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Research Article

A Nonuniform Bound to an Independent Test in High Dimensional Data Analysis via Stein's Method

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The Berry-Esseen bound for the random variable based on the sum of squared sample correlation coefficients and used to test the complete independence in high diemensions is shown by Stein's method. Although the Berry-Esseen bound can be applied to all real numbers in $\mathbb{R}$, a nonuniform bound at a real number z usually provides a sharper bound if z is fixed. In this paper, we present the first version of a nonuniform bound on a normal approximation for this random variable with an optimal rate of $(1/(0.5+|z|)) \cdot O(1/\sqrt{m})$ by using Stein's method.

1. Introduction

Frequently, the relations between many variables from an experiment or data are analyzed. For example, in financial analysis, the performance of various banking sectors are investigated or it is interesting to know how land use affects soil chemical properties in geology. For the traditional procedures in multivariate analysis based on asymptotic theory, the number of variables, say, m, is fixed and the size of a sample from population, say, n, tends to infinity. These inference procedures, however, may not be suitable when $m > n$. Many recent articles are focused on the assumption that $m > n$, or when both m and n go to infinity such as Schott [1, 2], Srivastava [3], Chen, Zhang and Zhong [4], Srivastava, Kollo, and Rosen [5], and Jiang and Yang [6].

Let $\{X_1, \ldots, X_n\}$ be a random sample of size n from the population. We attend a test for the independence of m variables illustrated by a random vector $X_i = (X_{1i}, \ldots, X_{mi})'$ with a covariance matrix Σ. Testing for complete independence in the population having a multivariate normal distribution is identical to testing $\Sigma = I_m$. In 2005, Schott [1] introduced a simple test procedure based on the sample correlation when m and n approach infinity. Let $R = (r_{ij}), 1 \leq i, j \leq m$ be the sample correlation matrix where

$$r_{ij} = \frac{\sum_{k=1}^{n} \left(X_{ik} - \overline{X}_i\right)\left(X_{jk} - \overline{X}_j\right)}{\sqrt{\sum_{k=1}^{n} \left(X_{ik} - \overline{X}_i\right)^2} \sqrt{\sum_{k=1}^{n} \left(X_{jk} - \overline{X}_j\right)^2}}, \tag{1}$$

$X_i = (X_{1i}, \ldots, X_{mi})'$ and $\overline{X}_i = (1/n) \sum_{k=1}^{n} X_{ik}$. Moreover, Schott defined

$$t_{n,m} = \sum_{i=2}^{m} \sum_{j=1}^{i-1} r_{ij}^2$$
$$\text{and } W_{n,m} = c_{n,m} \left(t_{n,m} - \frac{m(m-1)}{2(n-1)}\right), \tag{2}$$

where $c_{n,m} = n\sqrt{n+2}/\sqrt{m(m-1)(n-1)}$ and verified the central limit theorem

$$W_{n,m} \xrightarrow{d} \mathcal{N}(0,1) \tag{3}$$

when $m/n \in (0, \infty)$ and used $W_{n,m}$ to test the complete independence of the variables.

Let F and G be distribution functions. Recall that a constant C satisfying the condition

$$\sup_{x \in \mathbb{R}} |F(x) - G(x)| \leq C \tag{4}$$

is called a uniform bound and a constant $C(x_0)$ satisfying the conditon

$$\left|F(x_0) - G(x_0)\right| \le C(x_0) \tag{5}$$

is called a nonuniform bound. An advantage of a uniform bound is that it can be applied to all of z in $\mathbb{R}$. Nevertheless, if we have a real number x_0, a nonuniform bound at x_0 is alway sharper than a uniform bound.

When the central limit theorem is derived, it is usual to investigate a uniform bound, a nonuniform bound, or the rate of convergence in such result. Stein's method [7] is a powerful technique for this estimating. There are many works in many fields of mathematics that applied this method. For example, a uniform and a nonuniform bound on a normal approximation of randomized orthogonal array sampling design were obtained by using the exchangeable pair technique of Stein's method (see [8–12]), and Poisson and normal limit laws for fringe subtrees were proved by using the coupling technique of Stein's method (see [13]).

As an application of Stein's method for a normal approximation, Chen and Shao [14] established the first version of the Berry-Esseen bound for $W_{n,m}$ under the condition $\{X_{ij}, 1 \le i \le m, 1 \le j \le n\}$ are i.i.d. random variables and $m = O(n)$. If $E(X_{11}^{24}) < \infty$, their result is as follows:

$$\sup_{z\in\mathbb{R}} \left|P(W_{m,n} < z) - \Phi(z)\right| = O\left(\frac{1}{\sqrt{m}}\right) \tag{6}$$

where Φ is the standard normal distribution.

In this present work, we investigate the first version of a nonuniform bound for $W_{n,m}$ with an optimal rate of $(1/(0.5+|z|)) \cdot O(1/\sqrt{m})$, under the same conditions as the previous article [14], by using Stein's method.

2. Auxiliary Results

In this section, we give some lemmas which will be useful for the next section. Throughout this article, the constant C is an absolute constant and it may be different in each situation. To find a nonuniform bound of $W_{n,m}$, we truncate r_{ij} with the condition $0.5 + z$. Let $z > 0$ and define

$$\widehat{Y} = c_{n,m}\left(\widehat{t}_{n,m} - \frac{m(m-1)}{2(n-1)}\right) \tag{7}$$

where

$$\begin{aligned} \widehat{t}_{n,m} &= \sum_{i=2}^{m}\sum_{j=1}^{i-1} \widehat{r}_{ij,z}^2 \\ \widehat{r}_{ij,z} &= r_{ij}\mathbb{I}\left(r_{ij}^2 \le 0.5 + z\right), \\ r_{ij,z} &= r_{ij}\mathbb{I}\left(r_{ij}^2 > 0.5 + z\right) \end{aligned} \tag{8}$$

and $\mathbb{I}$ is the indicator function.

Clearly, $E(\widehat{r}_{ij,z}^p) \le E r_{ij}^p$ when $p \in \mathbb{N}$. For each $i \in \mathbb{N}$, $1 \le i \le n$, let X_i^* be an independent copy of X_i and let I be a random index uniformly distributed over $\{1, 2, \ldots, m\}$ where I is independent of $\{X_i^*, X_i, 1 \le i \le n\}$. Define

$$\begin{aligned} W^* &= c_{n,m}\left(t_{n,m}^* - \frac{m(m-1)}{2(n-1)}\right) \\ \text{and } \widehat{Y}^* &= c_{n,m}\left(\widehat{t}_{n,m}^* - \frac{m(m-1)}{2(n-1)}\right) \end{aligned} \tag{9}$$

where

$$\begin{aligned} \widehat{t}_{n,m}^* &= \widehat{t}_{n,m} - \sum_{\substack{j=1\\ j\ne I}}^{m} \widehat{r}_{Ij,z}^2 + \sum_{\substack{j=1\\ j\ne I}}^{m} \widehat{r}_{I^*j,z}^2 \\ \widehat{r}_{i^*j,z} &= r_{i^*j}\mathbb{I}\left(r_{i^*j}^2 \le 0.5 + z\right), \\ r_{i^*j,z} &= r_{i^*j}\mathbb{I}\left(r_{i^*j}^2 > 0.5 + z\right), \end{aligned} \tag{10}$$

and

$$r_{i^*j} = \frac{\sum_{k=1}^{n}\left(X_{ik}^* - \overline{X}_i^*\right)\left(X_{jk} - \overline{X}_j\right)}{\sqrt{\sum_{k=1}^{n}\left(X_{ik}^* - \overline{X}_i^*\right)^2}\sqrt{\sum_{k=1}^{n}\left(X_{jk} - \overline{X}_j\right)^2}}. \tag{11}$$

It is easy to see that $(\widehat{t}_{n,m}, \widehat{t}_{n,m}^*)$ is an exchangeable pair and so is $(\widehat{Y}, \widehat{Y}^*)$. Let $\mathcal{X}_n = \{X_1, \ldots, X_n\}$ and let $u_i = (u_{i1}, \ldots, u_{in})'$ where $u_{ik} = (X_{ik} - \overline{X}_i)/\sqrt{\sum_{l=1}^{n}(X_{il} - \overline{X}_i)^2}$. In 2012, Chen and Shao [14] showed that

$$\begin{aligned} E\left(r_{ij}^2\right) &= \frac{1}{n-1}, \\ E\left(r_{ij}^4\right) &= \frac{3}{n^2} + O\left(\frac{1}{n^3}\right) \quad \text{if } E\left(X_{11}^6\right) < \infty, \\ E\left(r_{ij}^8\right) &= O\left(\frac{1}{n^4}\right) \quad \text{if } E\left(X_{11}^{12}\right) < \infty, \\ \text{and } E\left(r_{ij}^{16}\right) &= O\left(\frac{1}{n^8}\right) \quad \text{if } E\left(X_{11}^{24}\right) < \infty. \end{aligned} \tag{12}$$

Denote $\widehat{p}_{ij,z} = P(r_{ij}^2 \le 0.5 + z) = P(r_{i^*j}^2 \le 0.5 + z)$. Recall that, for $i \ne j$,

$$\begin{aligned} E\left(r_{ij}^2\right) &= E\left(u_i' u_j u_j' u_i\right) = E\left(E\left(u_i' u_j u_j' u_i \mid u_i\right)\right) \\ &= E\left(u_i'\left(\frac{1}{n-1}I_n - \frac{1}{n(n-1)}\mathbf{1}_n\right)u_i\right) = \frac{1}{n-1} \end{aligned} \tag{13}$$

(see [14], pp.20). Then, for $i \ne j$,

$$\begin{aligned} E\left(\widehat{r}_{ij,z}^2\right) &= E r_{ij}^2 \mathbb{I}\left(r_{ij}^2 \le 0.5 + z\right) \\ &= E\left[E\left(u_i' u_j u_j' u_i \mid u_i\right)\mathbb{I}\left(r_{ij}^2 \le 0.5 + z\right)\right] \\ &= E\left[u_i'\left(\frac{1}{n-1}I_n - \frac{1}{n(n-1)}\mathbf{1}_n\right)u_i \mathbb{I}\left(r_{ij}^2 \le 0.5 + z\right)\right] \\ &= \frac{1}{n-1}P\left(r_{ij}^2 \le 0.5 + z\right) = \frac{\widehat{p}_{ij,z}}{n-1} \end{aligned} \tag{14}$$

where I_n is the $n \times n$ identity matrix and $\mathbf{1}_n$ is the $n \times n$ matrix with all entries 1. Similarly, if $j_1 \neq j_2$,

$$E\left(\widehat{r}_{ij_1,z}^2 \widehat{r}_{ij_2,z}^2\right) = \frac{\widehat{p}_{ij_1,z} \cdot \widehat{p}_{ij_2,z}}{(n-1)^2}. \tag{15}$$

Lemma 1. *If $E(X_{11}^6) < \infty$, then*

$$E\left(1 - \frac{m}{4} E\left(\left(\widehat{Y}^* - \widehat{Y}\right)^2 \mid \widehat{Y}\right)\right)^2 = O\left(\frac{1}{m}\right). \tag{16}$$

Proof. Define $u_i^* = (u_{i1}^*, u_{i2}^*, \ldots, u_{in}^*)'$ where $u_{ik}^* = (X_{ik}^* - \overline{X}_i^*)/\sqrt{\sum_{l=1}^n (X_{il}^* - \overline{X}_i^*)^2}$. Similar to (14), we have $E(\widehat{r}_{i^* j,z}^2) = \widehat{p}_{ij,z}/(n-1)$. By the same idea as (2.32) in [14],

$$\begin{aligned}
&\frac{1}{m}\sum_{i=1}^m E\left(\left(\sum_{\substack{j=1\\ j\neq i}}^m \left(\widehat{r}_{ij,z}^2 - \widehat{r}_{i^* j,z}^2\right)\right)^2 \mid \mathcal{X}_n\right) = \frac{1}{m} \\
&\cdot \sum_{i=1}^m E\left(\left[\sum_{\substack{j=1\\ j\neq i}}^m \left(\widehat{r}_{ij,z}^2 - \frac{\widehat{p}_{ij,z}}{n-1}\right)\right.\right. \\
&\left.\left. - \sum_{\substack{j=1\\ j\neq i}}^m \left(\widehat{r}_{i^* j,z}^2 - \frac{\widehat{p}_{ij,z}}{n-1}\right)\right]^2 \mid \mathcal{X}_n\right) = \frac{1}{m} \\
&\cdot \sum_{i=1}^m \left(\sum_{\substack{j=1\\ j\neq i}}^m \left(\widehat{r}_{ij,z}^2 - \frac{\widehat{p}_{ij,z}}{n-1}\right)\right)^2 + \frac{1}{m} \\
&\cdot \sum_{i=1}^m E\left(\left(\sum_{\substack{j=1\\ j\neq i}}^m \left(\widehat{r}_{ij,z}^2 - \frac{\widehat{p}_{i^* j,z}}{n-1}\right)\right)^2 \mid \mathcal{X}_n\right).
\end{aligned} \tag{17}$$

Then

$$\begin{aligned}
&E\left(1 - \frac{m}{4} E\left(\left(\widehat{Y}^* - \widehat{Y}\right)^2 \mid \widehat{Y}\right)\right)^2 = \frac{c_{n,m}^4 m^2}{16} \\
&\cdot E\left|\frac{1}{m}\sum_{i=1}^m \left(\sum_{\substack{j=1\\ j\neq i}}^m \left(\widehat{r}_{ij,z}^2 - \frac{\widehat{p}_{ij,z}}{n-1}\right)\right)^2\right. \\
&- \frac{2(m-1)(n-1)}{n^2(n+2)} \\
&+ \frac{1}{m}\sum_{i=1}^m E\left(\left(\sum_{\substack{j=1\\ j\neq i}}^m \left(\widehat{r}_{i^* j,z}^2 - \frac{\widehat{p}_{ij,z}}{n-1}\right)\right)^2 \mid \mathcal{X}_n\right) \\
&\left.- \frac{2(m-1)(n-1)}{n^2(n+2)}\right|^2 \leq \frac{c_{n,m}^4 m^2}{16} E\left|J_1 + J_2\right|^2 \\
&+ O\left(\frac{1}{n^2}\right) \leq c_{n,m}^4 m^2 C\left\{EJ_1^2 + EJ_2^2\right\} + O\left(\frac{1}{n^2}\right)
\end{aligned} \tag{18}$$

where

$$J_1 = \left|\frac{1}{m}\sum_{i=1}^m \left\{\sum_{\substack{j=1\\ j\neq i}}^m \left(\widehat{r}_{ij,z}^2 - \frac{\widehat{p}_{ij,z}}{n-1}\right)\right\}^2 - \frac{2(m-1)}{n^2}\right| \tag{19}$$

and

$$\begin{aligned}
J_2 = &\left|\frac{1}{m}\sum_{i=1}^m E\left(\left\{\sum_{\substack{j=1\\ j\neq i}}^m \left(\widehat{r}_{i^* j,z}^2 - \frac{\widehat{p}_{ij,z}}{n-1}\right)\right\}^2 \mid \mathcal{X}_n\right)\right. \\
&\left.- \frac{2(m-1)}{n^2}\right|.
\end{aligned} \tag{20}$$

From (14) and (15), we have

$$\begin{aligned}
&\frac{1}{m}\sum_{i=1}^m E\left\{\sum_{\substack{j=1\\ j\neq i}}^m \left(\widehat{r}_{ij,z}^2 - \frac{\widehat{p}_{ij,z}}{n-1}\right)\right\}^2 = \frac{1}{m} \\
&\cdot \sum_{i=1}^m E\left\{\sum_{\substack{j=1\\ j\neq i}}^m \left(\widehat{r}_{ij,z}^2 - \frac{\widehat{p}_{ij,z}}{n-1}\right)^2\right. \\
&\left.+ \sum_{\substack{j_1=1\\ j_1\neq i}}^m \sum_{\substack{j_2=1\\ j_2\neq i\\ j_1\neq j_1}}^m \left(\widehat{r}_{ij_1,z}^2 - \frac{\widehat{p}_{ij_1,z}}{n-1}\right)\left(\widehat{r}_{ij_2,z}^2 - \frac{\widehat{p}_{ij_2,z}}{n-1}\right)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{m}\sum_{i=1}^{m}\left\{\sum_{\substack{j=1\\ j\neq i}}^{m} E\widehat{r}_{ij,z}^{4} - \frac{2}{n-1}\sum_{\substack{j=1\\ j\neq i}}^{m}\widehat{p}_{ij,z}E\widehat{r}_{ij,z}^{2}\right.\\
&\quad + \frac{1}{(n-1)^{2}}\sum_{\substack{j=1\\ j\neq i}}^{m}\widehat{p}_{ij,z}^{2} + \sum_{\substack{j_1=1\\ j_1\neq i}}^{m}\sum_{\substack{j_2=1\\ j_2\neq i\\ j_2\neq j_1}}^{m} E\widehat{r}_{ij_1,z}^{2}\widehat{r}_{ij_2,z}^{2}\\
&\quad - \frac{1}{n-1}\sum_{\substack{j_1=1\\ j_1\neq i}}^{m}\sum_{\substack{j_2=1\\ j_2\neq i\\ j_2\neq j_1}}^{m}\widehat{p}_{ij_2,z}E\widehat{r}_{ij_1,z}^{2}\\
&\quad - \frac{1}{n-1}\sum_{\substack{j_1=1\\ j_1\neq i}}^{m}\sum_{\substack{j_2=1\\ j_2\neq i\\ j_2\neq j_1}}^{m}\widehat{p}_{ij_1,z}E\widehat{r}_{ij_2,z}^{2}\\
&\quad \left.+ \frac{1}{(n-1)^{2}}\sum_{\substack{j_1=1\\ j_1\neq i}}^{m}\sum_{\substack{j_2=1\\ j_2\neq i\\ j_2\neq j_1}}^{m}\widehat{p}_{ij_1,z}\widehat{p}_{ij_2,z}\right\} = \frac{1}{m}\sum_{i=1}^{m}\left\{\sum_{\substack{j=1\\ j\neq i}}^{m} E\widehat{r}_{ij,z}^{4}\right.\\
&\quad \left.- \frac{1}{(n-1)^{2}}\sum_{\substack{j=1\\ j\neq i}}^{m}\widehat{p}_{ij,z}^{2}\right\} = (m-1)\,E\widehat{r}_{1j,z}^{4}\\
&\quad - \frac{(m-1)\,\widehat{p}_{1j,z}^{2}}{(n-1)^{2}}.
\end{aligned}
\tag{21}
$$

This implies that

$$
\begin{aligned}
EJ_1^2 &= E\left(\frac{1}{m}\sum_{i=1}^{m}\left\{\sum_{\substack{j=1\\ j\neq i}}^{m}\left(\widehat{r}_{ij,z}^{2} - \frac{\widehat{p}_{ij,z}}{n-1}\right)\right\}^{2} - \frac{2(m-1)}{n^{2}}\right)^{2} = \frac{1}{m^{2}}\sum_{i=1}^{m}E\left\{\sum_{\substack{j=1\\ j\neq i}}^{m}\left(\widehat{r}_{ij,z}^{2} - \frac{\widehat{p}_{ij,z}}{n-1}\right)\right\}^{4} + \frac{1}{m^{2}}\\
&\quad \cdot\sum_{i_1=1}^{m}\sum_{\substack{i_2=1\\ i_2\neq i_1}}^{m} E\left\{\sum_{\substack{j=1\\ j\neq i_1}}^{m}\left(\widehat{r}_{i_1j,z}^{2} - \frac{\widehat{p}_{i_1j,z}}{n-1}\right)\right\}^{2}\left\{\sum_{\substack{l=1\\ l\neq i_2}}^{m}\left(\widehat{r}_{i_2l,z}^{2} - \frac{\widehat{p}_{i_2l,z}}{n-1}\right)\right\}^{2} - \frac{4(m-1)}{n^{2}}\left\{(m-1)\,E\widehat{r}_{1j,z}^{4} - \frac{(m-1)\,\widehat{p}_{1j,z}^{2}}{(n-1)^{2}}\right\} + \frac{4(m-1)^{2}}{n^{4}}\\
&= \frac{1}{m^{2}}\sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m} E\left(\widehat{r}_{ij,z}^{2} - \frac{\widehat{p}_{ij,z}}{n-1}\right)^{4} + \frac{4}{m^{2}}\sum_{i=1}^{m}\sum_{\substack{j_1=1\\ j_1\neq i}}^{m}\sum_{\substack{j_2=1\\ j_2\neq i\\ j_2\neq j_1}}^{m} E\left(\widehat{r}_{ij_1,z}^{2} - \frac{\widehat{p}_{ij_1,z}}{n-1}\right)^{3}\left(\widehat{r}_{ij_2,z}^{2} - \frac{\widehat{p}_{ij_2,z}}{n-1}\right) + \frac{3}{m^{2}}\\
&\quad \cdot\sum_{i=1}^{m}\sum_{\substack{j_1=1\\ j_1\neq i}}^{m}\sum_{\substack{j_2=1\\ j_2\neq i\\ j_2\neq j_1}}^{m} E\left(\widehat{r}_{ij_1,z}^{2} - \frac{\widehat{p}_{ij_1,z}}{n-1}\right)^{2}\left(\widehat{r}_{ij_2,z}^{2} - \frac{\widehat{p}_{ij_2,z}}{n-1}\right)^{2} + \frac{6}{m^{2}}\\
&\quad \cdot\sum_{i=1}^{m}\sum_{\substack{j_1=1\\ j_1\neq i}}^{m}\sum_{\substack{j_2=1\\ j_2\neq i\\ j_2\neq j_1}}^{m}\sum_{\substack{j_3=1\\ j_3\neq i\\ j_3\neq j_1\\ j_3\neq j_2}}^{m} E\left(\widehat{r}_{ij_1,z}^{2} - \frac{\widehat{p}_{ij_1,z}}{n-1}\right)^{2}\left(\widehat{r}_{ij_2,z}^{2} - \frac{\widehat{p}_{ij_2,z}}{n-1}\right)\left(\widehat{r}_{ij_3,z}^{2} - \frac{\widehat{p}_{ij_3,z}}{n-1}\right) + \frac{1}{m^{2}}\\
&\quad \cdot\sum_{i=1}^{m}\sum_{\substack{j_1=1\\ j_1\neq i}}^{m}\sum_{\substack{j_2=1\\ j_2\neq i\\ j_2\neq j_1}}^{m}\sum_{\substack{j_3=1\\ j_3\neq i\\ j_3\neq j_1\\ j_3\neq j_2}}^{m}\sum_{\substack{j_4=1\\ j_4\neq i\\ j_4\neq j_1\\ j_4\neq j_2\\ j_4\neq j_3}}^{m} E\left(\widehat{r}_{ij_1,z}^{2} - \frac{\widehat{p}_{ij_1,z}}{n-1}\right)\left(\widehat{r}_{ij_2,z}^{2} - \frac{\widehat{p}_{ij_2,z}}{n-1}\right)\left(\widehat{r}_{ij_3,z}^{2} - \frac{\widehat{p}_{ij_3,z}}{n-1}\right)\left(\widehat{r}_{ij_4,z}^{2} - \frac{\widehat{p}_{ij_4,z}}{n-1}\right) + \frac{1}{m^{2}}\sum_{i_1=1}^{m}\sum_{\substack{i_2=1\\ i_2\neq i_1}}^{m} E\left(\widehat{r}_{i_1i_2,z}^{2} - \frac{\widehat{p}_{i_1i_2,z}}{n-1}\right)^{4}\\
&\quad + \frac{1}{m^{2}}\sum_{i_1=1}^{m}\sum_{\substack{i_2=1\\ i_2\neq i_1}}^{m}\sum_{j_1=1}^{m}\sum_{j_2=1}^{m}\mathbb{I}(j_1\neq i_2 \text{ or } j_2\neq i_1)\,E\left(\widehat{r}_{i_1j_1,z}^{2} - \frac{\widehat{p}_{i_1j_1,z}}{n-1}\right)^{2}\left(\widehat{r}_{i_2j_2,z}^{2} - \frac{\widehat{p}_{i_2j_2,z}}{n-1}\right)^{2} + \frac{4}{m^{2}}\\
&\quad \cdot\sum_{i_1=1}^{m}\sum_{\substack{i_2=1\\ i_2\neq i_1}}^{m}\sum_{\substack{j=1\\ j\neq i_1\\ j\neq i_2}}^{m}\left[E\left(\widehat{r}_{i_1i_2,z}^{2} - \frac{\widehat{p}_{i_1i_2,z}}{n-1}\right)^{2}\left(\widehat{r}_{i_1j,z}^{2} - \frac{\widehat{p}_{i_1j,z}}{n-1}\right)\left(\widehat{r}_{i_2j,z}^{2} - \frac{\widehat{p}_{i_2j,z}}{n-1}\right) + E\left(\widehat{r}_{i_1i_2,z}^{2} - \frac{\widehat{p}_{i_1i_2,z}}{n-1}\right)\left(\widehat{r}_{i_1j,z}^{2} - \frac{\widehat{p}_{i_1j,z}}{n-1}\right)^{2}\left(\widehat{r}_{i_2j,z}^{2} - \frac{\widehat{p}_{i_2j,z}}{n-1}\right)\right]
\end{aligned}
$$

$$-\frac{4(m-1)^2}{n^2}E\widehat{r}_{1j,z}^4+4\frac{(m-1)^2\widehat{p}_{1j,z}^2}{n^2(n-1)^2}+\frac{4(m-1)^2}{n^4}. \tag{22}$$

Note that

$$\begin{aligned}
&-\frac{4(m-1)^2}{n^2}E\widehat{r}_{1j,z}^4+4\frac{(m-1)^2\widehat{p}_{1j,z}^2}{n^2(n-1)^2}+\frac{4(m-1)^2}{n^4}\\
&=\frac{-4(m-1)^2}{n^2}E\left(r_{1j}-r_{1j,z}\right)^4+4\frac{(m-1)^2\widehat{p}_{1j,z}^2}{n^2(n-1)^2}\\
&+\frac{4(m-1)^2}{n^4}=\frac{-4(m-1)^2}{n^2}\\
&\cdot E\left(r_{1j}^4-4r_{1j}^3r_{1j,z}+6r_{1j}^2r_{1j,z}^2-4r_{1j}r_{1j,z}^3+r_{1j,z}^4\right)\\
&+4\frac{(m-1)^2\widehat{p}_{1j,z}^2}{n^2(n-1)^2}+\frac{4(m-1)^2}{n^4}=\frac{-4(m-1)^2}{n^2}\\
&\cdot Er_{1j}^4-\frac{4(m-1)^2}{n^2}\left(-Er_{1j}^4\mathbb{I}\left(r_{1j}^2>0.5+z\right)\right)+4\\
&\cdot\frac{(m-1)^2\widehat{p}_{1j,z}^2}{n^2(n-1)^2}+\frac{4(m-1)^2}{n^4}\\
&=\frac{-4(m-1)^2}{n^2}\left[\frac{3}{n^2}+O\left(\frac{1}{n^3}\right)\right]+\frac{4(m-1)^2}{n^2}\\
&\cdot Er_{1j}^4\mathbb{I}\left(r_{1j}^2>0.5+z\right)+4\frac{(m-1)^2\widehat{p}_{1j,z}^2}{n^2(n-1)^2}\\
&+\frac{4(m-1)^2}{n^4}\le\frac{4(m-1)^2}{n^2}Er_{1j}^4\mathbb{I}\left(r_{1j}^2>0.5+z\right)\\
&\le\frac{4(m-1)^2}{n^2}\left\{Er_{1j}^8\right\}^{1/2}\left\{P\left(r_{1j}^2>0.5+z\right)\right\}^{1/2}\\
&\le\frac{4(m-1)^2}{n^2}O\left(\frac{1}{m^2}\right)\left\{\frac{Er_{1j}^4}{(0.5+z)^2}\right\}^{1/2}=O\left(\frac{1}{n^3}\right).
\end{aligned} \tag{23}$$

Hence,

$$\begin{aligned}
EJ_1^2&=O\left(\frac{1}{n^4}\right)+0+O\left(\frac{1}{n^4}\right)+0+0+O\left(\frac{1}{n^4}\right)+m\\
&\cdot O\left(\frac{1}{n^4}\right)+O\left(\frac{1}{n^3}\right)+O\left(\frac{1}{n^3}\right)=O\left(\frac{1}{n^3}\right).
\end{aligned} \tag{24}$$

Similarly, $EJ_2^2\le O(1/n^3)$. Therefore, the proof is complete. □

Lemma 2. *Let $g:\mathbb{R}\longrightarrow\mathbb{R}$ be a measurable function. Then*

$$E\widehat{Y}g\left(\widehat{Y}\right)=\frac{m}{4}E\left(\widehat{Y}^*-\widehat{Y}\right)\left(g\left(\widehat{Y}^*\right)-g\left(\widehat{Y}\right)\right). \tag{25}$$

Proof. Similar to the proof of (2.17) in [14], it is easy to see that

$$\begin{aligned}
&E\left(\widehat{Y}^*-\widehat{Y}\mid\mathcal{X}_n\right)\\
&=\frac{c_{n,m}}{m}\sum_{i=1}^{m}E\left(\sum_{\substack{j=1\\j\neq i}}^{m}\left(-\widehat{r}_{ij,z}^2-\widehat{r}_{i^*j,z}^2\right)\mid\mathcal{X}_n\right)=\frac{-2}{m}\widehat{Y}.
\end{aligned} \tag{26}$$

Since $(\widehat{Y},\widehat{Y}^*)$ is an exchangeable pair and the fact that

$$F(y,\widehat{y}^*)=(y,\widehat{y}^*)\left(g(\widehat{y}^*)+g(y)\right) \tag{27}$$

is antisymmetric [15], we have

$$\begin{aligned}
0&=E\left(\widehat{Y}^*-\widehat{Y}\right)\left(g\left(\widehat{Y}^*\right)+g\left(\widehat{Y}\right)\right)\\
&=E\left(\widehat{Y}^*-\widehat{Y}\right)\left(2g\left(\widehat{Y}\right)\right)\\
&\quad+E\left(\widehat{Y}^*-\widehat{Y}\right)\left(g\left(\widehat{Y}^*\right)-g\left(\widehat{Y}\right)\right)\\
&=2Eg\left(\widehat{Y}\right)E\left(\widehat{Y}^*-\widehat{Y}\mid\mathcal{X}_n\right)\\
&\quad+E\left(\widehat{Y}^*-\widehat{Y}\right)\left(g\left(\widehat{Y}^*\right)-g\left(\widehat{Y}\right)\right)\\
&=\frac{-4}{m}E\widehat{Y}g\left(\widehat{Y}\right)+E\left(\widehat{Y}^*-\widehat{Y}\right)\left(g\left(\widehat{Y}^*\right)-g\left(\widehat{Y}\right)\right).
\end{aligned} \tag{28}$$

Thus $E\widehat{Y}g(\widehat{Y})=(m/4)E(\widehat{Y}^*-\widehat{Y})(g(\widehat{Y}^*)-g(\widehat{Y}))$. □

Lemma 3. *Let $g:\mathbb{R}\longrightarrow\mathbb{R}$ be a continuous and piecewise continuously differentiable function. Then*

$$E\widehat{Y}g\left(\widehat{Y}\right)=E\int_{-\infty}^{\infty}g'\left(\widehat{Y}+t\right)K(t)\,dt \tag{29}$$

where

$$K(t) = \frac{m}{4}\left(\widehat{Y}^* - \widehat{Y}\right) \cdot \left(\mathbb{I}\left(0 \le t \le \widehat{Y}^* - \widehat{Y}\right) - \mathbb{I}\left(\widehat{Y}^* - \widehat{Y} \le t \le 0\right)\right). \tag{30}$$

Proof. Note that

$$E^{\widehat{Y}}\left\{\int_{-\infty}^{\infty} K(t)\,dt\right\} = \frac{m}{4}E\left(\left(\widehat{Y}^* - \widehat{Y}\right) \mid \widehat{Y}\right). \tag{31}$$

and

$$\begin{aligned}
&\frac{m}{4}E\left(\widehat{Y}^* - \widehat{Y}\right)\left(g\left(\widehat{Y}^*\right) - g\left(\widehat{Y}\right)\right) \\
&\quad = \frac{m}{4}E\left(\widehat{Y}^* - \widehat{Y}\right)\int_0^{\widehat{Y}^* - \widehat{Y}} g'\left(\widehat{Y} + t\right)dt \\
&\quad = E\int_{-\infty}^{\infty} g'\left(\widehat{Y} + t\right)K(t)\,dt.
\end{aligned} \tag{32}$$

By Lemma 2, this lemma is proved. □

Lemma 4.

(1) *If* $EX_{11}^6 < \infty$, *then* $E|\widehat{Y}^* - \widehat{Y}|^2 = O(1/m)$.

(2) *If* $EX_{11}^{12} < \infty$, *then* $E|\widehat{Y}^* - \widehat{Y}|^4 = O(1/m^2)$.

(3) *If* $EX_{11}^{12} < \infty$, *then* $E|\widehat{Y}^* - \widehat{Y}|^3 = O(1/m^{3/2})$.

(4) *If* $EX_{11}^{24} < \infty$, *then* $E|\widehat{Y}^* - \widehat{Y}|^6 = O(1/m^3)$.

Proof. Since $(a-b)^2 \le 2(a^2+b^2)$ for all $a, b \in \mathbb{R}$,

$$\begin{aligned}
E\left|\widehat{Y}^* - \widehat{Y}\right|^2 &= \frac{c_{n,m}^2}{m}\sum_{i=1}^{m} E\left(\sum_{\substack{j=1\\ j\ne i}}^{m}\left(\widehat{r}_{ij,z}^2 - \widehat{r}_{i^*j,z}^2\right)\right)^2 \\
&= \frac{c_{n,m}^2}{m}\sum_{i=1}^{m} E\left(\sum_{\substack{j=1\\ j\ne i}}^{m}\left(\widehat{r}_{ij,z}^2 - \frac{p_{ij,z}}{n-1}\right)\right. \\
&\quad \left. - \sum_{\substack{j=1\\ j\ne i}}^{m}\left(\widehat{r}_{i^*j,z}^2 - \frac{p_{ij,z}}{n-1}\right)\right)^2 \le \frac{C\cdot c_{n,m}^2}{m} \\
&\quad \cdot \sum_{i=1}^{m} E\left(\sum_{\substack{j=1\\ j\ne i}}^{m}\left(\widehat{r}_{ij,z}^2 - \frac{p_{ij,z}}{n-1}\right)\right)^2 = \frac{C\cdot c_{n,m}^2}{m} \\
&\quad \cdot \sum_{i=1}^{m}\left[\sum_{\substack{j=1\\ j\ne i}}^{m} E\left(\widehat{r}_{ij,z}^2 - \frac{p_{ij,z}}{n-1}\right)^2\right. \\
&\quad \left. + 2\sum_{\substack{j_1=1\\ j_1\ne i}}^{m}\sum_{\substack{j_2=1\\ j_2\ne i\\ j_2\ne j_1}}^{m} E\left(\widehat{r}_{ij_1,z}^2 - \frac{p_{ij_1,z}}{n-1}\right)\left(\widehat{r}_{ij_2,z}^2 - \frac{p_{ij_2,z}}{n-1}\right)\right].
\end{aligned} \tag{33}$$

From (12) and (14), we have $E|\widehat{Y}^* - \widehat{Y}|^2 = O(1/m)$.

Similarly, we can show that

$$E\left|\widehat{Y}^* - \widehat{Y}\right|^4 \le \frac{C\cdot c_{n,m}^4}{m}\sum_{i=1}^{m} E\left(\sum_{\substack{j=1\\ j\ne i}}^{m}\left(\widehat{r}_{ij,z}^2 - \frac{p_{ij,z}}{n-1}\right)\right)^4. \tag{34}$$

By using the same argument of (2.33) in [14],

$$\begin{aligned}
&E\left(\sum_{\substack{j=1\\ j\ne i}}^{m}\left(\widehat{r}_{ij,z}^2 - \frac{p_{ij,z}}{n-1}\right)\right)^4 \\
&\quad = \sum_{\substack{j=1\\ j\ne i}}^{m} E\left(\widehat{r}_{ij,z}^2 - \frac{p_{ij,z}}{n-1}\right)^4 + 4\sum_{\substack{j_1=1\\ j_1\ne i}}^{m}\sum_{\substack{j_2=1\\ j_2\ne i\\ j_2\ne j_1}}^{m} E\left(\widehat{r}_{ij_1,z}^2 - \frac{p_{ij_1,z}}{n-1}\right)^3\left(\widehat{r}_{ij_2,z}^2 - \frac{p_{ij_2,z}}{n-1}\right) \\
&\quad + 3\sum_{\substack{j_1=1\\ j_1\ne i}}^{m}\sum_{\substack{j_2=1\\ j_2\ne i\\ j_2\ne j_1}}^{m} E\left(\widehat{r}_{ij_1,z}^2 - \frac{p_{ij_1,z}}{n-1}\right)^2\left(\widehat{r}_{ij_2,z}^2 - \frac{p_{ij_2,z}}{n-1}\right)^2
\end{aligned}$$

$$+6\sum_{\substack{j_1=1\\ j_1\neq i}}^{m}\sum_{\substack{j_2=1\\ j_2\neq i\\ j_2\neq j_1}}^{m}\sum_{\substack{j_3=1\\ j_3\neq i\\ j_3\neq j_1\\ j_3\neq j_2}}^{m} E\left(\widehat{r}_{ij_1,z}^2-\frac{p_{ij_1,z}}{n-1}\right)^2\left(\widehat{r}_{ij_2,z}^2-\frac{p_{ij_2,z}}{n-1}\right)\left(\widehat{r}_{ij_3,z}^2-\frac{p_{ij_3,z}}{n-1}\right)$$

$$+\sum_{\substack{j_1=1\\ j_1\neq i}}^{m}\sum_{\substack{j_2=1\\ j_2\neq i\\ j_2\neq j_1}}^{m}\sum_{\substack{j_3=1\\ j_3\neq i\\ j_3\neq j_1\\ j_3\neq j_2}}^{m}\sum_{\substack{j_4=1\\ j_4\neq i\\ j_4\neq j_1\\ j_4\neq j_2\\ j_4\neq j_3}}^{m} E\left(\widehat{r}_{ij_1,z}^2-\frac{p_{ij_1,z}}{n-1}\right)\left(\widehat{r}_{ij_2,z}^2-\frac{p_{ij_2,z}}{n-1}\right)\left(\widehat{r}_{ij_3,z}^2-\frac{p_{ij_3,z}}{n-1}\right)\left(\widehat{r}_{ij_4,z}^2-\frac{p_{ij_4,z}}{n-1}\right)=O\left(\frac{1}{m^2}\right). \tag{35}$$

Hence, $E|\widehat{Y}^*-\widehat{Y}|^4=O(1/m^2)$. This implies that $E|\widehat{Y}^*-\widehat{Y}|^3=O(1/m^{3/2})$ by applying Hölder's inequality. Moreover, by following the same method as above, $E|\widehat{Y}^*-\widehat{Y}|^6=O(1/m^3)$. □

Lemma 5.

(1) *If* $EX_{11}^6<\infty$, *then* $E\widehat{Y}^2=O(1)$.

(2) *If* $EX_{11}^{12}<\infty$, *then* $E\widehat{Y}^4=O(1)$.

Proof. (1) Let $g(x)=x$ in Lemma 2, and we obtain $E\widehat{Y}^2=(m/4)E(\widehat{Y}^*-\widehat{Y})^2$. By the previous lemma, we know that $E|\widehat{Y}^*-\widehat{Y}|^2=O(1/m)$. Then $E\widehat{Y}^2=O(1)$ as desired.

(2) Applying Lemma 3 with $g(x)=x^3$,

$$\begin{aligned}E\widehat{Y}^4&=3E\int_{-\infty}^{\infty}\left(\widehat{Y}+t\right)^2K(t)\,dt\\&=\frac{3m}{4}E\widehat{Y}^2\left(\widehat{Y}^*-\widehat{Y}\right)^2+\frac{m}{2}E\widehat{Y}\left(\widehat{Y}^*-\widehat{Y}\right)^3\\&\quad+\frac{m}{4}E\widehat{Y}\left(\widehat{Y}^*-\widehat{Y}\right)^4.\end{aligned} \tag{36}$$

Following the same technique as Neammanee, Laipaporn, and Sungkamongkol [10], we have $E\widehat{Y}^4=O(1)$. □

3. Main Result

In this section, we prove the first version of a nonuniform bound for $W_{m,n}$.

Theorem 6. *Let* $\{X_{ij},\ 1\le i\le m,\ 1\le j\le n\}$ *be i.i.d. random variables, and let* Φ *be the standard normal distribution. Assume* $m=O(n)$. *If* $E(X_{11}^{24})<\infty$, *then for every* $z\in\mathbb{R}$

$$\left|P\left(W_{m,n}<z\right)-\Phi(z)\right|\le\frac{1}{0.5+|z|}\cdot O\left(\frac{1}{\sqrt{m}}\right). \tag{37}$$

Proof. For convenience, we write W instead of $W_{m,n}$. To bound $|P(W\le z)-\Phi(z)|$, it suffices to consider $z>0$ since $\Phi(z)=1-\Phi(-z)$ and we can apply the result to $-W$ when $z\le 0$. Then, from now on, we assume $z>0$.

In the view of (6) and the fact that

$$1=\frac{0.5+z}{0.5+z}<\frac{1}{0.5+z}\quad\text{for } 0<z\le 0.25, \tag{38}$$

we have

$$|P(W\le z)-\Phi(z)|=O\left(\frac{1}{\sqrt{q}}\right)\le\frac{1}{0.5+z}O\left(\frac{1}{\sqrt{q}}\right)\quad\text{for } 0<z\le 0.25. \tag{39}$$

Then it suffices to prove Theorem 6 only in the case of $z>0.25$.

Let $z>0.25$. Thus

$$\begin{aligned}|P(W\le z)-\Phi(z)|&\le P\left(W\neq\widehat{Y}\right)\\&\quad+\left|P\left(\widehat{Y}\le z\right)-\Phi(z)\right|.\end{aligned} \tag{40}$$

Note that

$$\begin{aligned}P\left(W\neq\widehat{Y}\right)&=P\left(\sum_{i=2}^{m}\sum_{j=1}^{i-1}\mathbb{I}\left(r_{ij}^2>0.5+z\right)\ge 1\right)\\&\le E\left(\sum_{i=2}^{m}\sum_{j=1}^{i-1}\mathbb{I}\left(r_{ij}^2>0.5+z\right)\right)\\&\le\sum_{i=2}^{m}\sum_{j=1}^{i-1}\frac{E\left(r_{ij}^8\right)}{(0.5+z)^4}\\&=\frac{1}{(0.5+z)^4}\cdot O\left(\frac{1}{m^2}\right)\end{aligned} \tag{41}$$

when we apply Chebyshev's Inequality and (12) in the first and the second inequality, respectively. For the rest of this proof, we will bound $|P(\widehat{Y}\le z)-\Phi(z)|$. Let g_z be Stein's solution of Stein's equation $g_z'(w)-wg_z(w)=\mathbb{I}(w\le z)-\Phi(z)$. Applying Stein's method, we transform the problem of bounding the distance between $P(\widehat{Y}\le z)$ and $\Phi(z)$ into the problem of bounding suitably chosen differential operators; that is,

$$\left|P\left(\widehat{Y}\le z\right)-\Phi(z)\right|=\left|Eg_z'\left(\widehat{Y}\right)-E\widehat{Y}g_z\left(\widehat{Y}\right)\right|. \tag{42}$$

By Lemma 3, we have

$$\begin{aligned}
&\left|P\left(\widehat{Y} \le z\right) - \Phi(z)\right| = \left|Eg_z'\left(\widehat{Y}\right)\right. \\
&\quad \left. - E\int_{-\infty}^{\infty} g_z'\left(\widehat{Y}+t\right)K(t)\,dt\right| = \left|Eg_z'\left(\widehat{Y}\right)\right. \\
&\quad - E\int_{-\infty}^{\infty} g_z'\left(\widehat{Y}\right)K(t)\,dt + E\int_{-\infty}^{\infty} g_z'\left(\widehat{Y}\right)K(t)\,dt \\
&\quad \left. - E\int_{-\infty}^{\infty} g_z'\left(\widehat{Y}+t\right)K(t)\,dt\right| \\
&\quad \le \left|Eg_z'\left(\widehat{Y}\right)E^{\widehat{Y}}\left\{1-\int_{-\infty}^{\infty}K(t)\,dt\right\}\right| \\
&\quad + \left|E\int_{-\infty}^{\infty} g_z'\left(\widehat{Y}\right)K(t)\,dt\right. \\
&\quad \left. - E\int_{-\infty}^{\infty} g_z'\left(\widehat{Y}+t\right)K(t)\,dt\right| \\
&\quad \le E\left|g_z'\left(\widehat{Y}\right)\left[1-\frac{m}{4}E\left(\left(\widetilde{Y}-\widehat{Y}\right)^2 \mid \widehat{Y}\right)\right]\right| \\
&\quad + E\left|\int_{-\infty}^{\infty}\left\{g_z'\left(\widehat{Y}\right)-g_z'\left(\widehat{Y}+t\right)\right\}K(t)\,dt\right| := A_1 \\
&\quad + A_2
\end{aligned} \tag{43}$$

(see, e.g., [8, 12] for more techniques). To bound A_1, we recall that $|g_z'| \le 1$ and one can show that $E|g_z'(\widehat{Y})| \le C/(0.5+z)^2$ (see [16], pp.248, for more details). By Lemma 1, we have

$$\begin{aligned}
A_1 &= E\left|g_z'\left(\widehat{Y}\right)\left(1-\frac{m}{4}E\left(\left(\widehat{Y}^*-\widehat{Y}\right)^2 \mid \widehat{Y}\right)\right)\right| \\
&\le \left\{E\left|g_z'\left(\widehat{Y}\right)\right|^2\right\}^{1/2} \\
&\quad \cdot \left\{E\left(1-\frac{m}{4}E\left(\left(\widehat{Y}^*-\widehat{Y}\right)^2 \mid \widehat{Y}\right)\right)^2\right\}^{1/2} \\
&\le \left\{\frac{C}{(0.5+z)^2}\right\}^{1/2} \\
&\quad \cdot \left\{E\left(1-\frac{m}{4}E\left(\left(\widehat{Y}^*-\widehat{Y}\right)^2 \mid \widehat{Y}\right)\right)^2\right\}^{1/2} \\
&= \frac{C}{(0.5+z)}\left\{E\left(1-\frac{m}{4}E\left(\left(\widehat{Y}^*-\widehat{Y}\right)^2 \mid \widehat{Y}\right)\right)^2\right\}^{1/2} \\
&= \frac{C}{(0.5+z)}\cdot O\left(\frac{1}{\sqrt{m}}\right).
\end{aligned} \tag{44}$$

Next, we will bound A_2. By using the same argument as bounding T_1 in [8], we have

$$A_2 \le A_{2,1} + A_{2,2} + A_{2,3} \tag{45}$$

where

$$\begin{aligned}
A_{2,1} &= E\mathbb{I}\left(\left|\widehat{Y}^*-\widehat{Y}\right| < \frac{0.5+z}{2}\right) \\
&\quad \cdot \int_{-\infty}^{\infty}\mathbb{I}\left(z-\max(0,t) < \widehat{Y} < z-\min(0,t)\right) \\
&\quad \cdot K(t)\,dt, \\
A_{2,2} &= E\mathbb{I}\left(\left|\widehat{Y}^*-\widehat{Y}\right| < \frac{0.5+z}{2}\right) \\
&\quad \cdot \int_{-\infty}^{\infty}\int_{t}^{0} h\left(\widehat{Y}+u\right)K(t)\,du\,dt, \\
A_{2,3} &= E\mathbb{I}\left(\left|\widehat{Y}^*-\widehat{Y}\right| \ge \frac{0.5+z}{2}\right) \\
&\quad \cdot \int_{-\infty}^{\infty}\left\{g_z'\left(\widehat{Y}\right)-g_z'\left(\widehat{Y}+t\right)\right\}K(t)\,dt.
\end{aligned} \tag{46}$$

and, for each $z > 0$, function $h : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$h(w) = \left(wg_z(w)\right)'. \tag{47}$$

For $\delta > 0$, let $f_\delta : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$f_\delta(t) = \begin{cases} 0 & \text{if } t < z-2\delta, \\ (0.5+t+\delta)(t-z+2\delta) & \text{if } z-2\delta \le t \le z+2\delta, \\ 4\delta(0.5+t+\delta) & \text{if } t > z+2\delta. \end{cases} \tag{48}$$

Then f_δ is a nondecreasing function and

$$f_\delta'(t) \ge \begin{cases} 0.5+z-\delta & \text{if } z-2\delta \le t \le z+2\delta, \\ 0 & \text{otherwise.} \end{cases} \tag{49}$$

Moreover, we can show that

$$h(w) \le \begin{cases} C(0.5+z) & \text{if } \dfrac{z}{2} \le w \le z, \\ \dfrac{C}{(0.5+z)^2} & \text{if } \dfrac{z}{2} < w \text{ or } w > z \end{cases} \tag{50}$$

([15], pp.23). Similar to bounding T_1 in [8], we get that

$$\begin{aligned}
A_{2,1} &= \frac{2}{0.5+z}E\int_{|t|\le|\widehat{Y}^*-\widehat{Y}|} f'_{|\widehat{Y}^*-\widehat{Y}|}\left(\widehat{Y}+t\right) \\
&\quad \cdot \mathbb{I}\left(z-2\left|\widehat{Y}^*-\widehat{Y}\right| < \widehat{Y} < z+2\left|\widehat{Y}^*-\widehat{Y}\right|\right) \\
&\quad \cdot K(t)\,dt, \\
A_{2,2} &= \frac{mC}{(0.5+z)^2}E\left|\widehat{Y}^*-\widehat{Y}\right|^3 + mC(0.5+z)E\left|\widehat{Y}^*\right. \\
&\quad \left. -\widehat{Y}\right|^3\mathbb{I}\left(\widehat{Y}+\left|\widehat{Y}^*-\widehat{Y}\right| \ge \frac{z}{2}\right), \\
A_{2,3} &= E\mathbb{I}\left(\left|\widehat{Y}^*-\widehat{Y}\right| \ge \frac{0.5+z}{2}\right)\int_{-\infty}^{\infty}K(t)\,dt.
\end{aligned} \tag{51}$$

Lemma 3 implies that

$$E\int_{-\infty}^{\infty} f_\delta'\left(\widehat{Y}+t\right)K(t)\,dt = E\widehat{Y}f_\delta\left(\widehat{Y}\right). \tag{52}$$

Thus

$$A_{2,1} \le \frac{2}{0.5+z}\left\{E\left|\widehat{Y}f_{|\widehat{Y}-\widehat{Y}|}\left(\widehat{Y}\right)\right|\right\} \tag{53}$$

and

$$\begin{aligned} &E\left|\widehat{Y}f_{|\widehat{Y}^*-\widehat{Y}|}\left(\widehat{Y}\right)\right| \\ &\quad\le 4E\left|\widehat{Y}\right|\left|\widehat{Y}^*-\widehat{Y}\right|\left(1+\left|\widehat{Y}\right|+\left|\widehat{Y}^*-\widehat{Y}\right|\right) \\ &\quad\le 4\left\{E\widehat{Y}^2\right\}^{1/2}\left\{E\left|\widehat{Y}^*-\widehat{Y}\right|^2\right\}^{1/2} \\ &\qquad+4\left\{E\widehat{Y}^4\right\}^{1/2}\left\{E\left|\widehat{Y}^*-\widehat{Y}\right|^2\right\}^{1/2} \\ &\qquad+4\left\{E\widehat{Y}^2\right\}^{1/2}\left\{E\left|\widehat{Y}^*-\widehat{Y}\right|^4\right\}^{1/2}. \end{aligned} \tag{54}$$

By Lemmas 4 and 5, we obtain

$$\begin{aligned} A_{2,1} &= \frac{1}{0.5+z}\cdot O\left(\frac{1}{\sqrt{m}}\right), \\ A_{2,2} &\le \frac{C}{(0.5+z)^2}\cdot O\left(\frac{1}{\sqrt{m}}\right)+mC(0.5+z) \\ &\quad\cdot\left\{E\left|\widehat{Y}^*-\widehat{Y}\right|^6\right\}^{1/2}\left\{P\left(\widehat{Y}+\left|\widehat{Y}^*-\widehat{Y}\right|\ge\frac{z}{2}\right)\right\}^{1/2} \\ &\le \frac{C}{(0.5+z)^2}\cdot O\left(\frac{1}{\sqrt{m}}\right)+mC(0.5+z) \\ &\quad\cdot O\left(\frac{1}{m^{3/2}}\right)\left\{\frac{E\widehat{Y}^4+E\left|\widehat{Y}^*-\widehat{Y}\right|^4}{z^4}\right\}^{1/2} = \frac{1}{0.5+z} \\ &\quad\cdot O\left(\frac{1}{\sqrt{m}}\right), \\ A_{2,3} &\le mE\mathbb{I}\left(\left|\widehat{Y}^*-\widehat{Y}\right|\ge\frac{0.5+z}{2}\right)\left|\widehat{Y}^*-\widehat{Y}\right|^2 \\ &\le m\left\{P\left(\left|\widehat{Y}^*-\widehat{Y}\right|\ge\frac{0.5+z}{2}\right)\right\}^{1/2} \\ &\quad\cdot\left\{E\left|\widehat{Y}^*-\widehat{Y}\right|^4\right\}^{1/2}\le C\left\{\frac{E\left|\widehat{Y}^*-\widehat{Y}\right|^2}{(0.5+z)^2}\right\}^{1/2} \\ &= \frac{1}{0.5+z}\cdot O\left(\frac{1}{\sqrt{m}}\right). \end{aligned} \tag{55}$$

Hence, $A_2 = (1/(0.5+z))\cdot O(1/\sqrt{m})$, and, therefore, the proof is now complete. □

4. Conclusions

This paper presents the first version of a nonuniform bound on a normal approximation for the random variables used in an independent test in high dimensional data analysis. In order to obtain an optimal rate of $(1/(0.5+|z|))\cdot O(1/\sqrt{m})$, the finiteness of the 24th moment is assumed.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflicts of interest.

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