Research Article

Stiffness and Elastic Deformation of 4-DoF Parallel Manipulator with Three Asymmetrical Legs for Supporting Helicopter Rotor

Jialin Song,1 Yang Lu,2 Yongli Wang,3 and Yi Lu3

1College of Electrical Engineering, Yanshan University, Qinhuangdao, Hebei 066004, China
2Harbin Electric Corporation (Qinhuangdao) Heavy Equipment Company Limited, Qinhuangdao, Hebei 066004, China
3College of Mechanical Engineering, Yanshan University, Qinhuangdao, Hebei 066004, China

Correspondence should be addressed to Yi Lu; luyi@ysu.edu.cn

Received 21 July 2019; Revised 18 August 2019; Accepted 10 November 2019; Published 1 February 2020

Abstract

The stiffness and elastic deformation of a 4-DoF parallel manipulator with three asymmetrical legs are studied systematically for supporting helicopter rotor. First, a 4-DoF 2SPS + RRPR type parallel manipulator with two linear SPS type legs and one RRPR type composite leg is constructed and its constraint characteristics are analyzed. Second, the formulas for solving the elastic deformation and the stiffness matrix of the above mentioned three asymmetrical legs are derived. Third, the formulas for solving the total stiffness matrix and the elastic deformation of this manipulator are derived and analyzed. Finally, its finite element model is constructed and its elastic deformations are solved using both the derived theoretical formulas and the finite element model. The theoretical solutions of the elastic deformations are verified by that of the finite element model.

1. Introduction

Various less mobility (less than 6-DoF) parallel manipulators (PMs) have been applied widely due to their merits, such as good performances in accuracy, rigidity, ability to manipulate large loads, and they are simple in structure and easy to control [1, 2]. Among them, some less mobility PMs with composite active constrained legs attract more attention because they have larger workspace, better flexibility and fewer legs for avoiding interferences easily; the unnecessary tiny self-movement can be eliminated by the composite active constrained legs; more actuators can be installed onto the base for reducing vibration [1, 2]. Therefore, this type of PMs have potential applications for supporting helicopter rotor, airplane operation simulator, parallel machine tools, micro manipulators, sensors, surgical manipulators, tunnel borers, barbettes of war ship, and satellite surveillance platforms. Stiffness is one of the important performances of PMs, because higher stiffness allows larger variable load and higher speeds with higher precision of the end-effector [3]. Therefore, it is significant to analyze the stiffness and to evaluate elastic deformation of this type of PMs in the early design stage. Let (R, P, U, S) be (revolute, prismatic, universal, spherical) joint, respectively. In this aspect, Gosselin and Zhang developed virtual joint method allowed taking into account the links flexibility, which were presented as rigid beams supplemented by linear and torsional springs [3]. Zhang and Lang Sherman [4] established a stiffness modeling for PMs with one passive leg. Dong et al. [5] analyzed the stiffness modeling and stiffness distributions of a 5-DOF hybrid robot by considering the component compliances associated with the elements of both the PM and the wrist. Li and Xu [6] derived stiffness matrix of a 3-PUU PM based on an overall Jacobian using the screw theory by considering the effect of actuations and constraints. Yang et al. [7] studied elastostatic stiffness modeling of over constrained PMs. Zhou et al. [8] derived the stiffness matrix of a redundantly actuated parallel mechanism based on the overall Jacobian. Based on strain energy and Castigliano’s theorem, Enferadi and Tootoonchi [9] obtained mathematical model of the manipulator stiffness matrix. Pashkevich et al. [10] proposed a methodology to enhance the stiffness of serial and parallel manipulators with passive joints, the manipulator elements are presented as pseudo-rigid bodies separated by multidimensional virtual springs and perfect passive joints; they [11] also presented a stiffness modeling method for over-constrained PMs with flexible links and compliant actuating
Chapter 2 SPS + RRPR Type PM

A 2SPS + RRPR type PM for supporting helicopter rotor is shown in Figure 1(a). The 2SPS + RRPR PM is composed of a moving platform $m$, a fixed base $B$, and 2 SPS (spherical joint-active prismatic joint-spherical joint) type legs $r_i (i = 1, 3)$ with the linear actuator and one RRPR (active revolute joint-revolute joint-active prismatic joint-revolute joint) type composite active leg $r_2$ with a linear actuator and a rotational actuator, see Figure 1(b). Here, $m$ is an equilateral ternary link $\Delta b_1 b_2 b_3$, with 3 sides $l_i = l$, 3 vertices $b_i$, and a center point $o$. $B$ is an equilateral ternary link $\Delta B_1 B_2 B_3$, with 3 sides $L_i = L$, 3 vertices $B_i$, and a center point $O$. Each of $r_i (i = 1, 3)$ connects $m$ to $B$ by a spherical joint $S$ at $b_i$, a leg $r_i$ with active prismatic joint $P$, and $S$ at $B_i$. The RRPR-type constrained composite active leg $r_2$, connects $B$ to $m$ by a universal joint $U$ attached to $B$ at $B_2$, a constrained leg $r_2$ with

**Figure 1:** A 2SPS+RRPR PM for supporting helicopter rotor (a), 2SPS+RRPR type PM (b), and a prototype of reconfigurable 3SPS PM (c).
active prismatic joint \( P \), a revolute joint \( R \), attached to \( m \) at \( b \). The universal joint \( U \) at \( b \) is composed of two cross revolute joints \( R_1 \) and \( R_z \). Here, \( R \) is connected with a rotational actuator. Therefore, the moving platform \( m \) of the 2SPS + RRPR type PM has 4 DOFs corresponding to \( R_1 \), \( R_1 \), \( R \), and one translation along limb \( r_2 \). The degree of freedom of the 2SPS + RRPR type PM has been calculated and verified using its simulation mechanism in [23]. Since each of the SPS-type active legs \( r_i (i = 1, 3) \) only bears the active force along \( r_i \) it obviously has relative larger capacity of load bearing and is simple in structure. In addition, the unnecessary tiny self-movement of the 4-DOF 2SPS + RRPR PM can be eliminated effectively and its workspace can be enlarged by the RRPR-type constrained composite active leg \( r_2 \). Comparing with other 4-DOF PMs with four active legs, the 2SPS+RRPR PM with three active legs has merits as follows: (1) The interference among three active legs and the moving platform can be avoided easily. (2) Its whole mechanism is simplified. (3) Its moving platform provides more room for installing the helicopter rotor, finger mechanisms, tools.

A prototype of the reconfigurable 3SPS experimental model is built, see Figure 1(c). It includes a \( m \), a \( B \) and 3 reconfigurable SPS-type legs \( r_i (i = 1, 2, 3) \). Each of \( r_i \) connects \( B \) to \( m \) by a spherical joint \( S \) at \( B \), a reconfigurable leg \( r \) with active prismatic joint \( P \), and \( S \) at \( b \). Here, \( m \) and \( B \) are the same as that of the 2SPS+RRPR PM. Each of \( S \) joints is composed of three revolute joints \( R \). It can be transformed into a \( U \) joint by adding one pin or be transformed into a \( R \) joint by adding two pins. Thus, the 2SPS + RRPR PM can be constructed easily from the prototype of reconfigurable 3SPS model to transform the upper \( S \) joint of \( r_2 \) into a \( R \) joint by adding two pins, to transform the lower \( S \) joint of \( r_2 \) into \( U \) joint by adding one pin, and to add a rotational actuator onto the vertical revolute joint \( R \) of \( U \) joint.

Let \( \perp \) be a perpendicular constraint, \( \mid \mid \) be a parallel constraint. Several geometric constraints \( (R_1 \), being coincident with the axis of motor, \( R_1 \) being coincident with \( y, R_2 \mid \mid Z, R_1 \perp B, R_1 \perp R_2, R_2 \perp R_1, R_1 \perp r_1, R_1 \perp r_2 \) and \( R_2 \perp r_3 \) are satisfied in this PM. Let \( [m] \) be a coordinate frame \( o-xyz \) fixed on \( m \) at \( o \). \([B] \) be a coordinate frame \( O-XYZ \) fixed on \( B \). Let \( (\alpha, \beta, \lambda) \) be three Euler angles of \( m, \varphi \) be one of \((\alpha, \beta, \lambda) \). Set \( s_p = \sin \varphi, c_p = \cos \varphi \) and \( r_\varphi = \tan \varphi \). The position vectors \( B \), of \( B \) on \( B \) in \([B]\), the position vectors \( b \) of \( b \) on \( m \) in \([m]\), the position vectors \( b \) of \( b \) on \( m \) in \([B]\), and the position vector \( o \) of \( o \) on \( m \) in \([B]\), the unit vectors \( \mathbf{r}_i \) of \( r_i \) and the vector \( e_i \) of the line \( e_i \) in \([B]\) can be expressed as follows: [23]

\[
\begin{align*}
B_i & = \left( X_{B_i}, Y_{B_i}, Z_{B_i} \right), \quad b_i = \left( x_{b_i}, y_{b_i}, z_{b_i} \right), \quad B & = \left( X_B, Y_B, Z_B \right), \\
o & = \left( X_o, Y_o, Z_o \right), \quad R & = \left( x_m, y_m, z_m \right), \quad m = \left( x, y, z \right), \\
b_i & = R b_i + o, \quad r_i = \left| b_i - B \right|, \quad \delta_i = \frac{b_i - B}{r_i}, \quad e_i = b_i - o.
\end{align*}
\]

(1)

Figure 2: Kinematics and statics model of 2SPS+RRPR PM.

Here, \( (X_o, Y_o, Z_o) \) are the components of \( o \) in \([B]\); \( B \) is a rotational transformation matrix from \([m]\) to \([B]\); \( \left( x_1, x_m, x_o, y_1, y_m, y_o, z_1, z_m, z_o \right) \) are nine orientation parameters of \( m \).

The formulas for solving \( m b_i (i = 1, 2, 3) \), \( B \), and \( B \), are derived from Equation (1) and represented as follows:

\[
\begin{align*}
\left. \begin{array}{lll}
\left( \begin{array}{l}
\text{b}_1 \\
\text{b}_2 \\
\text{b}_3
\end{array} \right) & = & \frac{e}{2} \left( \begin{array}{ccc}
\pm q & -1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array} \right) \left( \begin{array}{l}
\text{b}_1 \\
\text{b}_2 \\
\text{b}_3
\end{array} \right), \\
\text{b}_1 & = & \frac{e}{2} \left( \begin{array}{ccc}
\pm q x_1 & -y_1 + 2X_o & -Z_o \\
\pm q x_m & -y_m + 2Y_o & -Z_o \\
\pm q x_o & -y_o + 2Z_o & -Z_o
\end{array} \right) \left( \begin{array}{l}
\text{b}_1 \\
\text{b}_2 \\
\text{b}_3
\end{array} \right), \\
q & = & \sqrt{3}; i = 1, \pm = +; i = 3, \pm = -
\end{array} \right)
\end{align*}
\]

(2)

Under the geometric constraints of the RRPR-type constraining active leg \( r \), \( R \) is formed by 3 rotations of \( (Z, X_1, Y_1) \), namely, a rotation of \( \alpha \) about \( Z \)-axis i.e., \( R_1 \), followed by a rotation of \( \beta \) about \( X_1 \)-axis i.e., \( R_2 \), and a rotation of \( \lambda \) about \( Y_2 \)-axis i.e., \( R_3 \). Here, \( X_1 \) is formed by \( X \) rotating about \( z \), and \( Y_1 \) is formed by \( Y \) rotating about \( X_1 \) by \( \beta \), see Figure 2. Each of \( (x_1, x_m, x_o, y_1, y_m, y_o, z_1, z_m, z_o) \) can be expressed by \( (\alpha, \beta, \lambda) \) from Equations (1) and (2) as follows:

\[
\begin{align*}
x_1 & = c_\alpha c_\beta s_\lambda - s_\alpha s_\beta s_\lambda, \quad x_m = s_\alpha c_\beta + c_\alpha s_\beta s_\lambda, \quad x_o = -c_\beta s_\lambda, \\
y_1 & = -s_\alpha c_\beta y_m + c_\alpha s_\beta y_m = s_\beta, \\
x_o & = -y_1(e + Z_o s_\beta), \quad y_m = \frac{y_1 + 2Y_o}{\sqrt{1 + y_2^2}}, \quad y_o = \frac{y_2 + 2Y_o}{\sqrt{1 + y_2^2}}, \\
y_o & = E - \frac{y_m(e + Z_o y_m)}{\beta} = E - \frac{c_\beta}{c_\alpha}(e + Z_o s_\beta).
\end{align*}
\]

(3)
Let \( \mathbf{c} \) be the unit vector of an arm \( \mathbf{F}_{c_i} \), \( \mathbf{d}_i \) be the arm vector from \( \mathbf{F}_{c_i} \) to \( \mathbf{F}_{ci} \). Let \( v \) and \( \omega \) be the translational and angular velocities. Since \( \mathbf{F}_{c1}(j=1,2) \) limits the movement of PMs, based on principle of virtual work in [23], it is known that \( \mathbf{F}_{c1}(j=1,2) \) does not produce any power. Thus, there are

\[
\mathbf{F}_{c1} \cdot \mathbf{v} + (\mathbf{d}_i \times \mathbf{F}_{c_i}) \cdot \mathbf{\omega} = 0 \quad (j=1,2)
\]

Thus, the geometric constrains of \( \mathbf{F}_{c1} \) are determined in [23] as follows:

1. Let \( \mathbf{v}_2 \) be a velocity along revolute joint \( P \) in \( r_2 \), \( \mathbf{F}_{c1} \mathbf{v}_2 = 0 \) i.e., \( \mathbf{F}_{c1} \perp \mathbf{r}_2 \) must be satisfied.
2. Let \( \mathbf{R}_i \) be a unit vector of revolute joints \( R_i \) in \( r_2 \). Let \( \mathbf{p}_j \times \mathbf{F}_{cj} \) be a torque of \( \mathbf{F}_{c1} \) about \( \mathbf{R}_i \), \( \mathbf{p}_j \times \mathbf{F}_{cj} = 0 \) must be satisfied. Thus, each of \( \mathbf{F}_{c1} \) must either intersect or be parallel with all the revolute joints \( R_i \) in \( r_3 \). Thus, the geometric constrained conditions \( \{\mathbf{F}_{c1}\}|\mathbf{R}_i|\mathbf{F}_{c1}, \mathbf{F}_{c2} \) intersecting with both \( R_j \) and \( R_k \) at point \( a, \mathbf{F}_{c1}|\mathbf{R}_i|\mathbf{F}_{c1}, \mathbf{F}_{c2} \) intersecting with both \( R_j \) and \( R_k \) at point \( B_j \) are satisfied.

From the geometric constrains of \( \mathbf{F}_{c1} \), it leads to

\[
\mathbf{c}_1 = \mathbf{R}_2 = \begin{pmatrix} c_{\alpha} \\ s_{\alpha} \\ 0 \end{pmatrix}, \mathbf{c}_2 = \mathbf{R}_3 = \begin{pmatrix} -c_{\beta} \\ c_{\beta} s_{\alpha} \\ s_{\beta} \end{pmatrix}, \mathbf{a} = \begin{pmatrix} 0 \\ E \\ z_{\alpha}(z_{\alpha}+z_{\alpha}z_{\beta}) \end{pmatrix}, \mathbf{b}_1 = \mathbf{t}_p(z_{\alpha}+z_{\alpha}z_{\beta}) \begin{pmatrix} -s_{\alpha} \\ c_{\alpha} \\ s_{\beta} \end{pmatrix}, \mathbf{d}_1 = \mathbf{a} - \mathbf{b}_1 = \begin{pmatrix} -E \\ z_{\alpha}(z_{\alpha}+z_{\alpha}z_{\beta}) \end{pmatrix}, \mathbf{b}_2 = \mathbf{t}_p(z_{\alpha}+z_{\alpha}z_{\beta}) \begin{pmatrix} -s_{\alpha} \\ c_{\alpha} \\ s_{\beta} \end{pmatrix}.
\]

The general input velocity \( \mathbf{V}_i \), the general output velocity \( \mathbf{V} \) in \( \{\mathbf{B}\} \), \( \mathbf{F}_{ci}(i=1,2,3) \), \( \mathbf{T}_n \) and \( \mathbf{F}_{c1}(j=1,2) \) have been derived based on Equations (4)–(6) as follows:
\[ V_r = J_{axa} V, V = J^{-1} V_r, \alpha' = (0_{3 \times 3} s^T) V, \]
\[ V = \begin{pmatrix} v \\ \omega \end{pmatrix}, s = \begin{pmatrix} s_{12}^T \phi \\ 1 - c_{12} s_{12}^T \phi \end{pmatrix}, \]
\[ F = \begin{pmatrix} F_x \\ F_y \end{pmatrix}, T = \begin{pmatrix} T_x \\ T_y \end{pmatrix}, \]
\[ J = \begin{pmatrix} \delta_1^v \\ \delta_2^v \\ \delta_3^v \\ 0_{12}^v \\ c_1^v \\ c_2^v \\ (d_1 \times c_1)^v \\ (d_2 \times c_2)^v \end{pmatrix}, \]
\[ \alpha = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \alpha' = \begin{pmatrix} \alpha_1' \\ \alpha_2' \end{pmatrix}, \]
\[ (F) = (F_1) \begin{pmatrix} F_11 & F_12 & F_13 \\ F_21 & F_22 & F_23 \\ F_31 & F_32 & F_33 \end{pmatrix} \]
\[ (T) = (T_1) \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} \]

(7)

here, \( J \) is a 6\times6 Jacobian matrix of the 2SPS+RRPR PM, \( \alpha' \) is an angular velocity of \( R_1 \) (motor 1).

### 3. Stiffness Matrix and Elastic Deformation of SPS-Type Legs and RRPR-Type Leg

Suppose that the rigid platform \( m \) is elastically suspended and by 3 elastic active legs \( r_i \) and constrained by one elastic constrained leg \( r_i \). If only small displacements from its unpreloaded equilibrium position are considered, the overall wrench–deflection relation of the mechanism is linear elasticity. Based on the constructed workspace, each of length of piston/cylinder for active legs and constrained leg can be determined. Let \( r_{ij} \), \( A_{ij} \), \( I_{ij} \) and \( J_{ij} \) be the length, the section of a piston, the moment of inertia, and the rotational moment of inertia of leg \( r_i \), respectively. Let \( (r_i - r_{ij}) \), \( A_{2ij} \), \( I_{2ij} \), and \( J_{2ij} \) be the length, the section, the moment of inertia, and the rotational moment of inertia of a cylinder of \( r_i \), respectively. Let \( E_i \) and \( G_i \) be the modulus of elasticity and the rotational modulus of elasticity for leg \( r_i \) (\( i = 1, 2, 3 \)). When each of the active forces \( F_{ai}(i = 1, 2, 3) \) applies onto the SPS-type active leg \( r_i(i = 1, 3) \) and the RRPR-type constrained active leg \( r_i(i = 2) \) and along \( r_i \), the longitudinal elastic differential deformation \( dr_i \) of leg \( r_i \) see Figure 3(a).

The longitudinal elastic differential deformations of the SPS-type active leg \( r_i(i = 1, 3) \) and the RRPR-type composite active leg \( r_i(i = 2) \) under \( F_{ai} \) (a), The transverse elastic differential deformations of the RRPR-type composite leg \( r_2 \) under \( F_{cij} \) (b, c) and \( T_{au} \) (d). When each of the active forces \( F_{ai}(i = 1, 2, 3) \) applies onto the SPS-type active leg \( r_i(i = 1, 3) \) and the RRPR-type constrained active leg \( r_i(i = 2) \) and along \( r_i \), the longitudinal elastic differential deformation \( dr_i \) of leg \( r_i \) (see Figure 3(a)) can be solved as below [24]

\[ dr_i = \frac{F_{ai}}{k_{ai}} F_{ai} = k_{ai} dr_i, k_{ai} = -\frac{E_i}{(\frac{r_{ai} - r_{ai}}{A_{ai}}) + (\frac{r_{ai}}{A_{ai}})}, \]

(8)

here, \( k_{ai} \) is a longitudinal stiffness of SPS active leg \( r_i \) and RRPR-type constrained active leg \( r_i(i = 2) \).

The active torque \( T_a \) consists of a component \( T_{au} \) along \( r_2 \) and a component \( T_{av} \) perpendicular to \( r_2 \) (see Figure 3(b)). They can be expressed as follows:

\[ T_a = T_{au} Z, T_{au} = T_{au} \cdot \delta_2 = T_{av} \cdot \delta_2 \times (\delta_2 \times R_2) = T_{av} s_{\beta}. \]

(9)

When \( T_{au} \) is exerted onto leg \( r_2 \) at universal joint and \( T_{av} \perp r_2 \) is satisfied, the transverse elastic differential deflection \( da_i \) of \( r_2 \) at its end (Figure 3(b)) can be solved in [24] as follows:

\[ da_i = -\frac{T_{au}}{2E} \left[ \frac{(r_2 - r_1)^2}{I_{12}} \right] = \frac{T_{au}}{k_{au}}, \]

\[ k_{av} = \frac{-2E}{\left[ \left( \frac{r_{12}}{I_{12}} \right) + \left( \frac{r_{22}}{I_{22}} \right) \right]}. \]

(10)

When \( T_{au} \) is exerted onto leg \( r_2 \) at universal joint, the elastic rotational differential deformation \( d\theta_i \) of leg \( r_2 \) at its end can be solved based on the elastic deformation formula in [24] as below

\[ d\theta_i = \frac{T_{au}}{G} \left[ \frac{(r_2 - r_1)^2}{I_{12}} + \frac{(r_2 - r_1)^2}{I_{22}} + \frac{(r_2 - r_{12})^2}{I_{12}} \right] = \frac{T_{au}}{k_{ar}}, \]

(11)

here, \( k_{av} \) and \( k_{ar} \) are the transverse and rotational stiffness of leg \( r_2 \) vs. \( T_{av} \) and \( T_{au} \) respectively.

From Equations (10)–(12), it leads to

\[ da_i + d\theta_i = \left( \frac{1}{k_{au}} + \frac{1}{k_{ar}} \right) T_{au}, \]

\[ k_i = \left( \frac{1}{k_{au}} + \frac{1}{k_{ar}} \right). \]

(12)

When \( F_{cij} \) is exerted onto leg \( r_2 \) at point \( a \) and \( F_{cij} \) \( r_2 \) is satisfied, the transverse elastic differential deflection \( dc_i \) of \( r_2 \) at its end (see Figure 3(c)) can be solved in [24] as follows:

\[ dc_i = \frac{F_{cij}}{k_{cij}} \frac{dc_i}{c_{i2}} = \frac{-3E}{\left[ \frac{(r_2 - r_1)^2}{I_{12}} + \frac{(r_2 - r_1)^2}{I_{22}} + \frac{(r_2 - r_{12})^2}{I_{12}} \right]}, \]

(13)

here \( k_{cij} \) is a transverse stiffness of \( r_2 \) vs. \( F_{cij} \).

When \( F_{ci} \) is exerted onto leg \( r_2 \) at point \( a \) and \( F_{ci} \) \( r_2 \) is satisfied, the elastic rotational differential deformation \( d\theta_i \) of leg \( r_2 \) at its end can be solved in [24] as follows:

\[ d\theta_i = \left( \frac{a - b_2}{G} \right) \left[ \left( \frac{(r_2 - r_1)^2}{I_{12}} + \frac{(r_2 - r_{12})^2}{I_{12}} \right) \right] = \frac{F_{ci}}{k_{cij}}, \]

(14)

Similarly, from Equations (15) and (16), it leads to
Equation (8) to Equation (16) as follows:

\[ F_{c1} = k_2 (dc_1 + d\theta_2), \]

According to the principle of virtual work in [22], it is known that when a deformed mechanical system keeps a static balance under all external wrenches, the sum of the work generated by all external wrenches along virtual displacements of the same mechanical system and the work produced by all internal wrenches along virtual deformations of the same mechanical system must be zero. Therefore, the sum of the work generated by \((F_{a1}, F_{a2}, F_{a3}, T_o, F_{c1}, F_{c2})\) along deformations of the 2SPS + RRPR PM and the work produced by \((F, T)\) along

\[ dc = da_1 + dc_1, \quad \theta = \theta_1 + d\theta_2, \quad k_1 = \begin{pmatrix} k_{a1} & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{a2} & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{a3} & 0 & 0 & 0 \\ 0 & 0 & 0 & k_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_{c2} \end{pmatrix} \]

\[ K_r = \begin{pmatrix} F_{a1} \\ F_{a2} \\ F_{a3} \\ T_o \\ F_{c1} \\ F_{c2} \end{pmatrix}, \quad K^{-1}_r = \begin{pmatrix} dc_1 \\ dc_2 \end{pmatrix}, \quad K = \begin{pmatrix} F_{a1}^T & F_{a2}^T & F_{a3}^T & T_o^T & F_{c1}^T & F_{c2}^T \end{pmatrix} \]

\[ dX_o = \begin{pmatrix} dc_1 \\ dc_2 \end{pmatrix}, \quad dY_o = \begin{pmatrix} dc_1 \\ dc_2 \end{pmatrix}, \quad \begin{pmatrix} dc_1 \\ dc_2 \end{pmatrix} = \begin{pmatrix} k_{a1} & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{a2} & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{a3} & 0 & 0 & 0 \\ 0 & 0 & 0 & k_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_{c2} \end{pmatrix} \]

4. Total Stiffness Matrix and the Elastic Deformation of 2SPS + RRPR PM

Based on principle of virtual work in [22], it is known that when a deformed mechanical system keeps a static balance under all external wrenches, the sum of the work generated by all external wrenches along virtual displacements of the mechanical system and the work produced by all internal wrenches along virtual deformations of the same mechanical system must be zero. Therefore, the sum of the work generated by \((F_{a1}, F_{a2}, F_{a3}, T_o, F_{c1}, F_{c2})\) along deformations of the 2SPS + RRPR PM and the work produced by \((F, T)\) along...
the displacements of point o in \{B\} must be zero. Let \((dX_o, dY_o, dZ_o)\) be 3 translational components of the elastic differential deformation of \(m\) at \(o\) in \{B\}; \((d\varphi_o, d\varphi_y, d\varphi_z)\) be 3 rotational components of the elastic differential deformation of \(m\) in \{B\}. Thus, based on the theorem of work and energy equal to each other, from Equation (7) to Equation (17), it leads to

\[
\begin{pmatrix}
    d\alpha_1 \\
    d\alpha_2 \\
    d\alpha_3 \\
    d\theta \\
    d\varphi_x \\
    d\varphi_y \\
    d\varphi_z \\
\end{pmatrix}
= \begin{pmatrix}
    F_{a1} \\
    F_{a2} \\
    F_{a3} \\
    T_a \\
    F_{c1} \\
    F_{c2} \\
    F_{c3} \\
\end{pmatrix}
\]

\[
= \begin{pmatrix}
    dX_o \\
    dY_o \\
    dZ_o \\
    d\varphi_x \\
    d\varphi_y \\
    d\varphi_z \\
\end{pmatrix}
\]

\[
\Rightarrow \begin{pmatrix}
    d\alpha_1 \\
    d\alpha_2 \\
    d\alpha_3 \\
    d\theta \\
    d\varphi_x \\
    d\varphi_y \\
    d\varphi_z \\
\end{pmatrix} \cdot [-J^{-1} ] = \begin{pmatrix}
    dX_o \\
    dY_o \\
    dZ_o \\
    d\varphi_x \\
    d\varphi_y \\
    d\varphi_z \\
\end{pmatrix}
\]

\[
\Rightarrow \begin{pmatrix}
    d\alpha_1 \\
    d\alpha_2 \\
    d\alpha_3 \\
    d\theta \\
    d\varphi_x \\
    d\varphi_y \\
    d\varphi_z \\
\end{pmatrix} = \begin{pmatrix}
    dX_o \\
    dY_o \\
    dZ_o \\
    d\varphi_x \\
    d\varphi_y \\
    d\varphi_z \\
\end{pmatrix}
\]

Thus, from Equations (7), (14), and (15), it leads to

\[
\begin{pmatrix}
    dX_o \\
    dY_o \\
    dZ_o \\
    d\varphi_x \\
    d\varphi_y \\
    d\varphi_z \\
\end{pmatrix} = J^{-1} \begin{pmatrix}
    d\alpha_1 \\
    d\alpha_2 \\
    d\alpha_3 \\
    d\theta \\
    d\varphi_x \\
    d\varphi_y \\
\end{pmatrix}
\]

\[
= J^{-1} K_r^{-1} \begin{pmatrix}
    d\alpha_1 \\
    d\alpha_2 \\
    d\alpha_3 \\
    d\theta \\
\end{pmatrix}
\]

\[
= -J^{-1} K_r^{-1} \begin{pmatrix}
    F \\
    T \\
\end{pmatrix}
\]

\[
= -(J^{-1} K_r^{-1}) \begin{pmatrix}
    F \\
    T \\
\end{pmatrix}
\]

\[
= -J^{-1} K_r^{-1} \begin{pmatrix}
    F \\
    T \\
\end{pmatrix}
\]

Here

\[
K = \left[ J^{-1} K_r^{-1} (J^{-1})^{-1} \right]^{-1}
\]

\[
= -(J^T K J)_{6 \times 6} = \begin{pmatrix}
    k_{11} & k_{12} & \cdots & k_{16} \\
    k_{21} & k_{22} & \cdots & k_{26} \\
    \vdots & \vdots & \ddots & \vdots \\
    k_{61} & k_{62} & \cdots & k_{66} \\
\end{pmatrix},
\]

5. Analytic Solved Example of Elastic Deformation for 2SPS+RRPR PM

In the 2SPS + RRPR type PM, let initial independent pose variables vary vs. time \(t\) when given pose parameters \((\alpha, \beta, \gamma, \zeta)\), see Figures 4(a) and 4(b). Set \(L = 1.2\ m, l = 0.6\ m; F = [20 \ 30 \ 60]^{T} \ N\) and \(T = [-0.3 \ -0.3]^{T} \ Nm, E_i = 2.11 \times 10^9\ \ Pa\), the diameter of piston and cylinder for active legs \(r_i (i = 1, 2, 3)\) are \(D_{a1} = D_{a2} = 0.04\ m, E_i l_i = E_i l_2 = 26502\ N \cdot m^2, A_{a1} = A_{a2} = 0.0013\ m^2\). By using the relevant theoretical equations and Matlab, the extensions of \(r_i\) and \(\alpha\) are solved, see Figure 4(a). The position components \((X_o, Y_o, Z_o)\) of the moving platform are solved, see Figure 4(b). Three active forces \(F_{a}(i = 1, 2, 3)\), one active torque \(T_{a}\), two constrained forces \(F_{c1}\) and \(F_{c2}\) are solved, see Figures 4(c) and 4(d). The longitudinal deformations \(d r_i (i = 1, 2, 3)\) of \(r_i\) are solved, see Figure 4(e). The position deformations of \(m\) at \(o\) are solved, see Figure 4(f). The angular deformations of \(m\) are solved, see Figure 4(g). The transverse deformations \(dc_j (j = 1, 2)\) and \(dc_a\), and the rotational deformations \(d\theta_1\) and \(d\theta_2\) of \(r_i\) are solved, see Figure 4(h).

When \(r_1 = 1.8, r_2 = 1.7, r_2 = 1.66\ m\). \(\alpha = 0\), \(K\) is solved from Equations (7) and (20) as follows:

\[
K = \begin{pmatrix}
0.0923 & 0.0130 & 0.0266 & 0.0038 & 0.1466 & -0.0279 \\
0.0130 & 0.2417 & 0.7799 & -0.1465 & 0.0052 & -0.0042 \\
0.0266 & 0.7799 & 4.2941 & 0.0905 & -0.0097 & -0.0090 \\
0.0038 & -0.1465 & 0.0905 & 0.2721 & 0.0150 & -0.0010 \\
0.1466 & 0.0052 & -0.0097 & 0.0150 & 0.2338 & -0.0446 \\
-0.0279 & -0.0042 & -0.0009 & 0.0446 & 0.0084 & 0.0084
\end{pmatrix}
\]

6. A FE Model of 2SPS + RRPR PM and Its Solutions

A 3D assembly mechanism of the 2SPS + RRPR PM is constructed in SolidWorks [25]. Next, its finite element (FE) model is generated in ANSYS, see Figure 5. All relative geometry and material parameters of the 3D simulation assembly mechanism are the same as that in Section 5. The 3 equivalent revolute joints for 3 actuated revolute joints and 4 equivalent spherical joints for 4 actuated spherical joints are constructed, see Figure 5(a). The applied loads are shown in Figure 5(b). The boundary condition is explained as follows:
Figure 5: Simulation solutions of elastic deformations of EF model of the 2SPS + RRPR PM. (a) Equivalent spherical joint $S$ and revolute joints $R_1$ and $R_2$, (b) load condition, (c) FE model of 2SPS + UPR PM and its elastic deformation, (d) elastic deformation of $\alpha$, (e) elastic deformation $dX_\alpha$, (f) elastic deformation $dY_\alpha$, and (g) elastic deformation $dZ_\alpha$. 
TABLE 1: Simulation solved results of elastic deformations of EF model of 2SPS+RRPR PM.

<table>
<thead>
<tr>
<th>Elastic deformation of o, mm</th>
<th>FE model position of No.</th>
<th>Position of m (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>FE model</td>
<td>Theoretical</td>
<td>X₀</td>
</tr>
<tr>
<td>do</td>
<td>0.9058</td>
<td>0.8934</td>
</tr>
<tr>
<td>dX₀</td>
<td>−0.8387</td>
<td>−0.8550</td>
</tr>
<tr>
<td>dY₀</td>
<td>−0.3434</td>
<td>−0.2547</td>
</tr>
<tr>
<td>dZ₀</td>
<td>0.0258</td>
<td>0.0534</td>
</tr>
</tbody>
</table>

(1) If no setting is given, all the assembly parts in the FE model may constitute the same elastic body. Therefore, each of the assembly spherical joints in FE model constitutes the same elastic body. The simulation 3D assembly of spherical joints is used only for varying the pose of PM and the workload applied on m at o.

(2) All the relative geometry parameters of the 3D simulation assembly mechanism are the same as that in Section 5. The material parameters of r₁ (i = 1, 2, 3) are set as the same as that in Section 5.

(3) Construct 3 equivalent spherical joints for 4 actuated spherical joints, see Figure 5(a). Here, the diameters at the two ends of the SPS-type legs r₁ (i = 1, 3) are reduced sharply.

(4) Construct 3 equivalent revolute joints for the actuated revolute joints R₁, R₂, R₃, see Figures 5(a) and 5(b). Here, two holes for each of equivalent revolute joints are constructed and kept coincident with each other; the rotational stiffness and the axial stiffness are set as 0 and 1 × 10⁶ N/mm, respectively, according to the requirement for revolute joint in software.

(5) Each of the 3 linear active legs with prismatic joints is formed using the elastic linear rod, which is assigned by the alloy steel. Set SPS leg r₁ = 1.8, SPS leg r₂ = 1.7, RRPR leg r₂ = 1.66 m.

(6) A fixed constraint is added onto the base, which is assigned by the alloy steel with rigid body.

(7) The workload wrench \( F = [-20 30 60]^T \) N and \( T = [-0.3 -0.31]^T \) N m are applied onto m at o, which is assigned by the alloy steel with the rigid body, see Figure 5(b).

Some solved results of the elastic deformations are shown in Figures 5(c)–5(g) and Table 1.

An existing CAD software provides a function for automatically optimal mesh in order to avoid singularity element and to obtain the suitable results of finite element analysis (FEM). Therefore, the 3D assembly mechanism of the 2SPS+RRPR PM is automatically meshed by the function for automatically optimal mesh.

7. Analysis of Stiffness and Elastic Deformation of 2SPS+RRPR PM

Several conclusions are obtained from theoretical and simulation solutions as follows:

(1) The solved results of FE model in most cases are approximate numerical results which depend on some key factors such as finite element dimension and type, equivalence between actual joints and simulation joints, selected material parameter, solver, reasonable boundary constraints and connection constraints [23].

(2) It is known from Table 1 that the elastic deformations of FE model of this PM are basically coincident with that of theoretical ones in Section 5.

(3) It is known from Figures 4(e) and 4(h) that the transverse elastic deformations \( d_{c_1} (j = 1, 2) (0.5 \rightarrow 1.2 \times 10^{-4}) \) of \( r_z \) due to the constrained forces \( F_{c_j} \) are greatly larger than the longitudinal elastic deformation \( d_r (0.5 \rightarrow 2.5 \times 10^{-4}) \) of \( r_z \) due to active forces \( F_{a_j} \). It implies that the constrained wrench has great influence on the elastic deformation of this PM.

(4) It is known from Figure 4(c), h that the transverse elastic deformations \( d_{c_1} (j = 1, 2) \) of \( r_z \) due to the constrained forces \( F_{c_j} \) is larger than the transverse elastic deformation \( d_{a_z} \) of \( r_z \) due to active torque \( T_{a_j} \). The transverse elastic deformations and elastic rotational deformation of the SPS-type legs \( r_z (i = 1, 3) \) is 0. Therefore, the diameter of piston and cylinder of \( r_z \) should be increased.

(5) It is known from Figure 4(h) that the elastic rotational deformation \( d\theta_z \) of \( r_z \) due to \( F_{c_z} \) and the elastic rotational deformation \( d\theta_z \) of \( r_z \) due to \( F_{a_z} \) are inversely proportional to each other.

8. Conclusions

A 2SPS + RRPR parallel manipulator with asymmetrical structure is suitable for the helicopter rotor supporting base.

The formulas for solving the stiffness matrix and the elastic deformation of its three asymmetrical legs are derived. The formulas for solving its total stiffness matrix and the elastic deformation are derived based on the Jacobian matrix and the stiffness matrix of three asymmetrical legs. Both the stiffness matrix of its three asymmetrical legs and its total stiffness matrix are \( 6 \times 6 \) symmetric matrices, although this manipulator has asymmetrical structure.

The constrained wrench must be taken into account when establishing its total stiffness matrix and solving its elastic deformation.

The proposed methodological results can be applied to other less mobility parallel manipulators with asymmetrical structure and active legs for solving the elastic deformations of asymmetrical active legs and the elastic deformations of moving platform.
Nomenclatures

DoF: Degree of freedom
PM: Parallel manipulator
\{m\}: Coordinate frame o-xyz fixed on m
\{B\}: Coordinate frame O-XYZ fixed on B
o: The center point of m
O: The center point of B
b, B: The vertices of m, B(j = 1, 2, 3)
l, L: The side of m, B
c: The distances from b to o
e: The distances from B to O
\(r_i\): Active leg and its length (i = 1, 2, 3)
\(R, R, R_c\): Three revolute joints
\(R, P\): Revolute joint, prismatic joint
\(U, S\): Universal joint and spherical joint
\(X_o, Y_o, Z_o\): Position components of o in \{B\}
\(\alpha, \beta, \lambda\): Three Euler angles of m
\(mR\): Rotational transformation matrix
\(J\): Jacobian matrix
\(E_i\): Modulus of elasticity for \(r_i\)
\(G_i\): Rotational modulus of elasticity for \(r_i\)
\(C\): The total complacence matrix of PM
\(T_o\): The active torque
\(I_o\): Linear inertia moment of \(r_i\)
\(J_o\): Rotational inertia moment of \(r_i\)
\(V_o\): The general input velocity
\(V\): The general output velocity in \{B\}
\(x_o, y_o, z_o\): Nine orientation parameters of \(mR\)
\(v_o\): The input velocity along active leg \(r_i\)
\(\omega\): The linear and angular velocities of m at o in \{B\}
\(F, T\): The concentrated force and torque applied on m at o
\(F_{ai}, \delta\): The active forces and their unit vectors
\(F_{ci}, \kappa\): The constrained forces and their unit vectors
\(F_{ai}, F_{ci}\): The components of \(F_{ai}, \delta\)
\(F_{ai}, F_{ci}\): The components of \(F_{ai}, \delta\)
\(T_{ai}, T_{ci}\): The components of \(T_{ai}, \delta\)
\(K_{ai}, K_{ci}\): 6 x 6 total stiffness matrix of the legs \(r_i\) for piston, \(j = 1\) for cylinder
\(k_{ai}\): A longitudinal stiffness of \(r_i\)
\(k_{ai}\): The transverse and rotational stiffness of \(r_i\)
\(k_{ai}\): Transverse stiffness of \(r_i\) in plane with \(F_{ci}\)
\(K\): The stiffness matrix of PM
\(dr_i\): The longitudinal elastic differential deformation of \(r_i\)
\(\theta_{ai}, \theta_{ci}\): Rotational deformations of \(r_i\)
\(a_i, \beta_i, \lambda_i\): Differential deformations of 3 Euler angles of \(m\)
\(\varphi:\) One of (\(\alpha, \beta, \gamma, \gamma_1, \gamma_2\),
\(s_\varphi = \sin \varphi, c_\varphi = \cos \varphi, t_\varphi = \tan \varphi\)
\(r_{ij}\): The length of \(r_i\), \(j = 1\) for piston, \(j = 2\) for cylinder
\(A_{ij}, D_{ij}\): The cross-section and the diameters of \(r_i\)
\(||l, l, l||\): Perpendicular, parallel collinear constraint.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors would like to acknowledge Major Research Project (91748125) supported by National Natural Science Foundation of China.

References


