$.

Generator Function (Clayton)

$$\varphi(t) = \frac{1}{\theta}\left(t^{-\theta} - 1\right), \quad \theta > 0. \tag{1}$$

Inverse Function (Clayton)

$$\varphi^{-1}(t) = \left[\left(t^{-\theta} - 1\right) + 1\right]^{-1/\theta}. \tag{2}$$

Copula (Clayton)

$$C_C(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta}. \tag{3}$$

Kendal's Tau (Clayton)

$$\tau = \frac{\theta}{\theta + 2}. \tag{4}$$

Generator Function (Frank)

$$\varphi(t) = -\log\left(\frac{\exp(-\omega t) - 1}{\exp(-\omega) - 1}\right), \quad \omega \in (-\infty, \infty). \tag{5}$$

Inverse Function (Frank)

$$\varphi^{-1}(t) = -\frac{1}{\omega} = \log\left[\exp(-t)\left(\exp(-\omega) - 1\right) + 1\right]. \tag{6}$$

Copula (Frank)

$$C_F(u,v) = -\frac{1}{\omega}\log\left[1 + \frac{\left(\exp(-\omega u) - 1\right)\left(\exp(-\omega v) - 1\right)}{\exp(-\omega) - 1}\right]. \tag{7}$$

Kendal's Tau (Frank)

$$\tau = 1 - \frac{4}{\theta} + \frac{4D_1(\theta)}{\theta}, \tag{8}$$

where $D_1(\theta) = \int_0^\theta ((x/\theta)/(\exp(x) - 1))dx$ (Debye function).

2.2. C-Convolution. Let F and H be continuous cumulative distribution functions and C be a copula function. The C-Convolution of F and H is defined by the cumulative distribution function (CDF)

$$H *^C F(t) = \int_0^1 D_1C\left(w, F\left(t - H^{-1}(w)\right)\right) dw, \tag{9}$$

where $D_1C(w, F(t - H^{-1}(w))) = (\partial/\partial w)C(w, F(t - H^{-1}(w)))$.

2.3. Convolution-Based Copulas

2.3.1. Proposition. Let X and Y be two real-valued random variables on the same probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ with a dependence structure represented by the copula function $C_{X,Y}$ and continuous marginal distributions F_X and F_Y. Then,

$$\begin{aligned} C_{X,X+Y}(u,v) &= \int_0^u D_1C_{X,Y}\left(w, F_Y\left(F_{X+Y}^{-1}(v) - F_X^{-1}(w)\right)\right) dw, \\ F_{X+Y}(t) &= \int_0^1 D_1C_{X,Y}\left(w, F_Y\left(t - F_X^{-1}(w)\right)\right) dw \\ &= F_X *^C F_Y(t). \end{aligned} \tag{10}$$

2.3.2. Proposition. Let F, G, and H be three continuous CDFs, $C(w,v)$ be a copula function, and

$$\widehat{C}(u,v) = \int_0^u D_1C\left(w, F\left(G^{-1}(v) - H^{-1}(w)\right)\right) dw. \tag{11}$$

$\widehat{C}(u,v)$ is a copula iff $G = H *^C F$.

The C-Convolution operator is closed with respect to mixtures of copula functions.

Let A and B be bivariate copula functions.

$$C(u,v) = \lambda A(u,v) + (1-\lambda)B(u,v). \tag{12}$$

For all $\lambda \in [0,1]$ and for all CDFs H and F,

$$H *^C F = H *^{\lambda A + (1-\lambda)B} F = \lambda H *^A F + (1-\lambda) H *^B F. \tag{13}$$

2.4. Proposed Convolution-Based Hybrid Clayton-Frank Copula. Using the fact that the C-Convolution operator is closed with respect to mixtures of copula functions, we propose a hybrid Clayton-Frank copula as follows:

$$\begin{aligned} C_{CF}(u,v) = {} & \lambda\left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta} + (1-\lambda)\left(-\frac{1}{\omega}\right. \\ & \left. \cdot \log\left[1 + \frac{\left(\exp(-\omega u) - 1\right)\left(\exp(-\omega v) - 1\right)}{\exp(-\omega) - 1}\right]\right). \end{aligned} \tag{14}$$

2.4.1. Check for $C_{CF}(u,v)$ Being a Copula

(1) For It Being Grounded, That Is, $C(0,v)=C(u,0)=0$. From $C_{CF}(u,v)$ above,

$$\begin{aligned} C_{CF}(0,v) &= \lambda\left(0^{-\theta}+v^{-\theta}-1\right)^{-1/\theta}+(1-\lambda)\left(-\frac{1}{\omega}\right. \\ &\left.\cdot\log\left[1+\frac{(\exp(-\omega(0))-1)(\exp(-\omega v)-1)}{\exp(-\omega)-1}\right]\right) \\ &= \lambda\left(v^{-\theta}-1\right)^{-1/\theta}+(1-\lambda)\left(-\frac{1}{\omega}\right. \\ &\left.\cdot\log\left[1+\frac{(0)(\exp(-\omega v)-1)}{\exp(-\omega)-1}\right]\right)=\lambda\left(v^{-\theta}\right. \\ &\left.-1\right)^{-1/\theta}+(1-\lambda)\left(-\frac{1}{\omega}\log[1]\right)=\lambda\left(v^{-\theta}-1\right)^{-1/\theta} \\ &+(1-\lambda)\left(-\frac{1}{\omega}(0)\right)=\lambda\left(v^{-\theta}-1\right)^{-1/\theta}\longrightarrow 0, \end{aligned} \tag{15}$$

$$\text{since the largest value of } u \text{ will be one } (1),$$

$$\begin{aligned} C_{CF}(u,0) &= \lambda\left(u^{-\theta}+0^{-\theta}-1\right)^{-1/\theta}+(1-\lambda)\left(-\frac{1}{\omega}\right. \\ &\left.\cdot\log\left[1+\frac{(\exp(-\omega u)-1)(\exp(-\omega(0))-1)}{\exp(-\omega)-1}\right]\right). \end{aligned}$$

Following the steps in $C_{CF}(0,v)$, $C_{CF}(u,0)$ also gives zero (0).

(2) Uniform Marginals: $C(u,1)=u$, $C(1,v)=v$.

$$\begin{aligned} C_{CF}(u,1) &= \lambda\left(u^{-\theta}+1^{-\theta}-1\right)^{-1/\theta}+(1-\lambda)\left(-\frac{1}{\omega}\right. \\ &\left.\cdot\log\left[1+\frac{(\exp(-\omega u)-1)(\exp(-\omega(1))-1)}{\exp(-\omega)-1}\right]\right) \\ &= \lambda\left(u^{-\theta}\right)^{-1/\theta}+(1-\lambda)\left(-\frac{1}{\omega}\right. \\ &\left.\cdot\log\left[1+(\exp(-\omega u)-1)\right]\right), \end{aligned} \tag{16}$$

which is the marginal of u.

And

$$\begin{aligned} C_{CF}(1,v) &= \lambda\left(1^{-\theta}+v^{-\theta}-1\right)^{-1/\theta}+(1-\lambda)\left(-\frac{1}{\omega}\right. \\ &\left.\cdot\log\left[1+\frac{(\exp(-\omega(1))-1)(\exp(-\omega v)-1)}{\exp(-\omega)-1}\right]\right) \\ &= \lambda\left(v^{-\theta}\right)^{-1/\theta}+(1-\lambda)\left(-\frac{1}{\omega}\right. \\ &\left.\cdot\log\left[1+(\exp(-\omega v)-1)\right]\right), \end{aligned} \tag{17}$$

which is the marginal of v.

(3) 2-Increasing: $C(u_1,v_1)-C(u_1,v_2)-C(u_2,v_1)+C(u_2,v_2)\geq 0$, for $u_1>u_2$ and $v_1>v_2$.

$$\begin{aligned} C_{CF}(u_1,v_1) &= \lambda\left[u_1^{-\theta}+v_1^{-\theta}-1\right]^{-1/\theta}+(1-\lambda)\left[-\frac{1}{\omega}\right. \\ &\left.\cdot\log\left\{1+\frac{(\exp(-\omega u_1)-1)(\exp(-\omega v_1)-1)}{\exp(-\omega)-1}\right\}\right], \\ C_{CF}(u_1,v_2) &= \lambda\left[u_1^{-\theta}+v_2^{-\theta}-1\right]^{-1/\theta}+(1-\lambda)\left[-\frac{1}{\omega}\right. \\ &\left.\cdot\log\left\{1+\frac{(\exp(-\omega u_1)-1)(\exp(-\omega v_2)-1)}{\exp(-\omega)-1}\right\}\right], \\ C_{CF}(u_2,v_1) &= \lambda\left[u_2^{-\theta}+v_1^{-\theta}-1\right]^{-1/\theta}+(1-\lambda)\left[-\frac{1}{\omega}\right. \\ &\left.\cdot\log\left\{1+\frac{(\exp(-\omega u_2)-1)(\exp(-\omega v_1)-1)}{\exp(-\omega)-1}\right\}\right], \\ C_{CF}(u_2,v_2) &= \lambda\left[u_2^{-\theta}+v_2^{-\theta}-1\right]^{-1/\theta}+(1-\lambda)\left[-\frac{1}{\omega}\right. \\ &\left.\cdot\log\left\{1+\frac{(\exp(-\omega u_2)-1)(\exp(-\omega v_2)-1)}{\exp(-\omega)-1}\right\}\right]. \end{aligned} \tag{18}$$

The above ensures that the 2-increasing property is satisfied.

2.5. Conditional Distribution Function of v Given u. The conditional distribution function of v given u is obtained by finding the derivative of $C_{CF}(u,v)$ with respect to u; that is, $(\partial/\partial u)C_{CF}(u,v)$.

$$\begin{aligned} \frac{\partial}{\partial u}C_{CF}(u,v) &= \frac{\partial}{\partial u}\left[\lambda\left(u^{-\theta}+v^{-\theta}-1\right)^{-1/\theta}+(1-\lambda)\left(-\frac{1}{\omega}\log\left[1+\frac{(\exp(-\omega u)-1)(\exp(-\omega v)-1)}{\exp(-\omega)-1}\right]\right)\right], \\ \frac{\partial}{\partial u}C_{CF}(u,v) &= \lambda y^{-\theta-1}\left(u^{-\theta}+v^{-\theta}-1\right)^{-(\theta+1)/\theta}-\left[\frac{(\lambda-1)\exp(\omega)(\exp(u\omega)-1)}{-\exp(\omega(x+y))+\exp(u\omega+\omega)+\exp(v\omega+\omega)-\exp(\omega)}\right]. \end{aligned} \tag{19}$$

Table 1: Summary of results for lambda = 0.1.

Interval description	NLM ML estimate (mean)	AIC
Omega constant, theta varied from −4 to 1 by 0.5	1.797693e + 308	−1415.565
Omega constant, theta varied from 4 to 1 by 0.5	1.797693e + 308	−1415.565
Theta constant, omega varied from −4 to 1 by 0.5	62.07445	−4.256669
Theta constant, omega varied from 4 to 1 by 0.5	1.797693e + 308	−1415.565

Table 2: Summary of results for lambda = 0.5.

Interval description	NLM ML estimate (mean)	AIC
Omega constant, theta varied from −4 to 1 by 0.5	1.797693e + 308	−1415.565
Omega constant, theta varied from 4 to 1 by 0.5	1.797693e + 308	−1415.565
Theta constant, omega varied from −4 to 1 by 0.5	17.21489	−1.691549
Theta constant, omega varied from 4 to 1 by 0.5	1.797693e + 308	−1415.565

2.6. Joint Distribution Function of u and v. The joint distribution function is obtained from $(\partial^2/\partial u\partial v)C_{\mathrm{CF}}(u,v)$.

$$\frac{\partial^2}{\partial u\partial v}C_{\mathrm{CF}}(u,v) = \frac{(\theta+1)\lambda u^{\theta-1}v^{\theta-1}\left(u^{-\theta}+v^{-\theta}-1\right)^{-1/\theta}}{\left(v^{\theta}-u^{\theta}\left(v^{\theta}-1\right)\right)^2}\left[-\frac{(\lambda-1)\,\omega\left(\exp(\omega)-\exp(u\omega)\right)\left(\exp(u\omega)-1\right)\exp(v\omega+\omega)}{\left(-\exp(\omega(u+v))+\exp(u\omega+\omega)+\exp(v\omega+\omega)-\exp(\omega)\right)^2}\right]. \tag{20}$$

2.7. Parameter Estimation. In order to obtain theta (θ), Kendal's tau relationship with the traditional Clayton copula parameter (see (4)) is employed. Similarly, to obtain omega (ω), Kendal's tau relationship with the traditional Frank copula (see (8)) is used.

3. Results and Analysis

3.1. Comparison of Clayton-Frank Copula with Other Bivariate Copulas for Two Simulated Independent Random Variables. Assuming Independence of two randomly generated variables, various starting values for the maximum likelihood estimate of the hybrid Clayton-Frank model are considered. The simulation covers lengths $= 10^i$, $i = 2, 3, 4, 5$.

3.2. Results for $n = 10^i$, $i = 2$. Standard normal bivariate random variables of length $n = 100$ are generated and their CDF which lies in $(0, 1)$ is used for the copula analysis. The upper tail of the CDF is analysed to check whether the hybrid model works better in right skewed observations.

The interval −4 to 4 is considered for both parameters theta and omega from the hybrid model and pairs from this interval are considered as starting values for the optimization process in the maximum likelihood estimation.

Values of lambda (λ) $(0.1, 0.5, 0.8)$ are considered to see their individual effect on the hybrid Clayton-Frank copula model.

Using the same simulated pairs from the standard normal as used in the hybrid Clayton-Frank copula, the selected bivariate copula from a number of 40 bivariate copulas was the Rotated Tawn Type 2 (180 degrees) copula with an AIC value of −4.54. This means that, compared to the other bivariate copulas, the Rotated Tawn Type 2 (180 degrees) copula had the smallest AIC value. Comparing this value with the AIC values in Table 1, it can be observed that the hybrid Clayton-Frank copula outperforms all the 40 bivariate copulas under study when omega is kept constant and theta varied from −4 to 1, when omega is kept constant and theta varied from 4 to 1, and when theta is kept constant and omega varied from 4 to 1.

Using the same simulated pairs from the standard normal as used in the hybrid Clayton-Frank copula, the selected bivariate copula from a number of 40 bivariate copulas was the Rotated Tawn Type 2 (180 degrees) copula with an AIC value of −4.54. This means that, compared to the other bivariate copulas, the Rotated Tawn Type 2 (180 degrees) copula had the smallest AIC value. Comparing this value with the AIC values in Table 2, it can be observed that the hybrid Clayton-Frank copula outperforms all the 40 bivariate copulas under study when omega is kept constant and theta varied from −4 to 1, when omega is kept constant and theta varied from 4 to 1, and when theta is kept constant and omega varied from 4 to 1.

Table 3: Summary of results for lambda = 0.8.

Interval description	NLM ML estimate (mean)	AIC
Omega constant, theta varied from −4 to 1 by 0.5	$1.797693e+308$	−1415.565
Omega constant, theta varied from 4 to 1 by 0.5	$1.797693e+308$	−1415.565
Theta constant, omega varied from −4 to 1 by 0.5	3.554999	1.46329
Theta constant, omega varied from 4 to 1 by 0.5	$1.797693e+308$	−1415.565

Table 4: Summary of results for lambda = 0.1, $n = 10^i$, $i = 3$.

Interval description	NLM ML estimate (mean)	AIC
Omega constant, theta varied from −4 to 1 by 0.5	$1.797693e+308$	−1415.565
Omega constant, theta varied from 4 to 1 by 0.5	$1.797693e+308$	−1415.565
Theta constant, omega varied from −4 to 1 by 0.5	601.341	−8.798324
Theta constant, omega varied from 4 to 1 by 0.5	$1.797693e+308$	−1415.565

Table 5: Summary of results for lambda = 0.5, $n = 10^i$, $i = 3$.

Interval description	NLM ML estimate (mean)	AIC
Omega constant, theta varied from −4 to 1 by 0.5	$1.797693e+308$	−1415.565
Omega constant, theta varied from 4 to 1 by 0.5	$1.797693e+308$	−1415.565
Theta constant, omega varied from −4 to 1 by 0.5	194.1866	−6.537639
Theta constant, omega varied from 4 to 1 by 0.5	$1.797693e+308$	−1415.565

Using the same simulated pairs from the standard normal as used in the hybrid Clayton-Frank copula, the selected bivariate copula from a number of 40 bivariate copulas was the Rotated Tawn Type 2 (180 degrees) copula with an AIC value of −4.54. This means that, compared to the other bivariate copulas, the Rotated Tawn Type 2 (180 degrees) copula had the smallest AIC value. Comparing this value with the AIC values in Table 3, it can be observed that the hybrid Clayton-Frank copula outperforms all the 40 bivariate copulas under study when omega is kept constant and theta varied from −4 to 1, when omega is kept constant and theta varied from 4 to 1, and when theta is kept constant and omega varied from 4 to 1.

3.3. Results for $n = 10^i$, $i = 3$. Standard normal bivariate random variables generated of length $n = 1000$ are generated and their CDF which lies in $(0, 1)$ is used for the copula analysis. The upper tail of the CDF is analysed to check whether the hybrid model works better in right skewed observations.

The interval −4 to 4 is considered for both of the parameters theta and omega from the hybrid model too and pairs from this interval are considered as starting values for the optimization process in the maximum likelihood estimation.

Values of lambda $(0.1, 0.5, 0.8)$ are considered to see their individual effect on the overall hybrid model.

Using the same simulated pairs from the standard normal as used in the hybrid Clayton-Frank copula, the selected bivariate copula from a number of 40 bivariate copulas was the Frank Copula with an AIC value of −6.41. This means that, compared to the other bivariate copulas, the Frank copula had the smallest AIC value. Comparing this value with the AIC values in Table 4, it can be observed that the hybrid Clayton-Frank copula outperforms all the 40 bivariate copulas under study for combinations of theta and omega earlier mentioned.

Using the same simulated pairs from the standard normal as used in the hybrid Clayton-Frank copula, the selected bivariate copula from a number of 40 bivariate copulas was the Frank Copula with an AIC value of −6.41. This means that, compared to the other bivariate copulas, the Frank copula had the smallest AIC value. Comparing this value with the AIC values in Table 5, it can be observed that the hybrid Clayton-Frank copula outperforms all the 40 bivariate copulas under study for combinations of theta and omega earlier mentioned.

Using the same simulated pairs from the standard normal as used in the hybrid Clayton-Frank copula, the selected bivariate copula from a number of 40 bivariate copulas was the Frank Copula with an AIC value of −6.41. This means that, compared to the other bivariate copulas, the Frank copula had the smallest AIC value. Comparing this value with the AIC values in Table 6, it can be observed that the hybrid Clayton-Frank copula outperforms all the 40 bivariate copulas under study for combinations of theta and omega earlier mentioned except when omega is kept constant and theta varied from −4 to 1.

3.4. Results for $n = 10^i$, $i = 4$. Standard normal bivariate random variables of length $n = 10000$ are generated and their CDF which lies in $(0, 1)$ is used for the copula analysis. The upper tail of the CDF is analysed to check whether the hybrid model works better in right skewed observations.

Table 6: Summary of results for lambda = 0.8, $n = 10^i$, $i = 3$.

Interval description	NLM ML estimate (mean)	AIC
Omega constant, theta varied from −4 to 1 by 0.5	1.797693e + 308	−1415.565
Omega constant, theta varied from 4 to 1 by 0.5	1.797693e + 308	−1415.565
Theta constant, omega varied from −4 to 1 by 0.5	48.874	−3.778491
Theta constant, omega varied from 4 to 1 by 0.5	1.797693e + 308	−1415.565

Table 7: Summary of results for lambda = 0.1, $n = 10^i$, $i = 4$.

Interval description	NLM ML estimate (mean)	AIC
Omega constant, theta varied from −4 to 1 by 0.5	1.797693e + 308	−1415.565
Omega constant, theta varied from 4 to 1 by 0.5	1.797693e + 308	−1415.565
Theta constant, omega varied from −4 to 1 by 0.5	6360.052	−15.51558
Theta constant, omega varied from 4 to 1 by 0.5	1.797693e + 308	−1415.565

Table 8: Summary of results for lambda = 0.5, $n = 10^i$, $i = 4$.

Interval description	NLM ML estimate (mean)	AIC
Omega constant, theta varied from −4 to 1 by 0.5	1.797693e + 308	−1415.565
Omega constant, theta varied from 4 to 1 by 0.5	1.797693e + 308	−1415.565
Theta constant, omega varied from −4 to 1 by 0.5	2058.903	−13.25986
Theta constant, omega varied from 4 to 1 by 0.5	1.797693e + 308	−1415.565

Table 9: Summary of results for lambda = 0.8, $n = 10^i$, $i = 4$.

Interval description	NLM ML estimate (mean)	AIC
Omega constant, theta varied from −4 to 1 by 0.5	1.797693e + 308	−1415.565
Omega constant, theta varied from 4 to 1 by 0.5	1.797693e + 308	−1415.565
Theta constant, omega varied from −4 to 1 by 0.5	544.7788	−10.60076
Theta constant, omega varied from 4 to 1 by 0.5	1.797693e + 308	−1415.565

The interval −4 to 4 is considered for both of the parameters theta and omega from the hybrid model and pairs from this interval are considered as starting values for the optimization process in the maximum likelihood estimation.

Values of lambda $(0.1, 0.5, 0.8)$ are considered to see their individual effect on the overall hybrid model.

Using the same simulated pairs from the standard normal as used in the hybrid Clayton-Frank copula, the selected bivariate copula from a number of 40 bivariate copulas was the Independence Copula with an AIC value of 0. This means that, compared to the other bivariate copulas, the Independence Copula had the smallest AIC value. Comparing this value with the AIC values in Table 7, it can be observed that the hybrid Clayton-Frank copula outperforms all the 40 bivariate copulas under study for combinations of theta and omega earlier mentioned.

Using the same simulated pairs from the standard normal as used in the hybrid Clayton-Frank copula, the selected bivariate copula from a number of 40 bivariate copulas was the Independence Copula with an AIC value of 0. This means that, compared to the other bivariate copulas, the Independence copula had the smallest AIC value. Comparing this value with the AIC values in Table 8, it can be observed that the hybrid Clayton-Frank copula outperforms all the 40 bivariate copulas under study for combinations of theta and omega earlier mentioned.

Using the same simulated pairs from the standard normal as used in the hybrid Clayton-Frank copula, the selected bivariate copula from a number of 40 bivariate copulas was the Independence Copula with an AIC value of 0. This means that, compared to the other bivariate copulas, the Independence copula had the smallest AIC value. Comparing this value with the AIC values in Table 9, it can be observed that the hybrid Clayton-Frank copula outperforms all the 40 bivariate copulas under study for combinations of theta and omega earlier mentioned.

3.5. Results for $n = 10^i$, $i = 5$. Standard normal bivariate random variables of length $n = 100000$ are generated and their CDF which lies in $(0, 1)$ is used for the copula analysis. The upper tail of the CDF is analysed to check whether the hybrid model works better in right skewed observations.

Table 10: Summary of results for lambda = 0.1, $n = 10^i$, $i = 5$.

Interval description	NLM ML estimate (mean)	AIC
Omega constant, theta varied from −4 to 1 by 0.5	$1.797693e + 308$	−1415.565
Omega constant, theta varied from 4 to 1 by 0.5	$1.797693e + 308$	−1415.565
Theta constant, omega varied from −4 to 1 by 0.5	63509.4	−18.11789
Theta constant, omega varied from 4 to 1 by 0.5	$1.797693e + 308$	−1415.565

Table 11: Summary of results for lambda = 0.5, $n = 10^i$, $i = 5$.

Interval description	NLM ML estimate (mean)	AIC
Omega constant, theta varied from −4 to 1 by 0.5	$1.797693e + 308$	−1415.565
Omega constant, theta varied from 4 to 1 by 0.5	$1.797693e + 308$	−1415.565
Theta constant, omega varied from −4 to 1 by 0.5	20204.94	−15.82736
Theta constant, omega varied from 4 to 1 by 0.5	$1.797693e + 308$	−1415.565

Table 12: Summary of results for lambda = 0.8, $n = 10^i$, $i = 5$.

Interval description	NLM ML estimate (mean)	AIC
Omega constant, theta varied from −4 to 1 by 0.5	$1.797693e + 308$	−1415.565
Omega constant, theta varied from 4 to 1 by 0.5	$1.797693e + 308$	−1415.565
Theta constant, omega varied from −4 to 1 by 0.5	5175.958	−13.10356
Theta constant, omega varied from 4 to 1 by 0.5	$1.797693e + 308$	−1415.565

The interval −4 to 4 is considered for both of the parameters theta and omega from the hybrid model and pairs from this interval are considered as starting values for the optimization process in the maximum likelihood estimation.

Values of lambda $(0.1, 0.5, 0.8)$ are considered to see their individual effect on the overall hybrid model.

Using the same simulated pairs from the standard normal as used in the hybrid Clayton-Frank copula, the selected bivariate copula from a number of 40 bivariate copulas was the Independence Copula with an AIC value of 0. This means that, compared to the other bivariate copulas, the Independence Copula had the smallest AIC value. Comparing this value with the AIC values in Table 10, it can be observed that the hybrid Clayton-Frank copula outperforms all the 40 bivariate copulas under study for combinations of theta and omega earlier mentioned.

Using the same simulated pairs from the standard normal as used in the hybrid Clayton-Frank copula, the selected bivariate copula from a number of 40 bivariate copulas was the Independence Copula with an AIC value of 0. This means that, compared to the other bivariate copulas, the Independence copula had the smallest AIC value. Comparing this value with the AIC values in Table 11, it can be observed that the hybrid Clayton-Frank copula outperforms all the 40 bivariate copulas under study for combinations of theta and omega earlier mentioned.

Using the same simulated pairs from the standard normal as used in the hybrid Clayton-Frank copula, the selected bivariate copula from a number of 40 bivariate copulas was the Independence Copula with an AIC value of 0. This means that, compared to the other bivariate copulas, the Independence copula had the smallest AIC value. Comparing this value with the AIC values in Table 12, it can be observed that the hybrid Clayton-Frank copula outperforms all the 40 bivariate copulas under study for combinations of theta and omega earlier mentioned.

3.6. Results for $n = 10^i$, $i = 6$. Standard normal bivariate random variables of length $n = 1000000$ are generated and their CDF which lies in $(0, 1)$ is used for the copula analysis. The upper tail of the CDF is analysed to check whether the hybrid model works better in right skewed observations.

The interval −4 to 4 is considered for both of the parameters theta and omega from the hybrid model and pairs from this interval are considered as starting values for the optimization process in the maximum likelihood estimation.

Values of lambda $(0.1, 0.5, 0.8)$ are considered to see their individual effect on the overall hybrid model.

Using the same simulated pairs from the standard normal as used in the hybrid Clayton-Frank copula, the selected bivariate copula from a number of 40 bivariate copulas was the Independence Copula with an AIC value of 0. This means that, compared to the other bivariate copulas, the Independence Copula had the smallest AIC value. Comparing this value with the AIC values in Table 13, it can be observed that the hybrid Clayton-Frank copula outperforms all the 40 bivariate copulas under study for combinations of theta and omega earlier mentioned.

Using the same simulated pairs from the standard normal as used in the hybrid Clayton-Frank copula, the selected bivariate copula from a number of 40 bivariate copulas was the Independence Copula with an AIC value of 0. This means that, compared to the other bivariate copulas, the Independence Copula had the smallest

Table 13: Summary of results for lambda = 0.1, $n = 10^i$, $i = 6$.

Interval description	NLM ML estimate (mean)	AIC
Omega constant, theta varied from −4 to 1 by 0.5	$1.797693e + 308$	−1415.565
Omega constant, theta varied from 4 to 1 by 0.5	$1.797693e + 308$	−1415.565
Theta constant, omega varied from −4 to 1 by 0.5	676524.3	−22.84945
Theta constant, omega varied from 4 to 1 by 0.5	$1.797693e + 308$	−1415.565

Table 14: Summary of results for lambda = 0.5, $n = 10^i$, $i = 6$.

Interval description	NLM ML estimate (mean)	AIC
Omega constant, theta varied from −4 to 1 by 0.5	$1.797693e + 308$	−1415.565
Omega constant, theta varied from 4 to 1 by 0.5	$1.797693e + 308$	−1415.565
Theta constant, omega varied from −4 to 1 by 0.5	201250.4	−20.42461
Theta constant, omega varied from 4 to 1 by 0.5	$1.797693e + 308$	−1415.565

Table 15: Summary of results for lambda = 0.8, $n = 10^i$, $i = 6$.

Interval description	NLM ML estimate (mean)	AIC
Omega constant, theta varied from −4 to 1 by 0.5	$1.797693e + 308$	−1415.565
Omega constant, theta varied from 4 to 1 by 0.5	$1.797693e + 308$	−1415.565
Theta constant, omega varied from −4 to 1 by 0.5	51307.49	−17.68948
Theta constant, omega varied from 4 to 1 by 0.5	$1.797693e + 308$	−1415.565

Table 16: Bivariate copula families with which the Hybrid Clayton-Frank copula is compared.

Copula name	Number	Parameter 1	Parameter 2
Gaussian	1	$(-1, 1)$	-
Student t	2	$(-1, 1)$	$(2, \infty)$
Clayton (survival)	3, 13	$(0, \infty)$	-
Rotated Clayton (90 and 270 degrees)	23, 33	$(-\infty, 0)$	-
Gumbel (survival)	4, 14	$[1, \infty)$	-
Rotated Gumbel (90 and 270 degrees)	24, 34	$(-\infty, -1]$	-
Frank	5	$R \setminus \{0\}$	-
Joe (survival)	6, 16	$(1, \infty)$	-
Rotated Joe (90 and 270 degrees)	26, 36	$(-\infty, -1)$	-
Clayton-Gumbel (Survival)	7, 17	$(0, \infty)$	$[1, \infty)$
Rotated Clayton-Gumbel (90 and 270 degrees)	28, 38	$(-\infty, -1]$	$(-\infty, -1]$
Joe-Clayton (Survival)	9, 19	$[1, \infty)$	$(0, \infty)$
Rotated Joe-Clayton (90 and 270 degrees)	29, 39	$(-\infty, -1)$	$(-\infty, 0)$
Joe-Frank (Survival)	10, 20	$[1, \infty)$	$(0, 1]$
Rotated Joe-Frank (90 and 270 degrees)	30, 40	$(-\infty, -1)$	$[-1, 0)$
Tawn Type 1 (Survival)	104, 114	$[1, \infty)$	$[0, 1]$
Rotated Tawn Type 1 (90 and 270 degrees)	124, 134	$(-\infty, -1)$	$[0, 1]$
Tawn Type 2 (Survival)	204, 214	$[1, \infty)$	$[0, 1]$
Rotated Tawn Type 2 (90 and 270 degrees)	224, 234	$(-\infty, -1]$	$[0, 1]$

AIC value. Comparing this value with the AIC values in Table 14, it can be observed that the hybrid Clayton-Frank copula outperforms all the 40 bivariate copulas under study for combinations of theta and omega earlier mentioned.

Using the same simulated pairs from the standard normal as used in the hybrid Clayton-Frank copula, the selected bivariate copula from a number of 40 bivariate copulas was the Independence Copula with an AIC value of 0. This means that, compared to the other bivariate copulas, the Independence Copula had the smallest AIC value. Comparing this value with the AIC values in Table 15, it can be observed that the hybrid Clayton-Frank copula outperforms all the 40 bivariate copulas under study for combinations of theta and omega earlier mentioned.

Table 16 gives the various bivariate copulas (Archimedean and elliptical) with which the proposed hybrid Clayton-Frank copula is compared.

4. Conclusion

In summary, it can be observed that applying the closure property of the C-Convolution operator yielded a hybrid Clayton-Frank copula model. The joint distribution function and conditional distribution function were also obtained for the proposed hybrid Clayton-Frank copula model for a bivariate set of random variables. Using the multivariate normal distribution, simulations were performed testing the strength of the new model against about 40 existing bivariate copulas using the upper tail cumulative distribution. The proposed model outperformed most bivariate copulas (Archimedean and elliptical) for simulations that were considered. Finally, the new model eliminates restriction on copula parameters involved in the model, that is, $-\infty \leq \theta \leq \infty$ and $-\infty \leq \omega \leq \infty$, allowing for parameter values spanning the entire real line.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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