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Research Article

Output Feedback Nonlinear H_∞-Tracking Control of a Nonminimum-Phase 2-DOF Underactuated Mechanical System

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Nonlinear H_∞ synthesis is developed to solve the tracking control problem restricted to a two degrees-of-freedom (DOF) underactuated mechanical manipulator where position measurements are the only available information for feedback. A local H_∞ controller is derived by means of a certain perturbation of the differential Riccati equations, appearing in solving the H_∞ control problem for the linearized system. Stabilizability and detectability properties of the control system are thus ensured by the existence of the proper solutions of the unperturbed differential Riccati equations, and hence the proposed synthesis procedure obviates an extra verification work of these properties. Due to the nature of the approach, the resulting controller additionally yields the desired robustness properties against unknown but bounded external disturbances. The desired trajectory is centered at the upright position where the manipulator becomes a nonminimum-phase system. Simulation results made for a double pendulum show the effectiveness of the proposed controller.

1. Introduction

The focus of this paper is to solve the tracking control problem for a 2-DOF underactuated mechanism via nonlinear H_∞-control for time-varying systems [1] where joint position measurements are the only available information for feedback. Further research applications in the control of underactuated systems have gone in many directions, for example, fully actuated robots where it is required that motion continues in spite of a failure of any of its actuators. Other typical examples are the systems where its desired operation mode is oscillatory such as biped walking robots where a periodic trajectory is required to produce a coordinated motion (see, e.g., [2]); hopping robots where thrust, decompression, flight, and compression phases are also governed by a periodic motion (see, e.g., [3]); tracking control in drive systems with backlash where usually the position sensor is placed on the side of the motor instead of the side of the load (see, e.g., [4] and [5, page 456]); juggling systems [6]; among others.

Objective. In the present paper, we address the output tracking control problem in *nonminimum-phase* underactuated mechanical system. Representative works in this topic include orbital stabilization of underactuated systems by means of reference models as generator of limit cycles (see, e.g., [7–10]). In particular, this paper is devoted to the solution of a periodic balancing problem for a two-link underactuated mechanical manipulator introduced in [7], whose first link is not actuated whereas the second joint is actuated.

Contribution. For nonlinear mechanical systems, tracking control problem is known to be more difficult than stabilization mainly for underactuated systems whose initial conditions are close to an unstable equilibrium point. The central problem in nonminimum-phase underactuated systems, solved here, is the specification and design of output feedback inner-tracking controllers to drive the output (joint position) to a nontrivial reference trajectory in spite of external disturbances.

The prior work on the tracking control of nonminiminum-phase systems includes, among others, the results of Consolini and Tosques [11] and Berkemeier and Fearing [7] who developed an exact tracking control via state-feedback. Wang [12] partially addressed the above problem

by considering the regulation problem in linear systems. A unified treatment of the control of such systems via output feedback can be found in [13]. In the present paper, the nonlinear H_∞ control approach is extended for the time-varying nonlinear nonminimum-phase systems applied to tracking control problems for underactuated mechanical systems.

Methodology. The method we use for defining a desired trajectory for underactuated system is based on the work of Berkemeier and Fearing [7]. The method was successfully applied to derive a set of exact trajectories for the nonlinear equation which involve inverted periodic motion. This method was selected because the desired trajectories are at least twice-differentiable satisfying the smoothness assumption imposed on the system for the development of H_∞ control theory (see, e.g., [14]).

The above problem is locally resolved within the framework of nonlinear H_∞-control methods from [1, 14–16]. Those methods do not admit a straightforward application to the problem in question because in contrast to the standard case, a partial state stabilization (i.e., asymptotic stabilization of the output of the system) is only required provided that the complementary variables remain bounded. Their modification developed in the present paper is of the same level of simplicity, and it follows the common practice of proper solution to corresponding differential Riccati equations which is performed numerically.

The aforementioned H_∞ synthesis took its origins from game-theoretic approach from Basar and Bernhard [15], and the $\mathcal{L}_2$-gain analysis from Isidori and Astolfi [14]. It followed the line of reasoning, used in Orlov et al. [17], where the corresponding Hamilton-Jacobi-Isaacs expressions were required to be negative definite rather than semidefinite.

In contrast to the standard $\mathcal{L}_2$-gain analysis from Isidori and Astolfi [14] and Van der Shaft [18] the resulting H_∞ design procedure imposed the nonstabilizability-detectability conditions on the control systems. Under appropriate assumptions the existence of suitable solutions of Riccati differential equations, appearing in solving the H_∞ control problem for the linearized system, was shown to be necessary and sufficient condition for a local solution of the H_∞ control problem to exist. This mean that the verification of stabilizability and detectability conditions will be not required. A local solution was then derived by means of a certain perturbation of the Riccati equations when these unperturbed equations had bounded positive-semidefinite solutions. Thus, the local stabilizability and detectability properties of the control system were ensured by the existence of the proper solutions of the unperturbed Riccati equations, and hence the H_∞ synthesis obviated any extra work on verification of these properties.

Organization of the Paper. The paper is organized as follows. Background materials on time-varying H_∞-control synthesis are presented in Section 2. The tracking control problem of a 2-DOF underactuated system and its state equations are introduced in Section 3 while desired trajectory synthesis procedure is also discussed. A nonlinear H_∞-output control for time varying systems is also constructed. Performance issues of this controller are illustrated in a simulation study in Section 4. Finally, Section 5 presents conclusions.

2. Background Material on Nonlinear H_∞-Control of Time-Varying Systems

2.1. Basic Assumptions and Problem Statement. Consider a nonlinear system of the form

$$\begin{aligned} \dot{x} &= f(x,t) + g_1(x,t)w + g_2(x,t)u, \\ z &= h_1(x,t) + k_{12}(x,t)u, \\ y &= h_2(x,t) + k_{21}(x,t)w, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state space vector, $t \in \mathbb{R}$ is the time, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^r$ is the unknown disturbance, $z \in \mathbb{R}^l$ is the unknown output to be controlled, and $y \in \mathbb{R}^p$ is the only available measurement on the system. The following assumptions are assumed to hold.

(A1) The functions $f(x,t)$, $g_1(x,t)$, $g_2(x,t)$, $h_1(x,t)$, $h_2(x,t)$, $k_{12}(x,t)$, and $k_{21}(x,t)$ are piecewise continuous in t for all x and locally Lipschitz continuous in x for all t.

(A2) $f(0,t) = 0$, $h_1(0,t)$, and $h_2(0,t) = 0$ for all t.

(A3) $h_1^T(x,t)k_{12}(x,t) = 0$, $k_{12}(x,t)^T k_{12}(x,t) = I$, $k_{21}(x,t)g_1(x,t) = 0$, $k_{21}(x,t)k_{21}^T(x,t) = I$.

These assumptions are made for technical reasons. Assumption (A1) guarantees the well-posedness of the above dynamic system, while being enforced by integrable exogenous inputs. Assumption (A2) ensures that the origin is an equilibrium point of the nondriven ($u = 0$) disturbance-free ($w = 0$) dynamic system (1). Assumption (A3) is a simplifying assumption inherited from the standard H_∞-control problem.

A causal dynamic feedback compensator

$$u = \mathcal{K}(\xi, t), \tag{2}$$

$$\dot{\xi} = \eta(\xi, y, t) \tag{3}$$

with internal state $\xi \in \mathbb{R}^s$, is said to be globally (locally) admissible controller if the closed-loop systems (1)-(2) are globally (uniformly) asymptotically stable when $w = 0$.

Given a real number $\gamma > 0$, it is said that systems (1), (2) have $\mathcal{L}_2$-gain less than γ if the response z, resulting from w for initial state $x(t_0) = 0$, $\xi(t_0) = 0$, satisfies

$$\int_{t_0}^{t_1} \|z(t)\|^2 dt < \gamma^2 \int_{t_0}^{t_1} \|w(t)\|^2 dt \tag{4}$$

for all $t_1 > t_0$ and all piecewise continuous functions $w(t)$.

The time-varying H_∞-control problem is to find a globally admissible controller (2)-(3) such that $\mathcal{L}_2$-gain of the closed-loop systems (1), (2), (3) is less than γ. In turn, a locally admissible controller (2), (3) is said to be

a local solution of the H_∞-control problem if there exists a neighborhood U of the equilibrium such that inequality (4) is satisfied for all $t_1 > t_0$ and all piecewise continuous functions $w(t)$ for which the state trajectory of the closed-loop system starting from the initial point $(x(t_0), \xi(t_0)) = (0,0)$ remains in U for all $t \in [t_0, t_1]$.

2.2. Local State-Space Solution. Assumptions (A1)–(A3) allow one to linearize the corresponding Hamilton-Jacobi-Isaacs inequalities from [1] that arise in the state feedback and output-injection design thereby yielding a local solution of the time-varying H_∞-control problem. The subsequent local analysis involves the linear time-varying H_∞-control problem for the system

$$\begin{aligned} \dot{x} &= A(t)x + B_1(t)w + B_2(t)u, \\ z &= C_1(t)x + D_{12}(t)u, \\ y &= C_2(t)x + D_{21}(t)w, \end{aligned} \tag{5}$$

where

$$\begin{aligned} A(t) &= \frac{\partial f}{\partial x}(0,t), \qquad B_1(t) = g_1(0,t), \qquad B_2(t) = g_2(0,t), \\ C_1(t) &= \frac{\partial h_1}{\partial x}(0,t), \qquad D_{12}(t) = k_{12}(0,t), \\ C_2(t) &= \frac{\partial h_2}{\partial x}(0,t), \qquad D_{21}(t) = k_{21}(0,t). \end{aligned} \tag{6}$$

Such a problem is now well understood if the linear system (5) is stabilizable and detectable from u and y, respectively. Under these assumptions, the following conditions are necessary and sufficient for a solution to exist (see, e.g., [16]).

(C1) There exists a bounded positive semidefinite symmetric solution of the equation

$$\begin{aligned} -\dot{P}(t) &= P(t)A(t) + A^T(t)P(t) + C_1^T(t)C_1(t) \\ &\quad + P(t)\left[\frac{1}{\gamma^2}B_1(t)B_1^T(t) - B_2(t)B_2^T(t)\right]P(t), \end{aligned} \tag{7}$$

such that the system

$$\dot{x} = \left[A - \left(B_2B_2^T - \gamma^{-2}B_1B_1^T\right)P\right](t)x(t) \tag{8}$$

is exponentially stable. (Throughout, a time-dependent $n\times n$-matrix $P(t)$ is positive semidefinite if and only if $x^TP(t)x \geq 0$ for all n-vectors x and all time instants t whereas $P(t)$ is positive definite if and only if $x^TP(t)x \geq mx^Tx$ for all x and t, and some constant $m > 0$. Respectively, $P(t)$ is bounded if and only if $\|P(t)\| \leq m_0$ for all t and some constant $m_0 > 0$.)

(C2) There exists a bounded positive semidefinite symmetric solution to the equation

$$\begin{aligned} \dot{Z}(t) &= \widetilde{A}(t)Z(t) + Z(t)\widetilde{A}^T(t) + B_1(t)B_1^T(t) + Z(t) \\ &\quad \times \left[\frac{1}{\gamma^2}P(t)B_2(t)B_2^T(t)P(t) - C_2^T(t)C_2(t)\right]Z(t) \end{aligned} \tag{9}$$

specified with $\widetilde{A}(t) = A(t) + \gamma^{-2}B_1(t)B_1^T(t)P(t)$, such that the system

$$\dot{x} = \left[A - Z\left(C_2^TC_2 - \gamma^{-2}PB_2B_2^TP\right)\right](t)x(t) \tag{10}$$

is exponentially stable.

According to the time-varying bounded real lemma [17], conditions (C1) and (C2) ensure that there exists a positive constant ε_0 such that the system of the perturbed differential Riccati equations

$$\begin{aligned} -\dot{P}_\varepsilon &= P_\varepsilon(t)A(t) + A^T(t)P_\varepsilon(t) + C_1^T(t)C_1(t) \\ &\quad + P_\varepsilon(t)\left[\frac{1}{\gamma^2}B_1(t)B_1^T(t) - B_2(t)B_2^T(t)\right]P_\varepsilon(t) + \varepsilon I, \end{aligned} \tag{11}$$

$$\begin{aligned} \dot{Z}_\varepsilon &= \widetilde{A}(t)Z_\varepsilon(t) + Z\widetilde{A}^T(t) + B_1(t)B_1^T(t) + Z_\varepsilon(t) \\ &\quad \times \left[\frac{1}{\gamma^2}P(t)B_2(t)B_2^T(t)P(t) - C_2^T(t)C_2(t)\right]Z_\varepsilon(t) + \varepsilon I, \end{aligned} \tag{12}$$

has a unique positive definite symmetric solution $(P_\varepsilon(t), Z_\varepsilon(t))$ for each $\varepsilon \in (0, \varepsilon_0)$ where $\widetilde{A}(t) = A(t) + \gamma^{-2}B_1(t)B_1^T(t)P_\varepsilon(t)$.

Differential equations (11) and (12) are subsequently utilized to derive a local solution of the nonlinear H_∞-control problem for (1). The following resuls is extracted from [1].

Theorem 1. *Let conditions (C1) and (C2) be satisfied, and let $(P_\varepsilon(t), Z_\varepsilon(t))$ be the corresponding positive solution of* (11), (12) *under some $\varepsilon > 0$. Then the output feedback*

$$\begin{aligned} \dot{\xi} &= f(\xi,t) + \left[\frac{1}{\gamma^2}g_1(\xi,t)g_1^T(\xi,t) - g_2(\xi,t)g_2^T(\xi,t)\right] \\ &\quad \times P_\varepsilon(t)\xi + Z_\varepsilon(t)C_2^T(t)[y - h_2(\xi,t)], \\ u &= -g_2^T(\xi,t)P_\varepsilon(t)\xi \end{aligned} \tag{13}$$

is a local solution of the H_∞-control problem.

In what follows, Theorem 1 is used to design an H_∞ tracking controller for the underactuated system.

3. H_∞-Control of Underactuated System

3.1. Problem Statement. Consider the equation of motion of an underactuated mechanical system given by the Lagrange equation

$$M(q)\ddot{q} + N(q,\dot{q}) = B\tau + w_x(t), \tag{14}$$

where $q = [q_1, q_2] \in \mathbb{R}^2$ is a vector of generalized coordinates where q_1 and q_2 are the unactuated and actuated joints, respectively; $\tau \in \mathbb{R}$ is the vector of applied joint torques; $B = [0,1]^T$ is the input matrix that maps the torque input τ to the joint of coordinates space; $w_x(t) \in \mathbb{R}^2$ is the unknown disturbance vector to account for

destabilizing model discrepancies due to hard-to-model nonlinear phenomena such as friction and backlash, $t \in \mathbb{R}$ is the time; $M(q) \in \mathbb{R}^{2\times 2}$ is the symmetric positive-definite inertia matrix; $N(q,\dot{q}) = [N_1(q,\dot{q}), N_2(q,\dot{q})]^T \in \mathbb{R}^2$ is the vector that contains the Coriolis, centrifugal, and gravity torques. Appendix A presents the dynamic model of the double pendulum.

The *control objective* is to design a nonlinear H_∞ tracking controller that ensures

$$\lim_{t\to\infty} \|q(t) - q_d(t)\| = 0 \tag{15}$$

to be achieved asymptotically, while also attenuates the influence of external disturbances. Here, $q_d(t) \in \mathbb{R}^2$ is a continuously differentiable desired trajectory.

3.2. The Desired Trajectory. We point out that the present formulation is different from typical formulation of output tracking and regulation [1, 19], where the set point or the reference trajectory is a priori given because underactuated systems are not feedback or input-state linearizables due to its complexity. Therefore, special attention is required in the selection of the desired trajectory for the system under study. There are a few procedures to find desired trajectories for underactuated systems in literature [7, 8, 10, 20, 21], and under reasonable hypotheses all of them can be used to obtain a desired trajectory. The methodology from [7] is used here, where a set of exact trajectories is derived for the nonlinear equation of motion which involves inverted periodic motion. To this end, let us consider the desired trajectory which is solution of

$$\frac{d}{dt}\begin{bmatrix} q_d \\ \dot{q}_d \end{bmatrix} = \begin{bmatrix} \dot{q}_d \\ M^{-1}(q_d)\left[B\hat{\tau}(q_d,\dot{q}_d) - N(q_d,\dot{q}_d)\right] \end{bmatrix}, \tag{16}$$

where $q_d(t) \in \mathbb{R}^2$, $\dot{q}_d(t) \in \mathbb{R}^2$ are the desired joint positions and velocities, respectively, and

$$\hat{\tau} = N_2(q_d,\dot{q}_d) - \frac{2M_{22}(q_d) - M_{12}(q_d)}{2M_{12}(q_d) - M_{11}(q_d)} N_1(q_d,\dot{q}_d) \tag{17}$$

is the control input that makes the desired virtual output

$$y_d(t) = 2q_{d_1}(t) + q_{d_2}(t) - \phi \tag{18}$$

remains at zero for all $t \geq 0$ when $y_d(t)$ starts at $y_d(0) = \dot{y}_d(0) = 0, \phi$ is a constant parameter that parameterizes the equilibrium manifold of the pendulum, and the oscillations given by (16)–(18) are around this manifold. Throughout, we confine our research interest in desired oscillations around the upright position of the pendulum which correspond to the more difficult case due that the open-loop system has an unstable zero dynamics. Toward this end, we choose $\phi = \pi$ for all $t \geq 0$ in (18). It was shown in [7] that (16) and (17) generate a set of exact periodic trajectories given by

$$\ddot{q}_{d_1} = \frac{c_4 \sin(q_{d_1}) + c_5 \sin(\phi - q_{d_1})}{c_1 - c_2}, \tag{19}$$

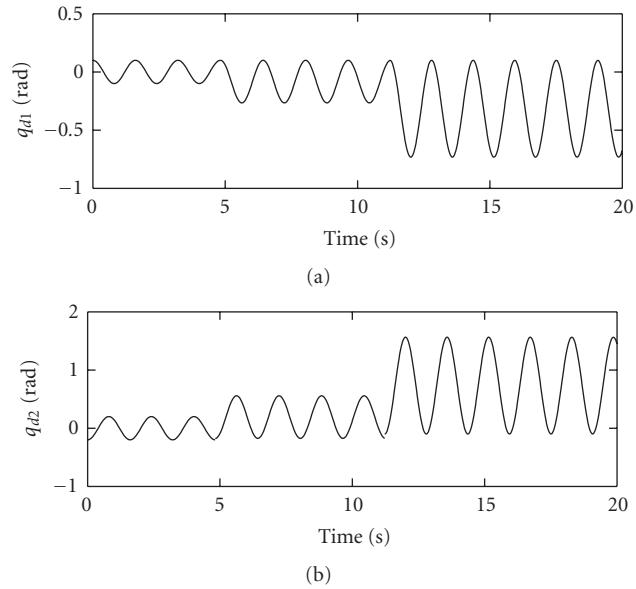


Figure 1: Plot of desired trajectories for Acrobot by selecting several values of ϕ.

that can be interpreted as the zero dynamics of the system (16) with respect to the output $y_d(t)$. Time evolution of the desired trajectory is illustrated in Figure 1 where the value of ϕ is modified along the time:

$$\phi = \begin{cases} 0, & \text{if } t < 5, \\ 0.025, & \text{if } 5 \leq t < 12, \\ 0.1, & \text{if } 12 \leq t \leq 20, \end{cases} \tag{20}$$

where $t \in \mathbb{R}$ is given in seconds. Notice that frequency and amplitude of oscillations change according to variations in ϕ. Figure 2 shows the profile of the frequency and amplitude of oscillations for several values of ϕ.

3.3. The Task. Our objective is to design a controller of the form

$$\tau = \hat{\tau}(q_d,\dot{q}_d) + u \tag{21}$$

with internal state $\xi(t) \in \mathbb{R}^4$, that ensures (15). Thus, the controller to be constructed consists of the trajectory compensator (17) and a disturbance attenuator u given in (2), (3) internally stabilizing the closed-loop system around the desired trajectory. In the sequel, we confine our investigation to the H_∞ tracking problem, where

(1) the output to be controlled is given by

$$z = \rho \begin{bmatrix} 0 \\ 2q_1 + q_2 - \pi \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \tag{22}$$

with a positive weight coefficient ρ;

(2) the joint position vector $q(t) \in \mathbb{R}^2$ is the only available measurement, and this measurement is corrupted by the error vector $w_o(t) \in \mathbb{R}^2$, that is,

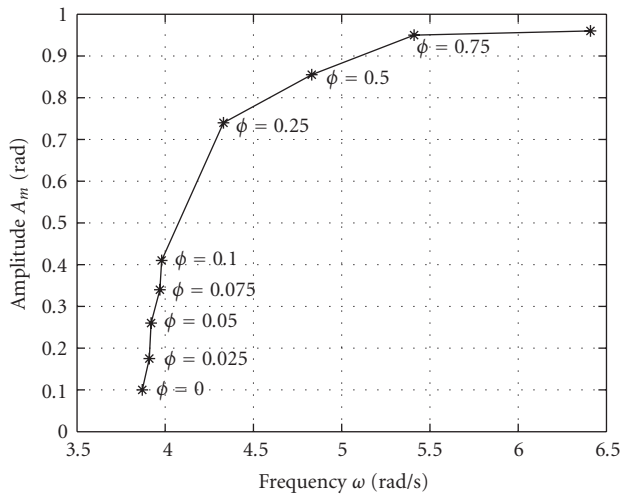


Figure 2: Profile of the frequency and amplitude of oscillations for several values of ϕ.

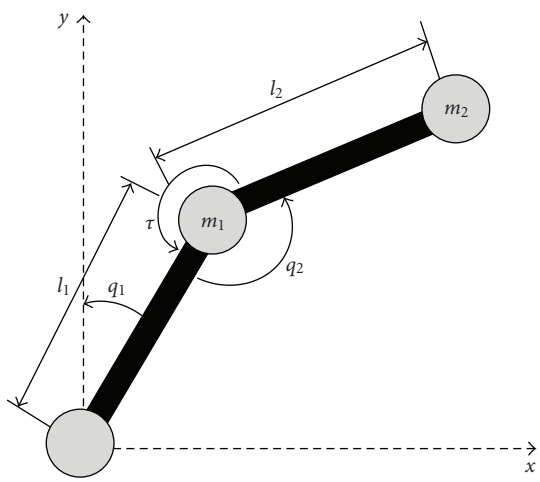
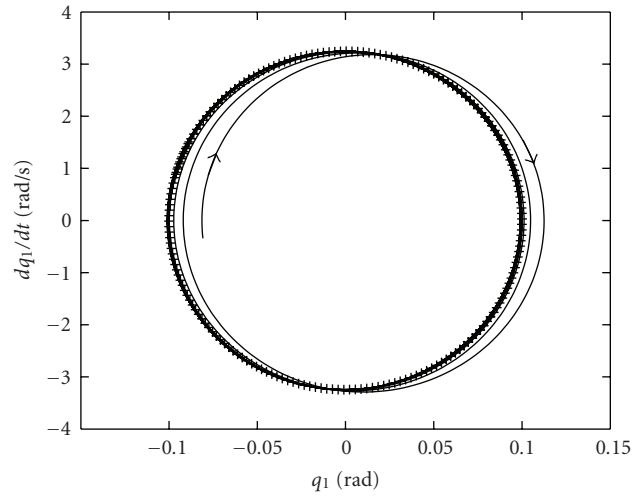
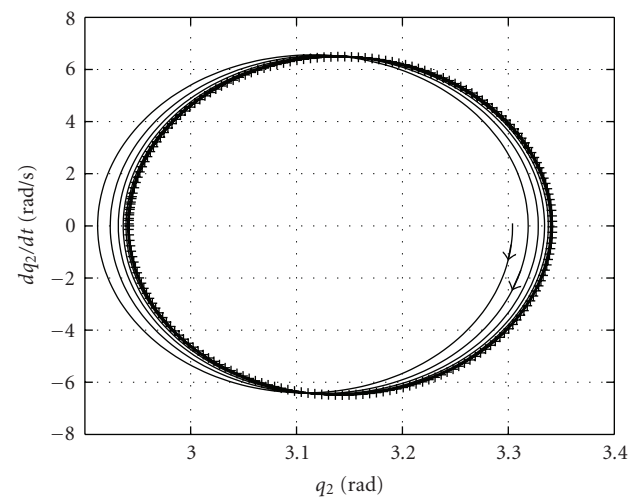


Figure 3: Schematic diagram of the acrobot where l_1 and l_2 denote the length of the nonactuated and actuated links, respectively; and m_1 and m_2 are the masses of each link.



(a)



(b)

Figure 4: Phase portrait of the first joint trajectory and desired trajectory (+) for the unperturbed case (a) and perturbed case (b).

$$y = q(t) + w_o(t). \tag{23}$$

The H_∞ control problem in question is thus stated as follows. Given the system representation (14)–(23), the desired trajectory $q_d(t) \in \mathbb{R}^2$, and a real number $\gamma > 0$, it is required to find (if any) a causal dynamic feedback controller (2), (3) such that the undisturbed closed-loop system is uniformly asymptotically stable around the origin, and its $\mathcal{L}_2$-gain is locally less than γ, that is, inequality (4) is satisfied for all $t_1 > t_0$ and all piecewise continuous functions $w(t) = [w_x(t), w_o(t)]^T$ for which the corresponding state trajectory of the closed-loop system, initialized at the origin, remains in some neighborhood of this point.

3.4. $\mathbf{H_\infty}$ *Synthesis.* To begin with, let us introduce the state deviation vector $x = (x_1, x_2)^T \in \mathbb{R}^4$ where $x_1 = (q_1 - q_{d1}, q_2 - q_{d2})^T$ and $x_2 = (\dot{q}_1 - \dot{q}_{d1}, \dot{q}_2 - \dot{q}_{d2})^T$. After that let us rewrite the system (14), the output to be controlled (22), and the output (23) in terms of the state vector x:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ M^{-1}(x_1 + q_d)N(x_1 + q_d, x_2 + \dot{q}_d) - \ddot{q}_d \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}(x_1 + q_d) \end{bmatrix} w_x + \begin{bmatrix} 0 \\ M^{-1}(x_1 + q_d)B \end{bmatrix} \tau, \tag{24}$$

$$z = \rho \begin{bmatrix} 0 \\ 2x_{1_1} + x_{1_2} + 2q_{d1} + q_{d2} - \pi \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \tag{25}$$

$$y = x_1 + q_d(t) + w_o(t).$$

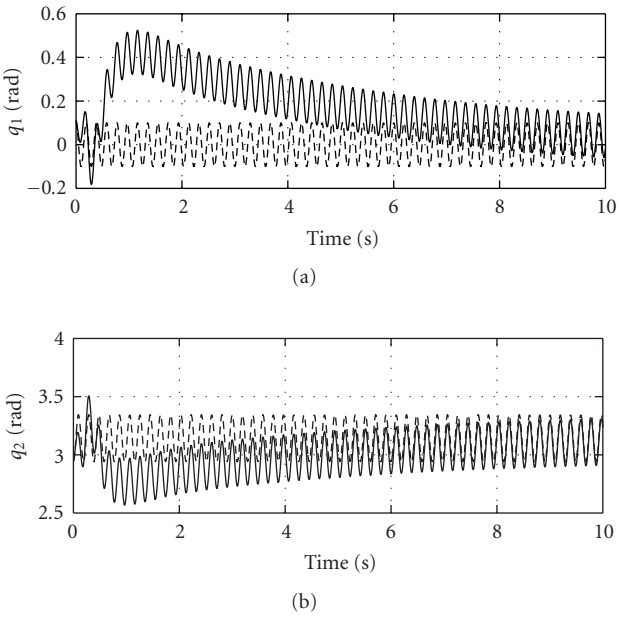


Figure 5: Time evolution of the output (*continuous line*) following the desired trajectory (*dashed line*) under perturbed torques.

Clearly, the above H_∞ tracking control problem is nothing else than a standard nonlinear H_∞ control problem from [1] stated for a time-varying nonlinear system (1) specified as

$$\begin{aligned}
f(x,t) &= \begin{bmatrix} x_2 \\ M^{-1}(x_1+q_d)N(x_1+q_d, x_2+\dot{q}_d) - \ddot{q}_d \end{bmatrix}, \\
g_1(x,t) &= \begin{bmatrix} 0_{2\times 2} & 0_{2\times 2} \\ M^{-1}(x_1+q_d) & 0_{2\times 2} \end{bmatrix}, \\
g_2(x,t) &= \begin{bmatrix} 0_{2\times 1} \\ M^{-1}(x_1+q_d)B \end{bmatrix}, \\
h_1(x,t) &= \rho \begin{bmatrix} 0 \\ 2x_{1_1} + x_{1_2} + 2q_{d1} + q_{d2} - \pi \end{bmatrix}, \\
h_2(x,t) &= x_1 + q_d, \qquad k_{12}(x,t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
k_{21}(x,t) &= \begin{bmatrix} 0_{2\times 2} & I_2 \end{bmatrix}.
\end{aligned} \tag{26}$$

Now by applying Theorem 1 to system (1) thus specified, we derive a local solution of the H_∞ tracking control problem.

Thus, the output feedback controller (13), specified according to (26), locally solves the H_∞ tracking control problem (4)–(24). Stabilizability and detectability properties of the control systems are ensured by the existence of the proper solutions of the unperturbed differential Riccati equations, and hence the corresponding synthesis procedure obviates an extra work (formidable in the nonlinear case) on verification of these properties.

4. Simulation Results

The controller performance was studied in simulation by applying the exposed ideas to the Acrobot, depicted in Figure 3, which is a two-link planar robot with no actuator at the shoulder (link 1) and actuator at the elbow (link 2). In the simulation, performed with MATLAB, the Acrobot was required to move from $[q_1(0), q_2(0)] = [-0.07, 3.3]$ to the desired trajectory $q_d(t) \in \mathbb{R}^2$ and $\phi = \pi$. The initial velocity $\dot{q}(0) \in \mathbb{R}^2$ and the initial compensator state $\xi(0) \in \mathbb{R}^4$ were set to zero for all the simulations. The matrices $M(q)$ and $N(q, \dot{q})$ for the Acrobot are given in Appendix A. We seek for orbital stabilization of the unactuated link q_2 around the equilibrium point $q^* = (0, \pi)$.

The control goal was achieved by implementing the nonlinear H_∞ controller with a weight parameter $\rho = 1$ on the Acrobot. By iterating on γ, we found the infimal achievable level $\gamma^\star \simeq 250$. However, in the subsequent simulations $\gamma = 2000$ was selected to avoid an undesirable high-gain controller design that would appear for a value of γ close to the optimum. With $\gamma = 2000$ we obtained that for $\varepsilon = 0.1$ the corresponding differential Riccati equations (11)-(12) with

$$\begin{aligned}
A(t) &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ A_{31}(t) & A_{32}(t) & A_{33}(t) & A_{34}(t) \\ A_{41}(t) & A_{42}(t) & A_{43}(t) & A_{44}(t) \end{bmatrix}, \\
B_1(t) &= \begin{bmatrix} 0_{2\times 2} & 0_{2\times 2} \\ M^{-1}(q_d) & 0_{2\times 2} \end{bmatrix}, \qquad B_2(t) = \begin{bmatrix} 0_{2\times 1} \\ M^{-1}(q_d) \end{bmatrix}, \\
C_1(t) &= \rho \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}, \qquad C_2(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \\
D_{12}(t) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad D_{22}(t) = \begin{bmatrix} 0_{2\times 2} & I_2 \end{bmatrix}
\end{aligned} \tag{27}$$

have positive-definite solutions. These solutions can be numerically found with MATLAB. Matrix $A(t)$ is given in Appendix B. It should be pointed out that the constant $\phi = \pi$ does not appear in $C_1(t)$ due to straightforward calculation of (6), but it is definitely required in (22) to improve the selection of γ which affects inequality (4) thus avoiding the synthesis of a high-gain controllers. Resulting trajectories is depicted in Figure 4. This figure demonstrates that the H_∞ controller does asymptotically stabilize the system motion around the desired trajectory. In addition, the H_∞ controller was successfully applied to the Acrobot under external disturbances

$$w_{x_i}(t) = b_i \cos(0.1t)\exp(-2t), \quad i = 1, 2, \tag{28}$$

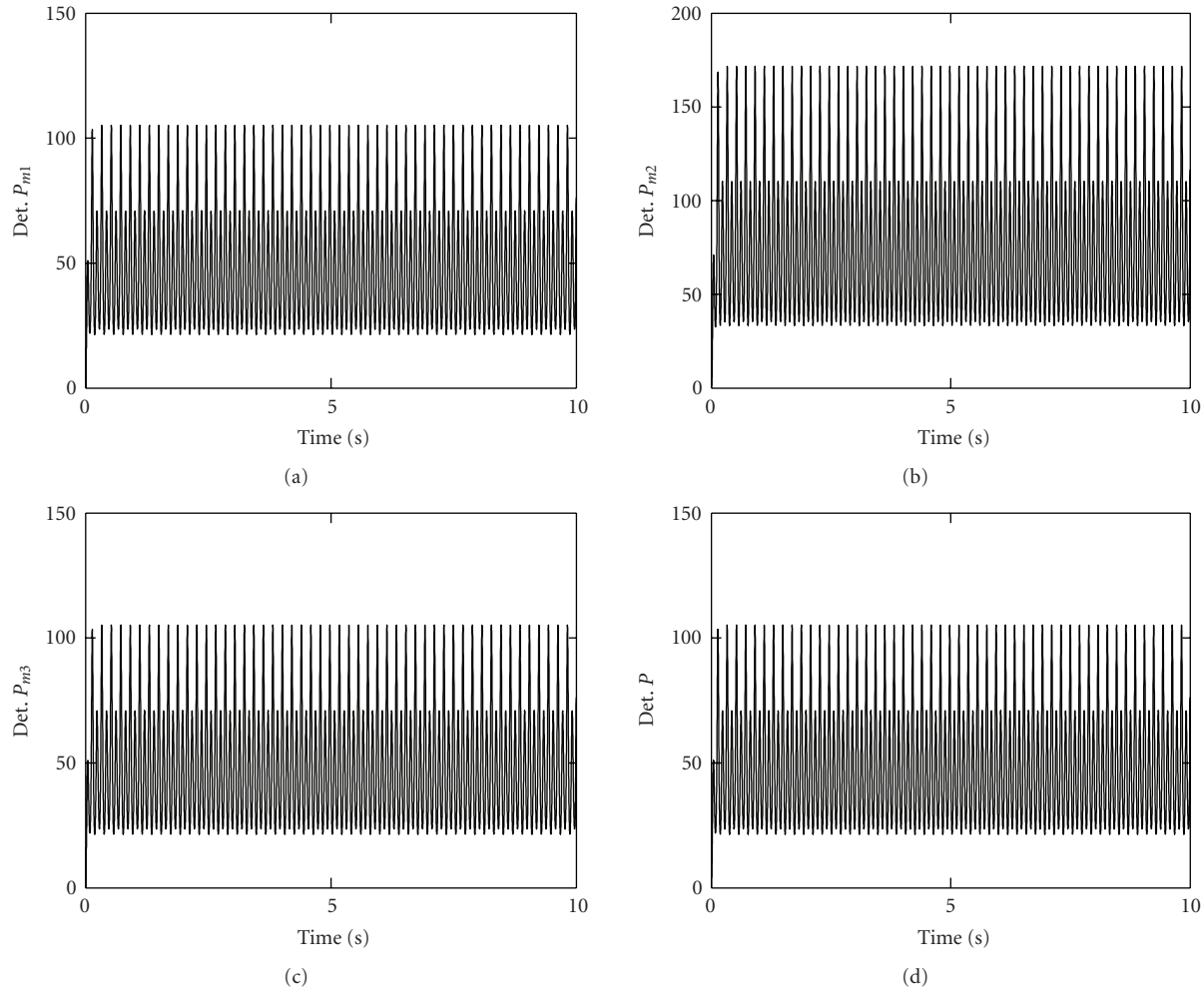


Figure 6: Time evolution of the determinants of the principal minors of the matrix $P \in \mathbb{R}^{4\times 4} P_{m1}, P_{m2}, P_{m3}, P_{m4}$.

where $b_1 = 1 \times 10^{-3}$ [N · m] and $b_2 = 2 \times 10^{-3}$ [N · m] are the disturbance levels at first and second joints, respectively. Resulting trajectories are depicted in Figure 5. Figures 6 and 7 show the time evolution of the determinant of minors of matrices $P(t) \in \mathbb{R}^{4\times 4}$ and $Z(t) \in \mathbb{R}^{4\times 4}$ denoted as

$$\begin{aligned} P_{m1}(t) &= P_{11}(t), \qquad P_{m2}(t) = \begin{pmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}(t) & P_{22}(t) \end{pmatrix}, \\ P_{m3}(t) &= \begin{pmatrix} P_{11}(t) & P_{12}(t) & P_{13}(t) \\ P_{12}(t) & P_{22}(t) & P_{23}(t) \\ P_{31}(t) & P_{32}(t) & P_{33}(t) \end{pmatrix}, \\ Z_{m1}(t) &= Z_{11}(t), \qquad Z_{m2}(t) = \begin{pmatrix} Z_{11}(t) & Z_{12}(t) \\ Z_{12}(t) & Z_{22}(t) \end{pmatrix}, \\ Z_{m3}(t) &= \begin{pmatrix} Z_{11}(t) & Z_{12}(t) & Z_{13}(t) \\ Z_{12}(t) & Z_{22}(t) & Z_{23}(t) \\ Z_{31}(t) & Z_{32}(t) & Z_{33}(t) \end{pmatrix}. \end{aligned} \tag{29}$$

These Figures highlight that matrices $P(t)$ and $Z(t)$, which are solution of (11) and (12), respectively, are bounded and positive definite for all $t \geq 0$.

5. Conclusions

The output feedback Nonlinear H_∞ tracking control problem is locally solved for an underactuaded mechanical system. The desired periodic orbit is centered at the upright position where the open-loop plant becomes a nonminimum-phase system. The developed controller drives the trajectories of the robot into a set of inverted exact desired trajectories governed by its zero dynamics. Simulation studies, made for the Acrobot, showed the effectiveness of the controller. The design of methods to generate reference trajectories evolving more frequencies and amplitudes in the upright position is in progress, and few results have been published for double-pendulums in [20, 22]. In future work there are two extensions of the result of the paper. First, one would like to synthesize the H_∞ control taking into account reference trajectories derived from alternative methods. The other

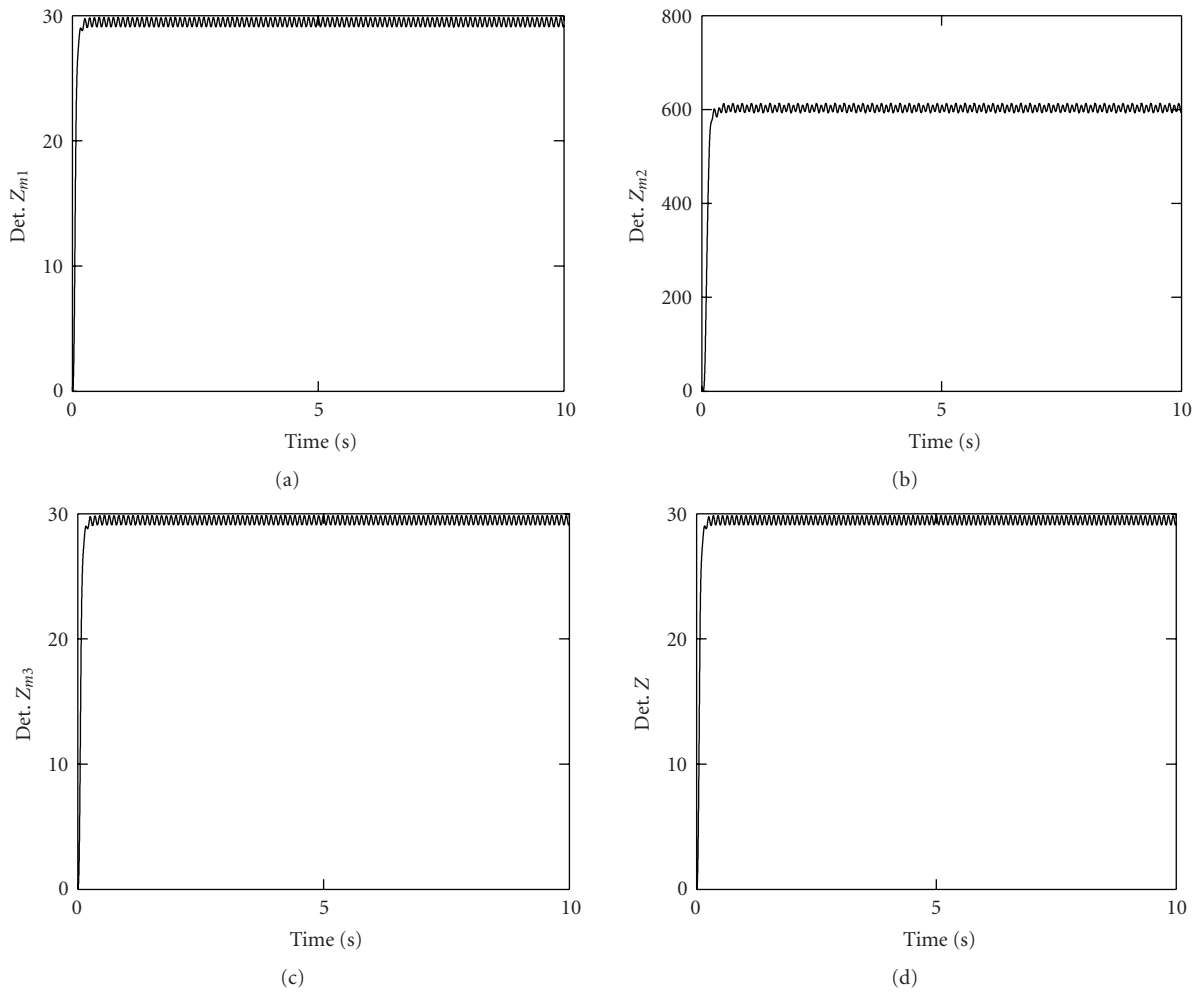


Figure 7: Time evolution of the determinants of the principal minors of the matrix $Z \in \mathbb{R}^{4\times 4} Z_{m1}, Z_{m2}, Z_{m3}, Z_{m4}$.

Table 1: Parameter values for the Acrobot.

c_1	c_2	c_3	c_4	c_5
0.0043	0.0051	0.0034	0.0494	0.038

would also like to extend the result of the paper for the nonsmooth case.

Appendices

A. Dynamic Model of Acrobot

The equation motion of Acrobot, described by (14), was specified by applying the Euler-Lagrange formulation [23] where

$$M(q) = \begin{bmatrix} M_{11}(q) & M_{12}(q) \\ M_{12}(q) & M_{22}(q) \end{bmatrix}, \qquad N(q,\dot{q}) = \begin{bmatrix} N_1(q,\dot{q}) \\ N_2(q,\dot{q}) \end{bmatrix} \tag{A.1}$$

with

$$\begin{aligned} M_{11}(q) &= c_1 + c_2 - 2c_3\cos(q_2), \\ M_{12}(q) &= c_2 - c_3\cos(q_2), \\ M_{22}(q) &= c_2, \\ N_1(q,\dot{q}) &= c_3\sin(q_2)\dot{q}_1\dot{q}_2 + c_3\dot{q}_2(\dot{q}_1+\dot{q}_2)\sin(q_2) \\ &\quad - c_4\sin(q_1) + c_5\sin(q_1+q_2), \\ N_2(q,\dot{q}) &= -c_3\sin(q_2)\dot{q}_1^2 + c_5\sin(q_1+q_2), \end{aligned} \tag{A.2}$$

where the values of c_i $(i = 1,\ldots,5)$, given in Table 1, were taken from the experimental Acrobot provided in [7].

B. Matrix $A(t)$

In this appendix we provide the computed matrix $A(t)$ for the Acrobot which was used in the solution of differential

Riccati equations (11) and (12):

$$A(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ A_{31}(t) & A_{32}(t) & A_{33}(t) & A_{34}(t) \\ A_{41}(t) & A_{42}(t) & A_{43}(t) & A_{44}(t) \end{bmatrix}, \tag{B.3}$$

where

$$\begin{aligned}
A_{31}(t) &= c_2 \Delta M(q_d)^{-1}[c_5 \cos(q_{d1} + q_{d2}) - c_4 \cos(q_{d1})] \\
&\quad - \Delta M(q_d)^{-1}[c_5 \cos(q_{d1} + q_{d2})] M_{12}(q_d), \\
A_{32}(t) &= c_2 \Delta M(q_d)^{-1}[c_3 \cos(q_{d2}) \dot{q}_{d1} \dot{q}_{d2} \\
&\quad + c_3 \dot{q}_{d2}(\dot{q}_{d1} + \dot{q}_{d2}) \cos(q_{d2}) + c_5 \cos(q_{d1} + q_{d2})] \\
&\quad + -2c_2 c_3 \Delta M(q_d)^{-2} \sin(q_{d2}) \cos(q_{d2}) N_1(q_d, \dot{q}_d) \\
&\quad - c_3 \Delta M(q_d)^{-1} N_2(q_d, \dot{q}_d) \sin(q_{d2}) \\
&\quad + 2c_3 \Delta M(q_d)^{-2} \sin(q_{d2}) \cos(q_{d2}) M_{12}(q_d) N_2(q_d, \dot{q}_d) \\
&\quad - \Delta M(q_d)^{-1}[-c_3 \cos(q_{d2}) \dot{q}_{d1} + c_5 \cos(q_{d1} + q_{d2})] \\
&\quad \times M_{12}(q_d), \\
A_{33}(t) &= 2c_2 c_3 \Delta M(q_d)^{-1} \sin(q_{d2}) \dot{q}_{d2} \\
&\quad + 2c_3 \Delta M(q_d)^{-1} M_{12}(q_d) \sin(q_{d2}) \dot{q}_{d1}, \\
A_{34}(t) &= 2c_2 c_3 \Delta M(q_d)^{-1}[\sin(q_{d2}) \dot{q}_{d1} + \sin(q_{d2}) \dot{q}_{d2}], \\
A_{41}(t) &= \Delta M(q_d)^{-1}[c_5 \cos(q_{d1} + q_{d2})] M_{11}(q_d) \\
&\quad - \Delta M(q_d)^{-1}[c_5 \cos(q_{d1} + q_{d2}) - c_4 \cos(q_{d1})] \\
&\quad \times M_{12}(q_d), \\
A_{42}(t) &= 2c_3 \Delta M(q_d)^{-1} N_2(q_d, \dot{q}_d) \sin(q_{d2}) + \Delta M(q_d)^{-1} \\
&\quad \times \left[c_5 \cos(q_{d1} + q_{d2}) - c_3 \cos(q_{d2}) \dot{q}_{d1}^2\right] M_{11}(q_d) \\
&\quad - 2c_3 \Delta M(q_d)^{-2} M_{11}(q_d) N_2(q_d, \dot{q}_d) \sin(q_{d2}) \cos(q_{d2}) \\
&\quad - c_3 \Delta M(q_d)^{-1} N_1(q_d, \dot{q}_d) \sin(q_{d2}) - \Delta M(q_d)^{-1} \\
&\quad \times \Big[2c_3 \cos(q_{d2}) \dot{q}_{d1} \dot{q}_{d2} + c_3 \cos(q_{d2}) \dot{q}_{d2}^2 \\
&\qquad + c_5 \cos(q_{d1} + q_{d2})\Big] M_{12}(q_d) \\
&\quad + 2c_3 \Delta M(q_d)^{-2} M_{12}(q_d) N_1(q_d, \dot{q}_d) \sin(q_{d2}) \cos(q_{d2}), \\
A_{43}(t) &= -2c_3 \Delta M(q_d)^{-1}[M_{11}(q_d) + M_{12}(q_d)] \sin(q_{d2}) \dot{q}_{d1}, \\
A_{44}(t) &= -2c_3 \Delta M(q_d)^{-1}[\dot{q}_{d1} + \dot{q}_{d2}] \sin(q_{d2}) M_{12}(q_d),
\end{aligned} \tag{B.4}$$

where $\Delta M(q_d) = M_{11}(q_d) M_{22}(q_d) - M_{12}^2(q_d)$.

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