Hindawi
Journal of Probability and Statistics
Volume 2020, Article ID 5497413, 9 pages
https://doi.org/10.1155/2020/5497413

Research Article

On the Maximum Likelihood Estimation of Extreme Value Index Based on k-Record Values

Abderrahim Louzaoui and Mohamed El Arrouchi

Ibn Tofail University, Faculty of Sciences Department of Mathematics, Kenitra, Morocco

Correspondence should be addressed to Mohamed El Arrouchi; elarrouchi.mohamed@uit.ac.ma

Received 5 May 2020; Accepted 26 May 2020; Published 24 June 2020

Academic Editor: Yaozhong Hu

In this paper, we study the existence and consistency of the maximum likelihood estimator of the extreme value index based on k-record values. Following the method used by Drees et al. (2004) and Zhou (2009), we prove that the likelihood equations, in terms of k-record values, eventually admit a strongly consistent solution without any restriction on the extreme value index, which is not the case in the aforementioned studies.

1. Introduction

Let $X_1, X_2, \ldots$, be a sequence of independent and identically distributed random variables (i.i.d.) having a continuous distribution function F. For each $n \geq 1$, denote by $X_{1,n} \leq \cdots \leq X_{n,n}$ the order statistics of the n-sample $(X_1, \ldots, X_n)$. We first recall some basic notions of the univariate extreme value theory. Assume that F belongs to the max-domain of attraction of an extreme value distribution, denoted by $F \in D(G_\gamma)$ with $\gamma \in \mathbb{R}$, i.e., there exist sequences $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$\lim_{n \longrightarrow \infty} \mathbb{P}\left(a_n^{-1}\left(X_{n,n} - b_n\right) \leq x\right) = G_\gamma(x) = \exp\left(-(1+\gamma x)^{-1/\gamma}\right), \tag{1}$$

for $1 + \gamma x > 0$. The parameter γ is called the extreme value index. The first-order condition $F \in D(G_\gamma)$ is equivalent to that there exists an auxiliary function $a > 0$ such that

$$\lim_{t \longrightarrow \infty} \frac{U(tx) - U(t)}{a(t)} = \frac{x^\gamma - 1}{\gamma}, \tag{2}$$

for all $x > 0$, where $U(t) = \inf\{x\colon F(x) \geq 1 - 1/t\}, t \geq 1$. For more details on the max-domain of attraction, see De Haan and Ferreira [1] and references therein.

The estimation of the extreme value index γ plays an important role in the classical extreme value theory, and many estimators have been proposed in the literature such as the Hill estimator [2], Pickands estimator [3], and moment estimator suggested by Dekkers et al. [4]. The books by Beirlant et al. [5] and De Haan and Ferreira [1] provide good reviews on this estimation problem.

Alternatively, condition (1) is equivalent to

$$\lim_{t \longrightarrow x^*} \mathbb{P}\left(X_1 - t \leq x\sigma(t) \,\middle|\, X_1 > t\right) = H_\gamma(x) = 1 - (1+\gamma x)^{-1/\gamma}, \tag{3}$$

for all $1 + \gamma x > 0$, where $\sigma(t) > 0$ is a positive function and x^* is the right endpoint of F, i.e., $x^* = \sup\{x\colon F(x) < 1\} \leq \infty$. H_γ is the so-called generalized Pareto distribution (GPD) function.

Based on (3), Smith [6] constructed the maximum likelihood (ML) estimator for (γ, σ) by solving two estimation equations, Drees et al. [7] derived its asymptotic normality for $\gamma > -1/2$ when the threshold is chosen as an upper order statistic, while Zhou [8] studied in detail its existence and consistency when $\gamma > -1$. On the contrary, the theory of record values is connected very closely to the extreme value theory through, like, for example, Resnick's duality theorem (see Theorem 2.3.3 in [9]) or the characterization of tail distributions (e.g., [10]). There are quite few publications which are devoted to the estimation of the extreme value index based on record values, see, for

example, Berred [11], Khaled et al. [12], and El Arrouchi and Imlahi [13]. We intend to investigate this problem in this paper, so we are interested here to propose an alternative of the above ML estimation based on the k-record values.

This paper is organized as follows. In Section 2, we give the likelihood equations based on k-record values. Section 3 is devoted to existence and consistency of the solutions of these equations, whose proofs will be given in Section 4.

2. Likelihood Equations Based on k-Record Values

Record values are of importance in many situations of real life as well as in many statistical applications involving data relating to natural phenomena, sports, economics, reliability, and life tests. Chandler [14] was the first to introduce the concept of record values, record times, and inter-record times in order to analyze weather data. We refer to Arnold et al. [9] and Nevzorov [15] and the references therein for a review of the general theory of records.

Let $k \geq 1$ be an integer. Define the sequences of k-record times $\{\nu_i^{(k)}, i \geq 1\}$ and k-record values $\{R_i^{(k)}, i \geq 1\}$ (see [16]) by

$$\begin{aligned} \nu_1^{(k)} &= k, \\ \nu_{i+1}^{(k)} &= \min\left\{ j > \nu_i^{(k)} \colon X_{j-k+1,j} > X_{\nu_i^{(k)}-k+1,\nu_i^{(k)}} \right\}, \\ R_i^{(k)} &= X_{\nu_i^{(k)}-k+1,\nu_i^{(k)}}, \quad i \geq 1. \end{aligned} \tag{4}$$

Similar to the conditional approach used for order statistics, our equations may be found by using the following lemma which will be proved at the end of Section 4.

Lemma 1. *For all integers $1 \leq k < n$, the conditional distribution of $(R_{n-k+1}^{(k)}, \ldots, R_n^{(k)})$, given $R_{n-k}^{(k)} = y$, is the same as the unconditional distribution of the k-record values $(S_1^{(k)}, \ldots, S_k^{(k)})$ arising from i.i.d. random variables $Z_1, Z_2, \ldots$, with the left-truncated distribution*

$$F_y(z) = \frac{(F(z) - F(y))}{(1 - F(y))}, \quad z > y. \tag{5}$$

Let $k = k_n$ be an intermediate sequence of integers satisfying $k_n \longrightarrow \infty$ and $k_n/n \longrightarrow 0$ as $n \longrightarrow \infty$, and let

$$(Y_0, Y_1, \ldots, Y_k) := \left(R_{n-k}^{(k)}, R_{n-k+1}^{(k)} - R_{n-k}^{(k)}, \ldots, R_n^{(k)} - R_{n-k}^{(k)}\right). \tag{6}$$

From Lemma 1, the conditional distribution of $(Y_1, \ldots, Y_k)$, given $Y_0 = y_0$, equals the unconditional distribution of the k-record values $(S_1^{(k)}, \ldots, S_k^{(k)})$ arising from i.i.d. random variables $Z_1, Z_2, \ldots$, with distribution $F_{y_0}(\cdot + y_0)$ which, in view of (3), can be approximated by the generalized Pareto distribution $H_\gamma(\cdot/\sigma)$ (see [7]). Using this information, one can construct an estimation of the unknown parameters γ and σ by a maximum likelihood estimation; that is, given the k-record values $(R_{n-k}^{(k)}, R_{n-k+1}^{(k)}, \ldots, R_n^{(k)})$, we maximize the likelihood function

$$L = k^k \left(1 - H_\gamma\left(\frac{Y_1}{\sigma}\right)\right)^k \prod_{i=1}^{k} \frac{h_{\gamma,\sigma}(Y_i)}{1 - H_\gamma(Y_i/\sigma)}, \tag{7}$$

with $Y_i = R_{n-i+1}^{(k)} - R_{n-k}^{(k)}$, $1 \leq i \leq k$, and $h_{\gamma,\sigma}(y) = \partial H_\gamma(y/\sigma)/\partial y$.

Remark 1. Observe that if $\gamma < -k$, $L \longrightarrow \infty$ when $\gamma/\sigma \downarrow -1/Y_1$, and so, the maximum of L does not exist. However, this case will be disregarded since k has been taken as a sequence k_n tending to infinity.

The likelihood equations are then given in terms of the partial derivatives:

$$\begin{aligned} \frac{\partial \log L}{\partial \gamma} &= -\sum_{i=1}^{k} \frac{Y_i/\sigma}{1 + (\gamma/\sigma)Y_i} + \frac{k}{\gamma^2} \log\left(1 + \frac{\gamma}{\sigma} Y_1\right) - \frac{k}{\gamma} \frac{Y_1/\sigma}{1 + (\gamma/\sigma)Y_1} = 0, \\ \frac{\partial \log L}{\partial \sigma} &= -\frac{k}{\sigma} + \sum_{i=1}^{k} \frac{(\gamma/\sigma^2)Y_i}{1 + (\gamma/\sigma)Y_i} + \frac{k}{\sigma} \frac{(\gamma/\sigma^2)Y_1}{1 + (\gamma/\sigma)Y_1} = 0. \end{aligned} \tag{8}$$

The maximum likelihood estimators for the extreme value index and the scale, $\widehat{\gamma}$ and $\widehat{\sigma}$, are obtained by solving the following likelihood equations:

$$\begin{cases} \dfrac{1}{\gamma} \log\left(1 + \dfrac{\gamma}{\sigma} Y_1\right) = 1, \\[2ex] \dfrac{1}{k} \displaystyle\sum_{i=1}^{k} \dfrac{1}{1 + (\gamma/\sigma)Y_i} = \dfrac{1}{\gamma} \dfrac{(\gamma/\sigma)Y_1}{1 + (\gamma/\sigma)Y_1}. \end{cases} \tag{9}$$

The equations for $\gamma = 0$ are defined by continuity. If $\gamma \neq 0$, they can be simplified to

$$\begin{cases} \log\left(1 + \dfrac{\gamma}{\sigma} Y_1\right) = \gamma, \\[2ex] \left(1 + \dfrac{1}{(\gamma/\sigma)Y_1}\right) \cdot \dfrac{1}{k} \displaystyle\sum_{i=1}^{k} \dfrac{1}{1 + (\gamma/\sigma)Y_i} = \dfrac{1}{\gamma}. \end{cases} \tag{10}$$

It follows that

$$\left(\left(1 + \frac{1}{(\gamma/\sigma)Y_1}\right) \log\left(1 + \frac{\gamma}{\sigma} Y_1\right)\right) \cdot \frac{1}{k} \sum_{i=1}^{k} \frac{1}{1 + (\gamma/\sigma)Y_i} = 1. \tag{11}$$

Put

$$\begin{aligned} f_n(t) &:= \log(1 + tY_1), \\ q_n(t) &:= \left(1 + \frac{1}{tY_1}\right) f_n(t), \\ g_n(t) &:= \frac{1}{k} \sum_{i=1}^{k} \frac{1}{1 + tY_i}, \end{aligned} \tag{12}$$

$$h_n(t) = q_n(t) \cdot g_n(t) - 1.$$

In view of (11), any root $(\widehat{\gamma}, \widehat{\sigma})$ of (10) satisfies $h_n(\widehat{\gamma}/\widehat{\sigma}) = 0$. Conversely, if t^* is a nonzero root of $h_n(t) = 0$,

we obtain $(\widehat{\gamma},\widehat{\sigma}) = (f_n(t^*), f_n(t^*)/t^*)$ as the solution of (10). We can readily check that $h_n(t) = 0$ has a trivial root $t^* = 0$ which must be omitted even if in reality, $\gamma = 0$.

3. Existence and Consistency

Our main results are the following theorems, stating the existence and the consistency of ML estimators.

Theorem 1. *Suppose (1) holds for $\gamma \neq 0$, and assume that, as $n \longrightarrow \infty$,*

$$k = k_n \longrightarrow \infty, \qquad \frac{k}{n} \longrightarrow 0, \qquad \frac{k}{\log n} \longrightarrow \infty. \tag{13}$$

Then, there exists a sequence of estimators $(\widehat{\gamma}_n, \widehat{\sigma}_n)$ and a random integer $N > 1$ such that

$$\mathbb{P}\left(\left\{\left(\widehat{\gamma}_n, \widehat{\sigma}_n\right) \text{ is a ML estimator for all } n \geq N\right\}\right) = 1, \tag{14}$$

and as $n \longrightarrow \infty$,

$$\widehat{\gamma}_n \xrightarrow{\text{a.s}} \gamma. \tag{15}$$

Moreover, if additionally, $(n \log\log n)^{1/2}/k \longrightarrow 0$ as $n \longrightarrow \infty$, then

$$\frac{\widehat{\sigma}_n}{a\left(e^{n/k-1}\right)} \xrightarrow{\text{a.s}} \sigma. \tag{16}$$

as $n \longrightarrow \infty$, where $a(t)$ is the auxiliary function in (2).

Theorem 2. *Suppose (1) holds for $\gamma = 0$. Assume that, as $n \longrightarrow \infty$,*

$$k = k_n \longrightarrow \infty, \qquad \frac{k}{n} \longrightarrow 0, \qquad \frac{k}{\log n} \longrightarrow \infty, \tag{17}$$

and with probability 1, the following relation does not hold for sufficiently large n:

$$R_n^{(k)} + R_{n-k}^{(k)} = \frac{2}{k}\sum_{i=1}^{k} R_{n-i+1}^{(k)}. \tag{18}$$

Then, there exists a sequence of estimators $(\widehat{\gamma}_n, \widehat{\sigma}_n)$ and a random integer $N > 1$ such that

$$\mathbb{P}\left(\left\{\left(\widehat{\gamma}_n, \widehat{\sigma}_n\right) \text{ is a ML estimator for all } n \geq N\right\}\right) = 1, \tag{19}$$

and as $n \longrightarrow \infty$,

$$\widehat{\gamma}_n \xrightarrow{\text{a.s}} 0. \tag{20}$$

Moreover, if additionally, $(n \log\log n)^{1/2}/k \longrightarrow 0$ as $n \longrightarrow \infty$, then

$$\frac{\widehat{\sigma}_n}{a\left(e^{n/k-1}\right)} \xrightarrow{\text{a.s}} \sigma, \quad n \longrightarrow \infty. \tag{21}$$

Remark 2. Extra condition (18) ensures the existence of a nonzero solution of the likelihood equations for $\gamma = 0$. Hence, the solution of the likelihood equations for $\gamma = 0$ will almost surely not be equal to 0 if, for example, F possesses a density.

4. Proofs

We first recall the following representation of the k-record values. Let $\{e_n, n \geq 1\}$ be an i.i.d sequence of standard exponential random variables, and denote by $S_n = e_1 + \cdots + e_n, n \geq 1$, their partial sums. Let $H(x) = -\log(1 - F(x))$ be the hazard function of F. It is easy to see that $U(x) = H^{\longleftarrow}(\log(x))$ for $x \geq 1$. Since F is continuous, the function $H^{\longleftarrow}$ is strictly increasing, and hence, we have the following representation (see relation (4.7), p. 167 in [17]):

$$\left\{R_n^{(k)}, n \geq 1\right\} \stackrel{d}{=} \left\{H^{\longleftarrow}\left(\frac{S_n}{k}\right), n \geq 1\right\} = \left\{U\left(e^{S_n/k}\right), n \geq 1\right\}. \tag{22}$$

So, from now on, we shall assume, without loss of generality, that $R_{n-i}^{(k)} = U(e^{S_{n-i}/k})$, for $n \geq 1$ and $0 \leq i \leq k$.

Before proving the above theorems, we need the following lemmas.

Lemma 2. *For a sequence $k = k_n \longrightarrow \infty$, $k/n \longrightarrow 0$, and $k/\log n \longrightarrow \infty$, we have as $n \longrightarrow \infty$, $(S_{n-[sk]} - S_{n-k})/k \xrightarrow{\text{a.s}} 1 - s$ uniformly on $s \in [0,1]$, where $[u]$ is the largest integer not exceeding u.*

Proof. First, we write, for $0 \leq s \leq 1$,

$$\begin{aligned} S_{n-[sk]} - S_{n-k} &= S_{n-[sk]} - (n - [sk]) - \left(S_{n-k} - (n-k)\right) + [(1-s)k] \\ &= S'_{n-[sk]} - S'_{n-k} + [(1-s)k], \end{aligned} \tag{23}$$

where $S'_i = S_i - i$. It follows that

$$\begin{aligned} \sup_{0\leq s\leq 1}\left|\frac{S_{n-[sk]} - S_{n-k}}{k} - (1-s)\right| &\leq \sup_{0\leq s\leq 1}\left|\frac{S'_{n-[sk]} - S'_{n-k}}{k}\right| \\ &\quad + \sup_{0\leq s\leq 1}\left|\frac{[(1-s)k]}{k} - (1-s)\right|. \end{aligned} \tag{24}$$

Since, for all $s \in [0,1]$, $(1-s)k - 1 < [(1-s)k] \leq (1-s)k$, we have

$$\sup_{0\leq s\leq 1}\left|\frac{[(1-s)k]}{k} - (1-s)\right| = o(1), \quad n \longrightarrow \infty. \tag{25}$$

By using the Komlós–Major–Tusnády approximation [18, 19], we can define Wiener processes $\{W(t), t \geq 0\}$ such that

$$\max_{1\leq i\leq N}\left|S'_i - W(i)\right| = {}^{\text{a.s.}}O(\log N), \quad N \longrightarrow \infty. \tag{26}$$

Next, observe that

$$\begin{aligned}\sup_{0\le s\le 1}\left|S_{n[sk]}{}' - S'_{n-k}\right| &\le \left|S_{n-k}{}' - W(n-k)\right| + \sup_{0\le s\le 1}\left|S_{n-[sk]}{}' - W(n-[sk])\right| \\ &\quad + \sup_{0\le s\le 1}\left|W(n-[sk]) - W(n-k)\right| \\ &\le \left|S_{n-k}{}' - W(n-k)\right| + \max_{0\le j\le k}\left|S_{n-j}{}' - W(n-j)\right| \\ &\quad + \max_{0\le s\le k}\left|W(n-k+s) - W(n-k)\right|.\end{aligned} \tag{27}$$

Note that, for the first terms,

$$\left|S'_{n-k} - W(n-k)\right| \overset{\text{a.s.}}{=} O(\log(n-k)), \quad n \longrightarrow \infty, \tag{28}$$

$$\max_{0\le j\le k}\left|S'_{n-j} - W(n-j)\right| \overset{\text{a.s.}}{=} O(\log n), \quad n \longrightarrow \infty. \tag{29}$$

For the last term, we use Theorem 3.2B in Hanson and Russo [20]. It implies that

$$\max_{0\le s\le k}\left|W(n-k+s) - W(n-k)\right| \overset{\text{a.s.}}{=} O\left(k\left(\log\left(\frac{n}{k}\right) + \log\log k\right)\right)^{1/2}, \quad n \longrightarrow \infty. \tag{30}$$

Combining this with (25), (28), (29), and the above conditions on k, we get

$$\sup_{0\le s\le 1}\left|\frac{S_{n-[sk]}{}' - S_{n-k}{}'}{k}\right| \overset{\text{a.s.}}{=} o(1), \quad n \longrightarrow \infty, \tag{31}$$

which completes the proof of lemma. □

Lemma 3. *Suppose (1) holds for $\gamma > 0$ and $k \longrightarrow \infty, k/n \longrightarrow 0, k/\log n \longrightarrow \infty$ as $n \longrightarrow \infty$. Let $T_n^{(\delta)} = (1+\delta)/R_{n-k}^{(k)}$. Then, for any $0 < \delta < 1$, we have as $n \longrightarrow +\infty$,*

$$\begin{aligned}
&f_n\left(T_n^{(\delta)}\right) \xrightarrow{\text{a.s}} f(\delta) := \log\left((1+\delta)e^{\gamma} - \delta\right), \\
&q_n\left(T_n^{(\delta)}\right) \xrightarrow{\text{a.s}} q(\delta) := \left(1 + \frac{1}{(1+\delta)(e^{\gamma}-1)}\right)\log\left((1+\delta)e^{\gamma} - \delta\right), \\
&g_n\left(T_n^{(\delta)}\right) \xrightarrow{\text{a.s}} g(\delta) := \int_0^1 \frac{\mathrm{d}s}{(1+\delta)e^{(1-s)\gamma} - \delta}, \\
&h_n\left(T_n^{(\delta)}\right) \xrightarrow{\text{a.s}} h(\delta) := q(\delta)g(\delta) - 1, \\
&f_n\left(T_n^{(-\delta)}\right) \xrightarrow{\text{a.s}} f(-\delta), \\
&q_n\left(T_n^{(-\delta)}\right) \xrightarrow{\text{a.s}} q(-\delta), \\
&g_n\left(T_n^{(-\delta)}\right) \xrightarrow{\text{a.s}} g(-\delta), \\
&h_n\left(T_n^{(\delta)}\right) \xrightarrow{\text{a.s}} h(-\delta).
\end{aligned} \tag{32}$$

In addition, if δ is close to 0 and n is large enough, we have

$$\begin{aligned}
h_n\left(T_n^{(\delta)}\right) &> 0 \text{ a.s.}, \\
h_n\left(T_n^{(-\delta)}\right) &< 0 \text{ a.s.}.
\end{aligned} \tag{33}$$

Proof. Write $f_n(T_n^{(\delta)}) = \log(1 + (1+\delta)((U(e^{S_n/k})/U(e^{S_{n-k}/k})) - 1))$, and note that when (1) is satisfied for $\gamma > 0$, it is well known that U is regularly varying at infinity with index γ so that $\lim_{t\longrightarrow\infty}(U(tx)/U(t)) = x^{\gamma}$ locally uniformly in $x > 0$. Next, from Lemma 2, $(S_n - S_{n-k})/k \xrightarrow{\text{a.s}} 1$ as $n \longrightarrow +\infty$; it follows readily that $f_n(T_n^{(\delta)}) \xrightarrow{\text{a.s}} f(\delta)$ as $n \longrightarrow +\infty$, and so, $q_n(T_n^{(\delta)}) \xrightarrow{\text{a.s}} q(\delta)$, $f_n(T_n^{(-\delta)}) \xrightarrow{\text{a.s}} f(-\delta)$, and $q_n(T_n^{(-\delta)}) \xrightarrow{\text{a.s}} q(-\delta)$ as $n \longrightarrow +\infty$.

Similarly, we observe that

$$g_n\left(T_n^{(\delta)}\right) = \int_0^1 \frac{1}{(1+\delta)U\left(e^{S_{n-[sk]}/k}\right)/U\left(e^{S_{n-k}/k}\right) - \delta}\,\mathrm{d}s. \tag{34}$$

From Lemma 2, $U(e^{S_{n-[sk]}/k})/U(e^{S_{n-k}/k}) \xrightarrow{\text{a.s}} e^{(1-s)\gamma}$ for all $s \in [0,1]$, and by the fact that U is an increasing function, we have, for all $0 < \delta < 1$ and $s \in [0,1]$,

$$\frac{1}{(1+\delta)U\left(e^{S_{n-[sk]}/k}\right)/U\left(e^{S_{n-k}/k}\right) - \delta} \le 1 \text{ a.s.}. \tag{35}$$

Hence, the dominated convergence theorem ensures that

$$g_n\left(T_n^{(\delta)}\right) \xrightarrow{\text{a.s}} g(\delta), \quad n \longrightarrow \infty. \tag{36}$$

By the same arguments, we have $g_n(T_n^{(-\delta)}) \xrightarrow{\text{a.s}} g(-\delta)$ as $n \longrightarrow \infty$.

Next, again by using the dominated convergence theorem and after straightforward calculations, we obtain

$$\begin{aligned}
q(0^+) &= \frac{\gamma}{1-e^{-\gamma}}, \\
q'(0^+) &= \frac{e^{\gamma} - \gamma - 1}{e^{\gamma} - 1}, \\
g(0^+) &= \frac{1-e^{-\gamma}}{\gamma}, \\
g'(0^+) &= -\frac{(1-e^{-\gamma})^2}{2\gamma}, \\
h(0^+) &= 0, \\
h'(0^+) &= \frac{2 - \gamma - (2+\gamma)e^{-\gamma}}{2\gamma}.
\end{aligned} \tag{37}$$

Put $P(\gamma) = 2 - \gamma - (2+\gamma)e^{-\gamma}$. Since, for all $\gamma > 0$, $1 + \gamma < e^{\gamma}$, then $P'(\gamma) = (1 + \gamma - e^{\gamma})/e^{\gamma} < 0$. Hence, for all

$\gamma > 0$, $P(\gamma) < 0$. Thus, $h'(0^+) < 0$. Consequently, there exists $\delta_0 > 0$, for any $0 < \delta < \delta_0$; when n is large enough,

$$h_n\left(T_n^{(\delta)}\right) < h\left(0^+\right) = 0 \text{ a.s.}. \tag{38}$$

The same arguments show that $h_n(T_n^{(-\delta)}) > 0$ a.s.. □

Remark 3. This lemma can be proved for $\delta = 0$. Indeed, Potter's inequality (see Proposition 0.8.(ii) in [17]) implies that, for any $\epsilon > 0$, there exists t_0 such that, for $t \geq t_0$ and $x \geq 1$, $(1-\epsilon)x^{\gamma-\epsilon} < U(tx)/U(t) < (1+\epsilon)x^{\gamma+\epsilon}$. Then, for all $\epsilon > 0$ small enough,

$$\begin{aligned} g_n\left(T_n^{(0)}\right) - \frac{(1+\epsilon)(1-e^{-\epsilon-\gamma})}{\epsilon+\gamma} &= \int_0^1 \frac{U\left(e^{S_{n-k}/k}\right)}{U\left(e^{S_{n-[sk]}/k}\right)} - (1+\epsilon)e^{-(\epsilon+\gamma)(1-s)}\mathrm{d}s \\ &\leq (1+\epsilon)(\epsilon+\gamma) \sup_{0\leq s\leq 1}\left|\frac{S_{n-[sk]} - S_{n-k}}{k} - (1-s)\right|. \end{aligned} \tag{39}$$

By Lemma 2, $\limsup_{n\longrightarrow\infty} g_n(T_n^{(0)}) \leq ((1+\varepsilon)(1-e^{-\varepsilon-\gamma}))/(\varepsilon+\gamma)$ a.s., which leads, when $\epsilon \longrightarrow 0$, to $\limsup_{n\longrightarrow\infty} g_n(T_n^{(0)}) \leq (1-e^{-\gamma})/\gamma$ a.s.. Similarly, $\liminf_{n\longrightarrow\infty} g_n(T_n^{(0)}) \geq (1-e^{-\gamma})/\gamma$ a.s..

Lemma 4. *Suppose (1) holds for $\gamma < 0$ and $k \longrightarrow \infty, k/n \longrightarrow 0, k/\log n \longrightarrow \infty$ as $n \longrightarrow \infty$. Let $T_n^{(\delta)} = -((1+\delta)/(U(\infty) - R_{n-k}^{(k)}))$. Then, for any $-1 < \delta < 0$, we have as $n \longrightarrow +\infty$,*

$$\begin{aligned} &f_n\left(T_n^{(\delta)}\right) \xrightarrow{\text{a.s}} f(\delta) := \log\left((1+\delta)e^{\gamma} - \delta\right), \\ &q_n\left(T_n^{(\delta)}\right) \xrightarrow{\text{a.s}} q(\delta) := \left(1 + \frac{1}{(1+\delta)(e^{\gamma}-1)}\right)\log\left((1+\delta)e^{\gamma} - \delta\right), \\ &g_n\left(T_n^{(\delta)}\right) \xrightarrow{\text{a.s}} g(\delta) := \int_0^1 \frac{\mathrm{d}s}{(1+\delta)e^{(1-s)\gamma} - \delta}, \\ &h_n\left(T_n^{(\delta)}\right) \xrightarrow{\text{a.s}} h(\delta) := q(\delta)g(\delta) - 1, \\ &q_n\left(T_n^{(-\delta)}\right) \xrightarrow{\text{a.s}} q(-\delta), \\ &g_n\left(T_n^{(-\delta)}\right) \xrightarrow{\text{a.s}} g(-\delta), \\ &h_n\left(T_n^{(\delta)}\right) \xrightarrow{\text{a.s}} h(-\delta). \end{aligned} \tag{40}$$

In addition, if δ is close to 0 and n is large enough, we have

$$\begin{aligned} h_n\left(T_n^{(\delta)}\right) &< 0 \text{ a.s.}, \\ h_n\left(T_n^{(-\delta)}\right) &> 0 \text{ a.s.}. \end{aligned} \tag{41}$$

Proof. This proof is similar to the previous proof with straightforward modifications. When (1) is satisfied for $\gamma < 0$, it is well known that $U(\infty) < \infty$ and $V(x) := U(\infty) - U(x)$ is regularly varying at infinity with index γ. Write

$$q_n\left(T_n^{(\delta)}\right) = \left(1 - \frac{1}{(1+\delta)\left(1 - \left(V\left(e^{S_n/k}\right)/V\left(e^{S_{n-k}/k}\right)\right)\right)}\right)\log\left(1 - (1+\delta)\left(1 - \frac{V\left(e^{S_n/k}\right)}{V\left(e^{S_{n-k}/k}\right)}\right)\right). \tag{42}$$

Again from Lemma 2, $(S_n - S_{n-k})/k \xrightarrow{\text{a.s}} 1$; it follows readily that $q_n(T_n^{(\delta)}) \xrightarrow{\text{a.s}} q(\delta)$, and so, $q_n(T_n^{(-\delta)}) \xrightarrow{\text{a.s}} q(-\delta)$.

Similarly, we write

$$g_n\left(T_n^{(\delta)}\right) = \int_0^1 \frac{1}{(1+\delta)\left(1 - V\left(e^{S_{n-[sk]}/k}\right)/V\left(e^{S_{n-k}/k}\right)\right) - \delta}\mathrm{d}s. \tag{43}$$

Since, from Lemma 2, $V(e^{S_{n-[sk]}/k})/V(e^{S_{n-k}/k}) \xrightarrow{\text{a.s}} e^{(1-s)\gamma}$ for all $s \in [0,1]$ and observe that, for all $-1<\delta<0$ and $s \in a[0,1]$,

$$\frac{1}{(1+\delta)\left(1-V\left(e^{S_{n-[sk]}/k}\right)/V\left(e^{S_{n-k}/k}\right)\right)-\delta} \leq -\frac{1}{\delta} \text{ a.s.}, \tag{44}$$

it implies by the dominated convergence theorem that

$$g_n\left(T_n^{(\delta)}\right) \longrightarrow^{\text{a.s}} g(\delta), \quad n \longrightarrow \infty. \tag{45}$$

By the same arguments, we have $g_n(T_n^{(-\delta)}) \xrightarrow{\text{a.s}} g(-\delta)$ as $n \longrightarrow \infty$.

Next, again by using the dominated convergence theorem and after straightforward calculations, we have

$$\begin{aligned} q(0^-) &= \frac{\gamma}{1-e^{-\gamma}}, \\ q'(0^-) &= \frac{e^{\gamma}-\gamma-1}{e^{\gamma}-1}, \\ g(0^-) &= \frac{1-e^{-\gamma}}{\gamma}, \\ g'(0^-) &= -\frac{(1-e^{-\gamma})^2}{2\gamma}, \\ h(0^-) &= 0, \\ h'(0^-) &= \frac{2-\gamma-(2+\gamma)e^{-\gamma}}{2\gamma}. \end{aligned} \tag{46}$$

Put $P(\gamma) = 2-\gamma-(2+\gamma)e^{-\gamma}$. Since, for all $\gamma<0$, $1+\gamma>e^{\gamma}$, then $P'(\gamma) = (1+\gamma-e^{\gamma})/e^{\gamma} > 0$. Hence, for all $\gamma<0$, $P(\gamma)<0$. Thus, $h'(0^-)>0$. Consequently, there exists $\delta_0<0$, for any $\delta_0<\delta<0$; when n is large enough,

$$h_n\left(T_n^{(\delta)}\right) < h(0^-) = 0 \text{ a.s.}. \tag{47}$$

The same arguments show that $h_n(T_n^{(-\delta)})>0$ a.s.. □

Lemma 5. *Suppose (1) holds for $\gamma=0$ and $k \longrightarrow \infty, k/n \longrightarrow 0, k/\log n \longrightarrow \infty$ as $n \longrightarrow \infty$. Let $T_n^{(\delta)} = \delta/(R_n^{(k)} - R_{n-k}^{(k)})$. Then, for any $0<\delta<1/2$ and as $n \longrightarrow +\infty$,*

$$\begin{aligned} g_n\left(T_n^{(\delta)}\right) &\xrightarrow{\text{a.s}} g(\delta) := \frac{\log(1+\delta)}{\delta}, \\ g_n\left(T_n^{(-\delta)}\right) &\xrightarrow{\text{a.s}} g(-\delta) := -\frac{\log(1-\delta)}{\delta}, \\ h_n\left(T_n^{(\delta)}\right) &\xrightarrow{\text{a.s}} h(\delta) := (1+\delta)\left(\frac{\log(1+\delta)}{\delta}\right)^2 - 1, \\ h_n\left(T_n^{(-\delta)}\right) &\xrightarrow{\text{a.s}} h(-\delta) := (1-\delta)\left(\frac{\log(1-\delta)}{\delta}\right)^2 - 1. \end{aligned} \tag{48}$$

Furthermore, for sufficiently large n, we have

$$\begin{aligned} h_n\left(T_n^{(\delta)}\right) &< 0 \text{ a.s.}, \\ h_n\left(T_n^{(-\delta)}\right) &< 0 \text{ a.s.}. \end{aligned} \tag{49}$$

Proof. For arbitrary $0<\delta<1/2$, let

$$\begin{aligned} T_n^{(\delta)} &= \frac{\delta}{R_n^{(k)} - R_{n-k}^{(k)}}, \\ T_n^{(-\delta)} &= \frac{-\delta}{R_n^{(k)} - R_{n-k}^{(k)}}. \end{aligned} \tag{50}$$

First, we have

$$\begin{aligned} q_n\left(T_n^{(\delta)}\right) &= \frac{(1+\delta)\log(1+\delta)}{\delta}, \\ q_n\left(T_n^{(-\delta)}\right) &= -\frac{(1-\delta)\log(1-\delta)}{\delta}, \\ g_n\left(T_n^{(\delta)}\right) &= \int_0^1 \frac{\mathrm{d}s}{1+\delta\left(\left(R_{n-[sk]}^{(k)} - R_{n-k}^{(k)}\right)/\left(R_n^{(k)} - R_{n-k}^{(k)}\right)\right)}, \\ g_n\left(T_n^{(\delta)}\right) &= \int_0^1 \frac{\mathrm{d}s}{1-\delta\left(\left(R_{n-[sk]}^{(k)} - R_{n-k}^{(k)}\right)/\left(R_n^{(k)} - R_{n-k}^{(k)}\right)\right)}. \end{aligned} \tag{51}$$

Suppose now (2) holds for $\gamma=0$, i.e., $\forall x>0$, and

$$\lim_{t \longrightarrow \infty} \frac{U(tx)-U(t)}{a(t)} = \log x. \tag{52}$$

Since U is monotone, this limit holds locally uniformly in $x>0$.

Next, observe that

$$\frac{R_{n-[sk]}^{(k)} - R_{n-k}^{(k)}}{R_n^{(k)} - R_{n-k}^{(k)}} = \frac{\left(U\left(e^{S_{n-[sk]}/k}\right) - U\left(e^{S_{n-k}/k}\right)\right)/a\left(e^{S_{n-k}/k}\right)}{\left(U\left(e^{S_n/k}\right) - U\left(e^{S_{n-k}/k}\right)\right)/a\left(e^{S_{n-k}/k}\right)}. \tag{53}$$

By using Lemma 2, it follows readily that, for all $s \in [0,1]$,

$$\frac{R_{n-[sh]}^{(k)} - R_{n-k}^{(k)}}{R_n^{(k)} - R_{n-k}^{(k)}} \xrightarrow{\text{a.s}} 1-s, \quad n \longrightarrow \infty. \tag{54}$$

Since, for any $0<\delta<1/2$,

$$0 < \frac{1}{1+\delta\left(\left(R_{n-[sk]}^{(k)} - R_{n-k}^{(k)}\right)/\left(R_n^{(k)} - R_{n-k}^{(k)}\right)\right)} < 1 \text{ a.s.}, \tag{55}$$

it follows by using the dominated convergence theorem that, as $n \longrightarrow \infty$,

$$g_n\left(T_n^{(\delta)}\right) \xrightarrow{\text{a.s}} \int_0^1 \frac{\mathrm{d}s}{1+\delta(1-s)} = \frac{\log(1+\delta)}{\delta}. \tag{56}$$

Similarly, for any $0<\delta<1/2$ and all $s \in [0,1]$,

$$0 < \frac{1}{1-\delta\left(\left(R_{n-[sk]}^{(k)} - R_{n-k}^{(k)}\right)/\left(R_n^{(k)} - R_{n-k}^{(k)}\right)\right)} < 2, \text{ a.s.}, \tag{57}$$

and so, as $n \longrightarrow \infty$,

$$g_n\left(T_n^{(-\delta)}\right) \xrightarrow{\text{a.s}} \int_0^1 \frac{\mathrm{d}s}{1-\delta(1-s)} = -\frac{\log(1-\delta)}{\delta}. \tag{58}$$

Hence, as $n \longrightarrow \infty$,

$$\begin{aligned} h_n\left(T_n^{(\delta)}\right) &\xrightarrow{\text{a.s}} h(\delta) = (1+\delta)\left(\frac{\log(1+\delta)}{\delta}\right)^2 - 1, \\ h_n\left(T_n^{(-\delta)}\right) &\xrightarrow{\text{a.s}} h(-\delta) = (1-\delta)\left(\frac{\ln(1-\delta)}{\delta}\right)^2 - 1. \end{aligned} \tag{59}$$

Note that, for $\varphi(x) = x/(\sqrt{x+1}) - \log(x+1)$, $\varphi\prime(x) = x^2 + 2x + 2/2(x+1)\sqrt{x+1} > 0$ for all $x > -1$, which implies that, for all $x > -1$ and $x \neq 0$, $(1+x)(\log(1+x)/x)^2 < 1$. Thus, $h(\delta) < 0$ and $h(-\delta) < 0$.

Consequently, for sufficiently large n, we have almost surely

$$\begin{aligned} h_n\left(T_n^{(\delta)}\right) &< 0, \\ h_n\left(T_n^{(-\delta)}\right) &< 0. \end{aligned} \tag{60}$$

□

Proof of Theorem 1. Here, we present the proof only for $\gamma > 0$. For $\gamma < 0$, the proof is essentially the same.

By choosing a suitable positive sequence $\delta_n \longrightarrow 0$ as $n \longrightarrow \infty$, there exists, from Lemma 3, a random integer $N > 1$ such that, for any $n \geq N$, $h_n(T_n^{(\delta_n)}) < 0$ a.s.. and $h_n(T_n^{(-\delta_n)}) > 0$ a.s.. This ensures, by the mean value theorem, the existence of a random variable $T_n^* \in [T_n^{(-\delta_n)}, T_n^{(\delta_n)}]$ a.s. such that $h_n(T_n^*) = 0$ a.s. when $n \geq N$.

Since f_n is an increasing function, we have almost surely

$$f_n\left(T_n^{(-\delta_n)}\right) \leq \widehat{\gamma}_n = f_n(T_n^*) \leq f_n\left(T_n^{(\delta_n)}\right). \tag{61}$$

From Lemma 3, $f_n(T_n^{(\delta_n)}) \xrightarrow{\text{a.s}} f(0) = \gamma$ and $f_n(T_n^{(-\delta_n)}) \xrightarrow{\text{a.s}} f(0) = \gamma$; this implies that $\widehat{\gamma}_n \xrightarrow{\text{a.s}} \gamma$, i.e., $\widehat{\gamma}_n$ is strongly consistent.

To prove the almost sure convergence of $\widehat{\sigma}_n$, we use the fact that, as $t \longrightarrow \infty$, $a(t) \sim \gamma U(t)$ (see Lemma 1.2.9, p. 22 in [1]).

So, as $n \longrightarrow \infty$,

$$\frac{\widehat{\sigma}_n}{a\left(e^{S_{n-k}/k}\right)} = \frac{f_n(T_n^*)}{T_n^* a\left(e^{S_{n-k}/k}\right)} \sim \frac{f_n(T_n^*)}{\gamma T_n^* U\left(e^{S_{n-k}/k}\right)} \text{ a.s..} \tag{62}$$

Since, for sufficiently large n, $T_n^{(-\delta_n)} \leq T_n^* \leq T_n^{(\delta_n)}$ a.s., we have eventually

$$1 - \delta_n \leq T_n^* U\left(e^{S_{n-k}/k}\right) \leq 1 + \delta_n \text{ a.s.,} \tag{63}$$

which leads to $T_n^* U(e^{S_{n-k}/k}) \xrightarrow{\text{a.s}} 1$ as $n \longrightarrow \infty$. Hence, as $n \longrightarrow \infty$,

$$\frac{\widehat{\sigma}_n}{a\left(e^{S_{n-k}/k}\right)} \overset{\text{a.s}}{\sim} 1. \tag{64}$$

By applying the law of the iterated logarithm, we have almost surely

$$\frac{S_{n-k}}{k} - \frac{(n-k)}{k} = O\left(\frac{(n\log\log n)^{1/2}}{k}\right), \quad n \longrightarrow \infty. \tag{65}$$

If $(n\log\log n)^{1/2}/k \longrightarrow 0$ as $n \longrightarrow \infty$, then $S_{n-k}/k - (n-k)/k \xrightarrow{\text{a.s}} 0$. Combining this with the fact that the function a is regularly varying at infinity with index γ, the consistency of $\widehat{\sigma}_n$ is proved for the positive case. □

Proof of Theorem 2. First, we choose a suitable positive sequence $\delta_n \longrightarrow 0$ as $n \longrightarrow \infty$. It follows from Lemma 4 that there exists a random integer $N > 1$ such that, for any $n \geq N$, $h_n(T_n^{(\delta_n)}) < 0$ a.s.. and $h_n(T_n^{(-\delta_n)}) < 0$ a.s.. Since, after straightforward calculations, we have almost surely

$$\begin{aligned} h_n(0) &= 0, \\ h_n'(0) &= \frac{R_n^{(k)} - R_{n-k}^{(k)}}{2} - \frac{1}{k}\sum_{i=1}^{k}\left(R_{n-i+1}^{(k)} - R_{n-k}^{(k)}\right) \neq 0. \end{aligned} \tag{66}$$

This ensures that when n is large enough, $h_n(t)$ changes the sign in the neighborhood of 0. Combining this with the fact that $h_n(T_n^{(\delta)})$ and $h_n(T_n^{(-\delta)})$ have the same sign, it is proved that almost surely, for sufficiently large n, there exists a nonzero root T_n^* of $h_n(t) = 0$ on $(T_n^{(-\delta)}, T_n^{(\delta)})$.

Recall that f_n is an increasing function. This implies almost surely

$$\log(1-\delta_n) = f_n\left(T_n^{(-\delta_n)}\right) \leq \widehat{\gamma}_n = f_n(T_n^*) \leq f_n\left(T_n^{(\delta_n)}\right) = \log(1-\delta_n). \tag{67}$$

Since $\delta_n \longrightarrow 0$ as $n \longrightarrow \infty$, $\widehat{\gamma}_n \xrightarrow{\text{a.s}} 0$, and the consistency is proved.

Now, we prove the almost sure convergence of $\widehat{\sigma}_n$. For this, we write

$$\frac{\widehat{\sigma}_n}{a\left(e^{S_{n-k}/k}\right)} = \frac{f_n(T_n^*)}{T_n^* a\left(e^{S_{n-k}/k}\right)} = \frac{\log(1+T_n^* Y_1)}{T_n^* Y_1}\frac{Y_1}{a\left(e^{S_{n-k}/k}\right)} \text{ a.s..} \tag{68}$$

Since, for sufficiently large n, $T_n^{(-\delta_n)} \leq T_n^* \leq T_n^{(\delta_n)}$ a.s., we have eventually

$$-\delta_n \leq T_n^* Y_1 \leq \delta_n \text{ a.s.,} \tag{69}$$

which leads to $T_n^* Y_1 \xrightarrow{\text{a.s}} 0$ as $n \longrightarrow \infty$. Hence, as $n \longrightarrow \infty$,

$$\frac{\log(1+T_n^* Y_1)}{T_n^* Y_1} \xrightarrow{\text{a.s}} 1. \tag{70}$$

Under (2), Lemma 2 ensures that

$$\frac{Y_1}{a\left(e^{S_{n-k}/k}\right)} = \frac{U\left(e^{S_n/k}\right) - U\left(e^{S_{n-k}/k}\right)}{a\left(e^{S_{n-k}/k}\right)} \xrightarrow{\text{a.s}} 1, \quad n \longrightarrow \infty. \tag{71}$$

Therefore,

$$\frac{\widehat{\sigma}_n}{a\left(e^{S_{n-k}/k}\right)} \xrightarrow{\text{a.s}} 1, \quad n \longrightarrow \infty. \tag{72}$$

Finally, if $(n\log\log n)^{1/2}/k \longrightarrow 0$ as $n \longrightarrow \infty$, we have, by applying the law of the iterated logarithm, $S_{n-k}/k - (n-k)/k \xrightarrow{\text{a.s}} 0$. Combining this with the fact that the function $a(t)$ is slowly varying at infinity, i.e., for all $x > 0$, $\lim_{t\longrightarrow\infty} a(tx)/a(t) = 1$ (see Lemma 1.2.9, p. 22 in [1]), the consistency of $\widehat{\sigma}_n$ is then proved for $\gamma = 0$. □

Proof of Lemma 1. Recalling the following representation,

$$\left(R_{n-k}^{(k)}, R_{n-k+1}^{(k)}, \ldots, R_n^{(k)}\right) \stackrel{d}{=} \left(H^{\longleftarrow}\left(\frac{S_{n-k}}{k}\right), H^{\longleftarrow}\left(\frac{S_{n-k+1}}{k}\right), \ldots, H^{\longleftarrow}\left(\frac{S_n}{k}\right)\right), \tag{73}$$

where $H(x) = -\log(1 - F(x))$ is the continuous hazard function of the distribution function F, $S_n = e_1 + \cdots + e_n$ and $e_1, e_2, \ldots$, be independent random variables having the standard exponential distribution.

It follows, without loss of generality, that

$$\begin{aligned}&\mathbb{P}\left(R_{n-k+1}^{(k)} > x_1, \ldots, R_n^{(k)} > x_k | R_{n-k}^{(k)} = y\right)\\ &= \mathbb{P}\left(H^{\longleftarrow}\left(\frac{S_{n-k+1}}{k}\right) > x_1, \ldots, H^{\longleftarrow}\left(\frac{S_n}{k}\right) > x_k | H^{\longleftarrow}\left(\frac{S_{n-k}}{k}\right) = y\right).\end{aligned} \tag{74}$$

We know that $H^{\longleftarrow}(t) > x \Longleftrightarrow t > H(x)$, and by continuity, $H^{\longleftarrow}(t) = x \Longrightarrow H(x) = t$. Then,

$$\mathbb{P}\left(R_{n-k+1}^{(k)} > x_1, \ldots, R_n^{(k)} > x_k | R_{n-k}^{(k)} = y\right) = \mathbb{P}\left(\frac{e_{n-k+1}}{k} > H(x_1) - H(y), \ldots, \frac{e_{n-k+1} + \cdots + e_n}{k} > H(x_n) - H(y) | H^{\longleftarrow}\left(\frac{S_{n-k}}{k}\right) = y\right), \tag{75}$$

which gives by independence

$$\begin{aligned}&\mathbb{P}\left(R_{n-k+1}^{(k)} > x_1, \ldots, R_n^{(k)} > x_k | R_{n-k}^{(k)} = y\right)\\ &= \mathbb{P}\left(e_{n-k+1} > k\left(H(x_1) - H(y)\right), \ldots, e_{n-k+1} + \cdots + e_n > k\left(H(x_n) - H(y)\right)\right)\\ &= \mathbb{P}\left(\frac{S_1}{k} > H(x_1) - H(y), \ldots, \frac{S}{k} > H(x_n) - H(y)\right).\end{aligned} \tag{76}$$

Observe that $H(z) - H(y) = -\log((1 - F(z))/(1 - F(y)))$ is the hazard function of the distribution function F_y. This, by using the above representation, proves the desired conclusion. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] L. De Haan and A. Ferreira, *Extreme Value Theory, An Introduction*, Springer, Berlin, Germany, 2006.

[2] B. M. Hill, "A simple general approach to inference about the tail of a distribution," *The Annals of Statistics*, vol. 3, no. 5, pp. 1163–1174, 1975.

[3] J. Pickands III, "Statistical inference using extreme order statistics," *The Annals of Statistics*, vol. 3, no. 1, pp. 119–131, 1975.

[4] A. L. M. Dekkers, J. H. J. Einmahl, and L. D. Haan, "A moment estimator for the index of an extreme-value distribution," *The Annals of Statistics*, vol. 17, no. 4, pp. 1833–1855, 1989.

[5] J. Beirlant, Y. Goegebeur, J. Segers, and J. Teugels, *Statistics of Extremes: Theory and Applications*, John Wiley & Sons, Hoboken, NJ, USA, 2004.

[6] R. L. Smith, "Estimating tails of probability distributions," *The Annals of Statistics*, vol. 15, no. 3, pp. 1174–1207, 1987.

[7] H. Drees, A. Ferreira, and L. de Haan, "On maximum likelihood estimation of the extreme value index," *The Annals of Applied Probability*, vol. 14, no. 3, pp. 1179–1201, 2004.

[8] C. Zhou, "Existence and consistency of the maximum likelihood estimator for the extreme value index," *Journal of Multivariate Analysis*, vol. 100, no. 4, pp. 794–815, 2009.
[9] B. C. Arnold, N. Balakrishnan, and H. N. Nagaraga, *Record*, John Wiley & Sons Inc., Hoboken, NJ, USA, 1998.
[10] M. El Arrouchi, "Characterization of tail distributions based on record values by using the Beurling's Tauberian theorem," *Extremes*, vol. 20, no. 1, pp. 111–120, 2017.
[11] A. M. Berred, "An estimator of the exponent of regular variation based on K-record values," *Statistics*, vol. 32, no. 2, pp. 93–109, 1998.
[12] M. Khaled, M. Dakkon, and M. El Arrouchi, "Prediction interval for future record from Weibull tailed distributions," *International Journal of Applied Mathematics and Statistics*, vol. 54, no. 2, pp. 83–94, 2016.
[13] M. El Arrouchi and A. Imlahi, "Optimal choice of kn-records in the extreme value index estimation," *Statistics & Desicions*, vol. 23, no. 2, pp. 101–115, 2005.
[14] K. N. Chandler, "The distribution and frequency of record values," *Journal of the Royal Statistical Society: Series B (Methodological)*, vol. 14, no. 2, pp. 220–228, 1952.
[15] V. B. Nevzerov, "On the kth record times and their generalizations," *Zapiski Nauchnykh Seminarov LOMI*, vol. 153, pp. 115–121, 1986.
[16] W. Dziubdziela and B. Kopociński, "Limiting properties of the k-th record values," *Applicationes Mathematicae*, vol. 15, no. 2, pp. 187–190, 1976.
[17] S. I. Resnick, *Extreme Values, Regular Variation and Point Processes*, Springer-Verlag, New York, NY, USA, 1987.
[18] J. Komlós, P. Major, and G. Tusnády, "An approximation of partial sums of independent RV's and the sample DF. I," *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, vol. 32, no. 1-2, pp. 111–131, 1975.
[19] J. Komlós, P. Major, and G. Tusnády, "An approximation of partial sums of independent RV's and the sample DF. II," *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, vol. 34, no. 1, pp. 34–58, 1976.
[20] D. L. Hanson and R. P. Russo, "Some results on increments of the wiener process with applications to lag sums of I.I.D. random variables," *The Annals of Probability*, vol. 11, no. 3, pp. 609–623, 1983.