Performance Analysis of AOA-Based Localization Using the LS Approach: Explicit Expression of Mean-Squared Error

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1. Introduction

There has been a great deal of research on the determination of emitter location. Localization consists of two parts: measuring localization parameters between nodes and with these parameters to estimate location. The localization parameters can be AOA and TOA (time of arrival). AOA-based localization schemes have been considered [1–9]. The AOA-based localization algorithm can be classified as follows: linear least-squared (LS) estimation [10], nonlinear least-squared estimation [11, 12], minimum mean-squared error (MMSE) estimation with Kalman filter [13], and maximum likelihood (ML) estimation [14].

In [10], least-squared (LS) algorithm for emitter localization is proposed. In the algorithm, the distances between the given bearing lines to emitter location estimate are calculated as a function of emitter location estimate. Cost function is defined from the sum of squares of the distances, and the location estimate is obtained from the location minimizing the cost function. In [11, 12], nonlinear least-squared algorithm is used for the emitter localization. In [13], since the bearing angle measurements are noisy, the measurements are combined using a nonlinear least squares filter or an extended Kalman filter to obtain the optimal filtered position estimate. The maximum likelihood (ML) and the Stansfield algorithm for AOA-based localization have been considered in [14], where performances in terms of the bias and the covariance matrix for these AOA-based localization algorithms have been presented analytically. In the proposed scheme, least-squared (LS) algorithm is employed to get location estimate using noisy AOA measurements. Explicit expressions of localization error and the mean-squared error (MSE) are derived.

Note that, in the LS-based linear bearing-based localization algorithm, the speed of the moving sensor is not necessarily constant. The assumption of constant moving speed is just for convenience in implementation of the algorithm in the numerical results. Gaussian random variable representing LOB measurement error at each sensor location is adopted. Means of these Gaussian random variables are set...
to be identically zero. Generally, it can be regarded as a thermal noise component generated inside the receiver. By exploiting this observation, we try to get an analytic expression of the location estimate and the analytic expression of the MSE of the location estimate.

AOA estimation errors as well as sensor location errors are responsible for the errors in the location estimate. In [16], it is assumed that sensor locations are available without uncertainty and that the AOA estimation errors can be modelled as Gaussian random variables. Explicit expression of the location estimate in these assumptions has been derived in [16]. In this paper, using the results presented in [16], an explicit expression of the MSE of the location estimate is derived. In [17], an iterative solution based on ML approach has been presented and its performance has been illustrated.

In this paper, we present more explicit expression of the MSE. Our expression is explicit and intuitive in that the estimation error and the MSE are expressed in terms of the AOA estimation error and the MSE are expressed in terms of the AOA estimation errors as well as sensor location errors. In [17], it is assumed that sensor locations are available without uncertainty, and that the AOA estimation errors can be modelled as Gaussian random variables. Explicit expression of the location estimate in these assumptions has been derived in [16]. In this paper, using the results presented in [16], an explicit expression of the MSE of the location estimate is derived. In [17], an iterative solution based on ML approach has been presented and its performance has been illustrated.

In this paper, we present more explicit expression of the MSE. Our expression is explicit and intuitive in that the estimation error and the MSE are expressed in terms of the AOA estimation error. Therefore, the contribution of this paper is to present an analytic expression of the MSE of the LS-based localization algorithm. In this paper, two approximations named K-approximation and L-approximation have been employed to derive the mean-squared error (MSE) of location estimate.

Taylor series expansion has been used in K-approximation. K-approximation has been adopted to get polynomial approximations of various sinusoids. Many studies have been conducted on error analysis due to various nonlinear approximations [18, 19].

2. LS-Based Location Estimate [10]

Let \((x_i, y_i)\) and \(\phi_i\) denote the \(i\)th sensor location and the AOA measurement at \((x_i, y_i)\). Given \(x = (x_i, y_i)^T\) and \(\phi_i\) for \(1 \leq i \leq N\), where \(N\) is the number of sensor coordinates, we are to estimate \(x_T = (x_T, y_T)\), which denotes the true emitter location. The estimation of the emitter location can be written as [10]

\[
Ax = b,
\]

(1)

where

\[
A = \begin{bmatrix}
\sin \phi_1 & -\cos \phi_1 \\
\sin \phi_2 & -\cos \phi_2 \\
\vdots & \vdots \\
\sin \phi_N & -\cos \phi_N \\
x_1 \sin \phi_1 - y_1 \cos \phi_1 \\
x_2 \sin \phi_2 - y_2 \cos \phi_2 \\
\vdots \\
x_N \sin \phi_N - y_N \cos \phi_N 
\end{bmatrix},
\]

(2)

\[
b = \begin{bmatrix}
\sin \phi_1 \\
\sin \phi_2 \\
\vdots \\
\sin \phi_N \\
x_1 \sin \phi_1 - y_1 \cos \phi_1 \\
x_2 \sin \phi_2 - y_2 \cos \phi_2 \\
\vdots \\
x_N \sin \phi_N - y_N \cos \phi_N 
\end{bmatrix}.
\]

The linear least-squared (LS) estimation algorithm for estimating the emitter location is briefly described. Since \(A\) is not invertible, the following normal equation is obtained from (1) for the LS estimate:

\[
(A^HA)x = A^Hb.
\]

(3)

The location estimate for the noiseless AOA measurements can be written as

\[
x = (A^HA)^{-1}A^Hb.
\]

(4)

Noiseless AOA is denoted by \(\phi_i\), and noisy AOA is denoted \(\phi_i + \delta \phi_i\), where \(\delta \phi_i\) denotes an error in AOA measurement. Under these conditions, \(x'\) denotes an estimate of emitter location. The location estimate can be obtained from the least squares solution of

\[
A'x' = b',
\]

(5)

where \(A'\) and \(b'\) are defined as

\[
A' = \begin{bmatrix}
\sin (\phi_1 + \delta \phi_1) & -\cos (\phi_1 + \delta \phi_1) \\
\sin (\phi_2 + \delta \phi_2) & -\cos (\phi_2 + \delta \phi_2) \\
\vdots & \vdots \\
\sin (\phi_N + \delta \phi_N) & -\cos (\phi_N + \delta \phi_N)
\end{bmatrix},
\]

(6)

\[
b' = \begin{bmatrix}
x_1 \sin (\phi_1 + \delta \phi_1) - y_1 \cos (\phi_1 + \delta \phi_1) \\
x_2 \sin (\phi_2 + \delta \phi_2) - y_2 \cos (\phi_2 + \delta \phi_2) \\
\vdots \\
x_N \sin (\phi_N + \delta \phi_N) - y_N \cos (\phi_N + \delta \phi_N)
\end{bmatrix}.
\]

(7)

The normal equation of (7) is given by

\[
A'^HA'x' = A'^Hb'.
\]

(8)

The location estimate is given by

\[
x' = (A'^HA')^{-1}A'^Hb'.
\]

(9)

3. Approximation of \(A'^HA'\) and \(A'^Hb'\)

From (6) and (7), it is easy to show that the entries of \(A'^HA'\) and \(A'^Hb'\) can be expressed as

\[
(A'^HA')_{11} = \sum_{i=1}^{N} \sin^2(\phi_i + \delta \phi_i),
\]

\[
(A'^HA')_{12} = (A'^HA')_{21} = -\frac{1}{2} \sum_{i=1}^{N} \sin 2(\phi_i + \delta \phi_i),
\]
where \((A^{H}A)^{(k)}\) and \((A^{H}b)^{(k)}\) denote the entry at the \(i\)th row and the \(j\)th column of \(A^{H}A\) and \(A^{H}b\), respectively. Note that \(A^{H}A\) is a 2 \(\times\) 2 matrix and that \(A^{H}b\) is a 2 \(\times\) 1 matrix.

Let \((A^{H}A)^{(k)}\) and \((A^{H}b)^{(k)}\) denote the approximations of \(A^{H}A\) and \(A^{H}b\) based on the \(k\)th-order Taylor series expansion, respectively:

\[
A^{H}A^{(k)} = (A^{H}A)^{(k)} = A^{H}A + \delta(A^{H}A)^{(k)},
\]

\[
A^{H}b^{(k)} = (A^{H}b)^{(k)} = A^{H}b + \delta(A^{H}b)^{(k)},
\]

where \(\delta(A^{H}A)^{(k)}\) and \(\delta(A^{H}b)^{(k)}\) are defined from

\[
\delta(A^{H}A)^{(k)} = (A^{H}A)^{(k)} - A^{H}A,
\]

\[
\delta(A^{H}b)^{(k)} = (A^{H}b)^{(k)} - A^{H}b.
\]

If the first-order \(K\)-approximation, corresponding to \(k = 1\), is applied, (11) and (12) can be written as

\[
A^{H}A^{(1)} = A^{H}A + \delta(A^{H}A)^{(1)},
\]

\[
A^{H}b^{(1)} = A^{H}b + \delta(A^{H}b)^{(1)}.
\]

The explicit expressions of \(\delta(A^{H}A)^{(1)}\) and \(\delta(A^{H}b)^{(1)}\) based on the first-order Taylor series are derived in [16]

\[
\delta(A^{H}A)^{(1)} = \sum_{i=1}^{N} \delta \phi\_i \sin 2 \phi\_i,
\]

\[
\delta(A^{H}A)^{(1)} = -\sum_{i=1}^{N} \delta \phi\_i \cos 2 \phi\_i,
\]

\[
\delta(A^{H}A)^{(1)} = -\sum_{i=1}^{N} \delta \phi\_i \sin 2 \phi\_i,
\]

\[
\delta(A^{H}b)^{(1)} = \sum_{i=1}^{N} (x_i \delta \phi\_i \sin 2 \phi\_i - y_i \delta \phi\_i \cos 2 \phi\_i),
\]

\[
\delta(A^{H}b)^{(1)} = \sum_{i=1}^{N} (-x_i \delta \phi\_i \cos 2 \phi\_i + y_i \delta \phi\_i \sin 2 \phi\_i).
\]

The explicit expressions of \(\delta(A^{H}A)^{(2)}\) and \(\delta(A^{H}b)^{(2)}\) and \(\delta(A^{H}b)^{(3)}\) are derived in Appendices A and B, where \(k = 2, 3\) denotes that the second- and third-order Taylor expansion has been adopted.

### 4. Error Bound for the \(K\)-Approximation

In this section, we describe the error bound due to various orders of \(K\)-approximation for each of the entries of \((A^{H}A)^{(k)}\) and \((A^{H}b)^{(k)}\). In Table 1, all the terms of Taylor series expansion for each entries of \((A^{H}A)^{(k)}\) and \((A^{H}b)^{(k)}\) are tabulated. Table 2 tabulates the first-order Taylor series, the second-order Taylor series, the third-order Taylor series, and the associated error bounds for each entry of \((A^{H}A)^{(k)}\)

Note that the upper bound of absolute value of the error between the original function value and the \(n\)th-order Taylor series is given by the \(n + 1\)th-order term of the Taylor series expansion. For example, the difference between the original function value and the first-order Taylor series is given by the second-order term of the Taylor series expansion, which is described in the second row of Table 2. The corresponding error bounds for the second-order Taylor series and the third-order Taylor series are given in the third row and the fourth row of Table 2, respectively.

Similarly, Tables 3–6 tabulate the same quantities for \((A^{H}A)^{(k)}\), \((A^{H}b)^{(k)}\), \((A^{H}A)^{(k)}\), and \((A^{H}b)^{(k)}\)

### 5. Approximation of \(x^{(k)}\)

Substituting (11) and (12) in (7) results in

\[
(x^{(k)}) = (A^{H}A + \delta(A^{H}A)^{(k)})^{-1}(A^{H}b + \delta(A^{H}b)^{(k)}).
\]

\[
x^{(k,j)} = x^{(k)} + (A^{H}A)^{-1}(A^{H}b)^{(k)}(A^{H}A)^{-1}(A^{H}b)^{(k)}x^{(k)} - x^{(k)}).
\]
Table 1: Explicit expressions of each term of the Taylor series expansion.

<table>
<thead>
<tr>
<th></th>
<th>Constant</th>
<th>First-order term</th>
<th>Second-order term</th>
<th>Third-order term</th>
<th>Fourth-order term</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (A^{TH}A')_{11}^{(k)} )</td>
<td>[ \sum_{i=1}^{N} \sin^2 \phi_i ]</td>
<td>[ \sum_{i=1}^{N} \sin 2 \phi_i \delta \phi_i ]</td>
<td>[ \sum_{i=1}^{N} \cos 2 \phi_i \delta \phi_i ]</td>
<td>[ -\frac{2}{3} \sum_{i=1}^{N} \sin 2 \phi_i \delta \phi_i^3 ]</td>
<td>[ \frac{1}{3} \sum_{i=1}^{N} \cos 2 \phi_i \delta \phi_i^4 ]</td>
</tr>
<tr>
<td>( (A^{TH}A')_{12}^{(k)} )</td>
<td>[ -\frac{1}{2} \sum_{i=1}^{N} \sin 2 \phi_i ]</td>
<td>[ -\frac{1}{2} \sum_{i=1}^{N} \cos 2 \phi_i \delta \phi_i ]</td>
<td>[ -\frac{1}{2} \sum_{i=1}^{N} \sin 2 \phi_i \delta \phi_i ]</td>
<td>[ \frac{2}{3} \sum_{i=1}^{N} \cos 2 \phi_i \delta \phi_i^3 ]</td>
<td>[ \frac{1}{3} \sum_{i=1}^{N} \sin 2 \phi_i \delta \phi_i^4 ]</td>
</tr>
<tr>
<td>( (A^{TH}A')_{22}^{(k)} )</td>
<td>[ \sum_{i=1}^{N} \cos^2 \phi_i ]</td>
<td>[ -\sum_{i=1}^{N} \sin 2 \phi_i \delta \phi_i ]</td>
<td>[ -\sum_{i=1}^{N} \cos 2 \phi_i \delta \phi_i ]</td>
<td>[ \frac{2}{3} \sum_{i=1}^{N} \sin 2 \phi_i \delta \phi_i^3 ]</td>
<td>[ \frac{1}{3} \sum_{i=1}^{N} \cos 2 \phi_i \delta \phi_i^4 ]</td>
</tr>
</tbody>
</table>

Table 2: Taylor series and error bound for \( (A^{TH}A')_{11}^{(k)} \).

<table>
<thead>
<tr>
<th></th>
<th>Constant</th>
<th>First-order term</th>
<th>Second-order term</th>
<th>Third-order term</th>
<th>Fourth-order term</th>
</tr>
</thead>
<tbody>
<tr>
<td>First-order Taylor series</td>
<td>[ \sum_{i=1}^{N} \sin^2 \phi_i + \sum_{i=1}^{N} \sin 2 \phi_i \delta \phi_i ]</td>
<td>[ \sum_{i=1}^{N} \sin 2 \phi_i \delta \phi_i ]</td>
<td>[ \sum_{i=1}^{N} \cos 2 \phi_i \delta \phi_i ]</td>
<td>[ \frac{2}{3} \sum_{i=1}^{N} \sin 2 \phi_i \delta \phi_i^3 ]</td>
<td>[ \frac{1}{3} \sum_{i=1}^{N} \cos 2 \phi_i \delta \phi_i^4 ]</td>
</tr>
<tr>
<td>Second-order Taylor series</td>
<td>[ \sum_{i=1}^{N} \sin^2 \phi_i + \sum_{i=1}^{N} \sin 2 \phi_i \delta \phi_i + \sum_{i=1}^{N} \cos 2 \phi_i \delta \phi_i^2 ]</td>
<td>[ \sum_{i=1}^{N} \sin 2 \phi_i \delta \phi_i ]</td>
<td>[ \sum_{i=1}^{N} \cos 2 \phi_i \delta \phi_i ]</td>
<td>[ \frac{2}{3} \sum_{i=1}^{N} \sin 2 \phi_i \delta \phi_i^3 ]</td>
<td>[ \frac{1}{3} \sum_{i=1}^{N} \cos 2 \phi_i \delta \phi_i^4 ]</td>
</tr>
<tr>
<td>Third-order Taylor series</td>
<td>[ \sum_{i=1}^{N} \sin^2 \phi_i + \sum_{i=1}^{N} \sin 2 \phi_i \delta \phi_i + \sum_{i=1}^{N} \cos 2 \phi_i \delta \phi_i^2 ]</td>
<td>[ \sum_{i=1}^{N} \sin 2 \phi_i \delta \phi_i ]</td>
<td>[ \sum_{i=1}^{N} \cos 2 \phi_i \delta \phi_i ]</td>
<td>[ \frac{2}{3} \sum_{i=1}^{N} \sin 2 \phi_i \delta \phi_i^3 ]</td>
<td>[ \frac{1}{3} \sum_{i=1}^{N} \cos 2 \phi_i \delta \phi_i^4 ]</td>
</tr>
</tbody>
</table>

Table 3: Taylor series and error bound for \( (A^{TH}A')_{12}^{(k)} \).

<table>
<thead>
<tr>
<th></th>
<th>Constant</th>
<th>First-order term</th>
<th>Second-order term</th>
<th>Third-order term</th>
<th>Fourth-order term</th>
</tr>
</thead>
<tbody>
<tr>
<td>First-order Taylor series</td>
<td>[ -\frac{1}{2} \sum_{i=1}^{N} \sin 2 \phi_i ]</td>
<td>[ -\frac{1}{2} \sum_{i=1}^{N} \cos 2 \phi_i \delta \phi_i ]</td>
<td>[ \sum_{i=1}^{N} \sin 2 \phi_i \delta \phi_i ]</td>
<td>[ \sum_{i=1}^{N} \cos 2 \phi_i \delta \phi_i^2 ]</td>
<td>[ \frac{2}{3} \sum_{i=1}^{N} \cos 2 \phi_i \delta \phi_i^3 ]</td>
</tr>
<tr>
<td>Second-order Taylor series</td>
<td>[ -\frac{1}{2} \sum_{i=1}^{N} \sin 2 \phi_i ]</td>
<td>[ -\frac{1}{2} \sum_{i=1}^{N} \cos 2 \phi_i \delta \phi_i ]</td>
<td>[ -\frac{1}{2} \sum_{i=1}^{N} \sin 2 \phi_i \delta \phi_i ]</td>
<td>[ \frac{2}{3} \sum_{i=1}^{N} \sin 2 \phi_i \delta \phi_i^3 ]</td>
<td>[ \frac{1}{3} \sum_{i=1}^{N} \cos 2 \phi_i \delta \phi_i^4 ]</td>
</tr>
<tr>
<td>Third-order Taylor series</td>
<td>[ -\frac{1}{2} \sum_{i=1}^{N} \sin 2 \phi_i ]</td>
<td>[ -\frac{1}{2} \sum_{i=1}^{N} \cos 2 \phi_i \delta \phi_i ]</td>
<td>[ -\frac{1}{2} \sum_{i=1}^{N} \sin 2 \phi_i \delta \phi_i ]</td>
<td>[ \frac{2}{3} \sum_{i=1}^{N} \sin 2 \phi_i \delta \phi_i^3 ]</td>
<td>[ \frac{1}{3} \sum_{i=1}^{N} \cos 2 \phi_i \delta \phi_i^4 ]</td>
</tr>
</tbody>
</table>

6. Analytic Expression of the MSE of the Location Estimate

In this section, we derive explicit closed-form expressions of the mean-squared error (MSE) of the location estimates in (18). Let \( x_T \) and \( y_T \) denote the x-coordinate and y-coordinate of \( \mathbf{x}_T \), respectively:

\[
\mathbf{x}_T = [ x_T \ y_T ]^T
\]  
(19)

Similarly, \( x^{(k=1)} \) and \( y^{(k=1)} \) denote the x-coordinate
Similarly, the distance between \( x' \) and \( x_T \) is given by
\[
d(x', x_T) = \| x' - x_T \|_2.
\] (21)

Similarly, the distance between \( x^{(k,j=1)} \) and \( x_T \) is written as
\[
d(x^{(k,j=1)}, x_T) = \| x^{(k,j=1)} - x_T \|_2.
\] (22)

Substituting (18) in (22) yields
\[
\| x^{(k,j=1)} - x_T \|_2 = \| (A^H A)^{-1} (\delta (A^H b) - \delta (A^H A) x_T) \|_2.
\] (23)

From (23), the MSE of the location estimate is
\[
E \left( \| x^{(k,j=1)} - x_T \|_2^2 \right) = E \left( \| (A^H A)^{-1} (\delta (A^H b) - \delta (A^H A) x_T) \|_2^2 \right).
\]

Let \( D \) and \( H \) be defined as
\[
D \equiv \delta (A^H b) - \delta (A^H A) x_T
\]
\[
= \begin{bmatrix}
\delta (A^H b)^{(k)} - \delta (A^H A) x_T \\
\delta (A^H b)^{(k)} - \delta (A^H A) x_T
\end{bmatrix},
\]

where \( D_i \) denote the \( i \)th entry of \( D \) and \( H_{ij} \) denote the entry at the \( i \)th row and \( j \)th column of \( H \), respectively.

\[
E(D_1^2) = E \left( \| \delta (A^H b) \|_2^2 \right) + x_T E \left( \| \delta (A^H A) \|_2^2 \right)
\]
\[
+ y_T E \left( \| \delta (A^H A) \|_2^2 \right)
\]
\[
+ 2x_T y_T E \left( \delta (A^H b)^{(k)} \| \delta (A^H A) \|_2^2 \right)
\]
\[
+ 2y_T E \left( \delta (A^H b)^{(k)} \| \delta (A^H A) \|_2^2 \right),
\]

Table 4: Taylor series and error bound for \( (A^H A)_2^{(k)} \).

<table>
<thead>
<tr>
<th>Constant term</th>
<th>First-order term</th>
<th>Second-order term</th>
<th>Third-order term</th>
<th>Fourth-order term</th>
<th>Error bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sum_{i=1}^{N} \cos^2 \phi_i )</td>
<td>( -\sum_{i=1}^{N} \sin 2\phi_i \delta \phi_i )</td>
<td>( -\sum_{i=1}^{N} \cos 2\phi_i \delta \phi_i^2 )</td>
<td>( \frac{2}{3} \sum_{i=1}^{N} \sin 2\phi_i \delta \phi_i^3 )</td>
<td>( \frac{1}{3} \sum_{i=1}^{N} \cos 2\phi_i \delta \phi_i^4 )</td>
<td>( -\sum_{i=1}^{N} \cos 2\phi_i \delta \phi_i^2 )</td>
</tr>
</tbody>
</table>
Table 5: Taylor series and error bound for \((A^Hb)^{(k)}\).

<table>
<thead>
<tr>
<th>Term</th>
<th>Expression</th>
<th>Error bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>[\sum_{i=1}^{N} \left( x_i \sin^2 \phi_i - \frac{y_i}{2} \sin 2\phi_i \right)]</td>
<td>[\sum_{i=1}^{N} \left( x_i \cos 2\phi_i \right) \delta \phi_i^2]</td>
</tr>
<tr>
<td>First-order term</td>
<td>[\sum_{i=1}^{N} \left( x_i \sin \phi_i - \frac{y_i}{2} \sin 2\phi_i \right)\delta \phi_i + \left( x_i \cos 2\phi_i + y_i \sin 2\phi_i \right) \delta \phi_i^2]</td>
<td>[\frac{2}{3} \sum_{i=1}^{N} \left( -y_i \cos 2\phi_i \right) \delta \phi_i^3]</td>
</tr>
<tr>
<td>Second-order term</td>
<td>[\sum_{i=1}^{N} \left( x_i \sin \phi_i - \frac{y_i}{2} \sin 2\phi_i \right)\delta \phi_i + \left( x_i \cos 2\phi_i + y_i \sin 2\phi_i \right) \delta \phi_i^2]</td>
<td>[\frac{2}{3} \sum_{i=1}^{N} \left( -y_i \cos 2\phi_i \right) \delta \phi_i^3]</td>
</tr>
<tr>
<td>Third-order term</td>
<td>[\sum_{i=1}^{N} \left( x_i \sin \phi_i - \frac{y_i}{2} \sin 2\phi_i \right)\delta \phi_i + \left( x_i \cos 2\phi_i + y_i \sin 2\phi_i \right) \delta \phi_i^2 + \frac{2}{3} \sum_{i=1}^{N} \left( -x_i \sin 2\phi_i + y_i \cos 2\phi_i \right) \delta \phi_i^3]</td>
<td>[\frac{2}{3} \sum_{i=1}^{N} \left( -y_i \cos 2\phi_i \right) \delta \phi_i^3]</td>
</tr>
</tbody>
</table>

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Table 6: Taylor series and error bound for $(A^{H}b)^{(k)}$. 

<table>
<thead>
<tr>
<th>Term</th>
<th>First-order Taylor series</th>
<th>Second-order Taylor series</th>
<th>Third-order Taylor series</th>
<th>Fourth-order Taylor series</th>
<th>Error bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>$-N \sum_{i=1}^{N} \left( \frac{x_i}{2} \sin 2\phi_i + y_i \cos^2 \phi_i \right)$</td>
<td>$-N \sum_{i=1}^{N} \left( x_i \cos 2\phi_i + y_i \sin 2\phi_i \right) \delta \phi_i$</td>
<td>$N \sum_{i=1}^{N} \left( -y_i \cos 2\phi_i \right) \delta \phi_i^2$</td>
<td>$-1 \sum_{i=1}^{N} \left( -y_i \cos 2\phi_i \right) \delta \phi_i^4$</td>
<td>$N \sum_{i=1}^{N} \left( x_i \sin 2\phi_i \right) \delta \phi_i^2$</td>
</tr>
<tr>
<td>First-order term</td>
<td>$-N \sum_{i=1}^{N} \left( \frac{x_i}{2} \sin 2\phi_i - y_i \cos^2 \phi_i \right)$</td>
<td>$-N \sum_{i=1}^{N} \left( x_i \cos 2\phi_i + y_i \sin 2\phi_i \right) \delta \phi_i$</td>
<td>$N \sum_{i=1}^{N} \left( x_i \sin 2\phi_i \right) \delta \phi_i^2$</td>
<td>$-1 \sum_{i=1}^{N} \left( x_i \sin 2\phi_i \right) \delta \phi_i^4$</td>
<td>$N \sum_{i=1}^{N} \left( -y_i \cos 2\phi_i \right) \delta \phi_i^2$</td>
</tr>
<tr>
<td>Second-order term</td>
<td>$-N \sum_{i=1}^{N} \left( \frac{x_i}{2} \sin 2\phi_i - y_i \cos^2 \phi_i \right)$</td>
<td>$-\frac{2}{3} N \sum_{i=1}^{N} \left( x_i \cos 2\phi_i + y_i \sin 2\phi_i \right) \delta \phi_i^2$</td>
<td>$N \sum_{i=1}^{N} \left( x_i \sin 2\phi_i \right) \delta \phi_i^2 + \frac{2}{3} N \sum_{i=1}^{N} \left( x_i \cos 2\phi_i + y_i \sin 2\phi_i \right) \delta \phi_i^3$</td>
<td>$-1 \sum_{i=1}^{N} \left( x_i \sin 2\phi_i \right) \delta \phi_i^4$</td>
<td>$N \sum_{i=1}^{N} \left( -y_i \cos 2\phi_i \right) \delta \phi_i^2$</td>
</tr>
</tbody>
</table>
Table 7: Summary of the performance analysis of the AOA-based localization algorithm using the LS approach.

<table>
<thead>
<tr>
<th>Data</th>
<th>Estimate of emitter location</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^H A A^H b$</td>
<td>$x_r = (A^H A)^{-1} A^H b$</td>
</tr>
<tr>
<td>$A^H A', A'^H b'$</td>
<td>$x' = (A'^H A')^{-1} A'^H b'$</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
E(D_1 D_2) &= E \left( \delta(A^H b)^{(k)} \right) + x_r E \left( \delta(A^H A)^{(k)} \right) \\
&\quad - x_T E \left( \delta(A^H b)^{(k)} \right) \delta(A^H A)^{(k)}_{21} \\
&\quad - y_T E \left( \delta(A^H b)^{(k)} \right) \delta(A^H A)^{(k)}_{22} \\
&\quad - x_T E \left( \delta(A^H b)^{(k)} \right) \delta(A^H A)^{(k)}_{11} \\
&\quad - y_T E \left( \delta(A^H b)^{(k)} \right) \delta(A^H A)^{(k)}_{12} \\
&\quad + x_T^2 E \left( \delta(A^H A)^{(k)}_{11} \right) \delta(A^H A)^{(k)}_{21} \\
&\quad + x_T y_T E \left( \delta(A^H A)^{(k)}_{11} \right) \delta(A^H A)^{(k)}_{22} \\
&\quad + x_T y_T E \left( \delta(A^H A)^{(k)}_{12} \right) \delta(A^H A)^{(k)}_{21} \\
&\quad + y_T^2 E \left( \delta(A^H A)^{(k)}_{12} \right) \delta(A^H A)^{(k)}_{22}, \\

E(D_2^2) &= E \left( \delta(A^H b)^{(k)} \right)^2 + x_T E \left( \delta(A^H A)^{(k)} \right)^2 \\
&\quad + y_T^2 E \left( \delta(A^H A)^{(k)} \right)^2 \\
&\quad + 2 x_T y_T E \left( \delta(A^H A)^{(k)}_{21} \right) \delta(A^H A)^{(k)}_{22} \\
&\quad - 2 x_T E \left( \delta(A^H b)^{(k)} \right) \delta(A^H A)^{(k)}_{21} \\
&\quad - 2 y_T E \left( \delta(A^H b)^{(k)} \right) \delta(A^H A)^{(k)}_{22}, \quad (30)
\end{align*}
\]

where the mean values in (28), (29), and (30) are derived in Appendices D, E, and F. Note that the first-order Taylor expansion, the second-order Taylor expansion, and third-order Taylor series have been employed in Appendices D, E, and F, respectively.

The summary of the localization algorithm is tabulated in Table 7.

7. Results and Discussion

Trajectory of sensor locations is given in Figure 1.

Simulation parameters are as follows:
- Speed of sensor: 0.280 km/sec
- Sampling interval: 2 sec
- Number of sensor locations: 100
- Number of Monte Carlo simulation: $S = 1000$

The empirical MSE of $x$, $x^{(k)}$, and $x^{(k=1)}$ is defined as

\[
\begin{align*}
\text{Simulation } E \left( \left\| x_r' - x_r \right\|^2 \right) \\
&= \frac{1}{S} \sum_{s=1}^{S} \left( \left| x_r(s) - x_T \right|^2 + \left| y_r(s) - y_T \right|^2 \right), \\
\text{Simulation } E \left( \left\| x^{(k)} - x_T \right\|^2 \right) \\
&= \frac{1}{S} \sum_{s=1}^{S} \left( \left| x^{(k)}(s) - x_T \right|^2 + \left| y^{(k)}(s) - y_T \right|^2 \right), \\
\text{Simulation } E \left( \left\| x^{(k=1)} - x_T \right\|^2 \right) \\
&= \frac{1}{S} \sum_{s=1}^{S} \left( \left| x^{(k=1)}(s) - x_T \right|^2 + \left| y^{(k=1)}(s) - y_T \right|^2 \right), \quad (31)
\end{align*}
\]

Figure 1: Geometry of the localization.
where the lower script \((s)\) denotes the estimate associated with the \(s\)th repetition out of \(S\) repetitions.

Similarly, the empirical MSEs of \(\mathbf{x}^{(k=2)}\), \(\mathbf{x}^{(k=2,l=1)}\), \(\mathbf{x}^{(k=3)}\), and \(\mathbf{x}^{(k=3,l=1)}\) are given by

\[
\text{Simulation} \ E \left( \left\| \mathbf{x}^{(k=2)} - \mathbf{x}_T \right\|^2 \right) = \frac{1}{S} \sum_{s=1}^{S} \left( \left\| \mathbf{x}^{(k=2)}(s) - \mathbf{x}_T \right\|^2 + \left\| \mathbf{y}^{(k=2)}(s) - \mathbf{y}_T \right\|^2 \right),
\]

(34)

\[
\text{Simulation} \ E \left( \left\| \mathbf{x}^{(k=2,l=1)} - \mathbf{x}_T \right\|^2 \right) = \frac{1}{S} \sum_{s=1}^{S} \left( \left\| \mathbf{x}^{(k=2,l=1)}(s) - \mathbf{x}_T \right\|^2 + \left\| \mathbf{y}^{(k=2,l=1)}(s) - \mathbf{y}_T \right\|^2 \right),
\]

(35)

\[
\text{Simulation} \ E \left( \left\| \mathbf{x}^{(k=3,l=1)} - \mathbf{x}_T \right\|^2 \right) = \frac{1}{S} \sum_{s=1}^{S} \left( \left\| \mathbf{x}^{(k=3,l=1)}(s) - \mathbf{x}_T \right\|^2 + \left\| \mathbf{y}^{(k=3,l=1)}(s) - \mathbf{y}_T \right\|^2 \right),
\]

(36)

\[
\text{Simulation} \ E \left( \left\| \mathbf{x}^{(k=3)} - \mathbf{x}_T \right\|^2 \right) = \frac{1}{S} \sum_{s=1}^{S} \left( \left\| \mathbf{x}^{(k=3)}(s) - \mathbf{x}_T \right\|^2 + \left\| \mathbf{y}^{(k=3)}(s) - \mathbf{y}_T \right\|^2 \right),
\]

(37)

\[\mathbf{x}^{(k)}\text{ in (31) is defined in (9), } \mathbf{x}^{(k=1)}\text{ and } \mathbf{x}^{(k=1,l=1)}\text{ in (32) and (33) are given by (17) and (18) with } k = 1. \text{ Similarly, } \mathbf{x}^{(k=2)}\text{ and } \mathbf{x}^{(k=2,l=1)}\text{ in (34) and (35) are given by (17) and (18) with } k = 2. \text{ Similarly, } \mathbf{x}^{(k=3)}\text{ and } \mathbf{x}^{(k=3,l=1)}\text{ in (36) and (37) are given by (17) and (18) with } k = 3.\]

The results with \(\mathbf{x}^{(k=1,l=1)}\) can be calculated using (9) and (18), respectively. In Figures 2(a)–2(c), we illustrate how the
location estimates are distributed with $k = 1$, $k = 2$, and $k = 3$ for the linear trajectory. It is clear that $x^{(k=2,l=1)}$ is more accurate than $x^{(k=1,l=1)}$ and that $x^{(k=3,l=1)}$ is more accurate than $x^{(k=2,l=1)}$. Similarly, $x'$ is closer to $x''$ than $x^{(k=1,l=1)}$ and $x^{(k=3,l=1)}$ is closer to $x'$ than $x^{(k=2,l=1)}$. Similar observations can be made in Figure 3, where the results for the circular trajectory have been illustrated.

Linear trajectory in Figure 1(a) is considered to get actual error and error bound in Figure 4. In Figure 4(a), the actual errors for $(A^H A')_{11}$ and their error bounds associated with the first-order Taylor series for 50 repetitions are illustrated. Note that the $x$-axis represents each independent trial. It is clearly shown that the errors for all the cases are actually smaller than the error bounds. Figure 4(b) represents corresponding results for $(A^H b')_{1}$, respectively.

The results validating the derived MSEs are illustrated in Figures 5 and 6. The results in Figures 5 and 6 correspond to the linear trajectory in Figure 1(a) and the circular trajectory in Figure 1(b), respectively.

Standard deviations of AOA estimation error are set to 10 logarithmically equally spaced values between 0.01 and 10 degrees. More specifically, the standard deviations are approximately given by $0.01, 0.0215, 0.0464, 0.1, 0.2154, 0.4642, 1.0, 2.1544, 4.6416, 10$.

The MSEs with respect to the standard derivation of the LOB error are illustrated in Figures 5(a) and 5(b) for the linear trajectory. The results in Figure 5(a) with 'Simulation $E(||x' - x||^2)$', 'Simulation $E(||x^{(k=1)} - x||^2)$', 'Simulation $E(||x^{(k=2,l=1)} - x||^2)$', and 'Analytic $E(||x^{(k=1,l=1)} - x||^2)$' are obtained from (31), (32), (33) and (27), respectively.

Figure 3: Distribution of estimates for the circular trajectory: (a) first-order $K$-approximation, (b) second-order $K$-approximation, and (c) third-order $K$-approximation.
It should be noted that \( \text{Simulation } E(\| \mathbf{x}^{(k=1,j=1)} - \mathbf{x} \|^2) \)' and \( \text{'Analytic } E(\| \mathbf{x}^{(k=1,j=1)} - \mathbf{x} \|^2) \)' show excellent agreements, which validates (18) with \( k = 1 \). Therefore, we can use analytically derived expression \( \text{'Analytic } E(\| \mathbf{x}^{(k=1,j=1)} - \mathbf{x} \|^2) \)' to see how \( \text{Simulation } E(\| \mathbf{x} - \mathbf{x} \|^2) \)'.

The same observations can be made in Figure 5(b). The results in Figure 5(b) with \( \text{'Simulation } E(\| \mathbf{x} - \mathbf{x} \|^2) \)' and \( \text{'Analytic } E(\| \mathbf{x} - \mathbf{x} \|^2) \)' show excellent agreements, which validates (18) with \( k = 1 \). Therefore, we can use analytically derived expression \( \text{'Analytic } E(\| \mathbf{x}^{(k=1,j=1)} - \mathbf{x} \|^2) \)' to see how \( \text{Simulation } E(\| \mathbf{x} - \mathbf{x} \|^2) \)'.

The same observations can be made in Figure 5(b). The results in Figure 5(b) with \( \text{'Simulation } E(\| \mathbf{x} - \mathbf{x} \|^2) \)' and \( \text{'Analytic } E(\| \mathbf{x} - \mathbf{x} \|^2) \)' show excellent agreements, which validates (18) with \( k = 1 \). Therefore, we can use analytically derived expression \( \text{'Analytic } E(\| \mathbf{x}^{(k=1,j=1)} - \mathbf{x} \|^2) \)' to see how \( \text{Simulation } E(\| \mathbf{x} - \mathbf{x} \|^2) \)'.
Figure 5: MSE of target location estimate for the linear trajectory: (a) first-order $K$-approximation, (b) second-order $K$-approximation, and (c) third-order $K$-approximation.

Figure 6: MSE of target location estimate for the circular trajectory: (a) first-order $K$-approximation, (b) second-order $K$-approximation, and (c) third-order $K$-approximation.
The error due to $K$-approximation can be made smaller by adopting the second-order $K$-approximation, which is illustrated in Figure 6(c). In Figure 6(c), it is shown that 'Simulation $E(\|x'-x\|^2)$' and 'Simulation $E(\|x^{(k=3)}-x\|^2)$' show agreement for all standard deviation of AOA estimation, which is quite different from 'Simulation $E(\|x'-x\|^2)$' and 'Simulation $E(\|x^{(k=1)}-x\|^2)$' in Figure 6(a).

8. Conclusion

Performance analysis of AOA-based localization is considered in this paper. Monte Carlo-based performance analysis is computationally very intensive, especially for a large number of repetitions. Closed-form expression of the mean-squared error (MSE) of location estimate has been derived, and the validity is shown in the numerical results. The usefulness of the derivation lies in the fact that the MSE can be analytically obtained in a closed-form without computationally intensive Monte Carlo simulation. Since error due to $K$-approximation and $L$-approximation is highly dependent on the noise variance, the scheme is more useful for high SNR. To reduce the error due to $K$-approximation and $L$-approximation, higher order Taylor series can be adopted to reduce error at the expense of more computation in calculating the analytic MSE for taking higher order terms.

Appendix

A. Second-Order Approximation of $A^H A'$ and $A^H b'$

The second-order Taylor expansion-based approximation of the entries of $A^H A'$ and $A^H b'$ is presented.

$$A^H A'_{11} = \sum_{i=1}^{N} \sin^2(\phi_i + \delta \phi_i) = \sum_{i=1}^{N} \frac{1}{2} (1 - \cos (2\phi_i + 2\delta \phi_i))$$

$$= \sum_{i=1}^{N} \frac{1}{2} (1 - \cos 2\phi_i + \cos 2\phi_i + \sin 2\phi_i - \sin 2\phi_i)$$

$$\approx 2 \sin^2 \phi_i \cos 2\phi_i = \frac{(2\delta \phi_i)^2}{2} \sin 2\phi_i = 2\delta \phi_i$$

$$A^H b'_{11} = \sum_{i=1}^{N} (\sin^2 \phi_i) \delta,$$

$$A^H A'_{k=2} = \sum_{i=1}^{N} (\delta \phi_i^2 \cos 2\phi_i + \delta \phi_i \sin 2\phi_i).$$

(A.1)
\[
\left( A^H A' \right)_{12} = \left( A^H A' \right)_{21} = -\frac{1}{2} \sum_{i=1}^{N} \sin 2(\phi_i + \delta \phi_i) \\
= \sum_{i=1}^{N} \left( -\frac{1}{2} \sin 2\phi_i \cos 2\phi_i + \cos 2\phi_i \sin 2\delta \phi_i \right) \\
= \frac{N}{2} \left( \cos 2\delta \phi_i = \frac{(2\delta \phi_i)^2}{2} \sin 2\delta \phi_i = 2\delta \phi_i \right) \\
= \sum_{i=1}^{N} \left( -\frac{1}{2} \sin 2\phi_i - \frac{N}{2} \sum_{i=1}^{N} \frac{\cos 2\phi_i}{2} \right) \\
= \left. A^H A + \delta(A^H A) \right|_{12}^{(k-2)} , \\
\left( A^H A \right)_{22} = \sum_{i=1}^{N} \left( \cos^2 \phi_i \right) , \\
\delta(A^H A)_{22} = \sum_{i=1}^{N} \left( -\delta \phi_i^2 \cos 2\phi_i - \delta \phi_i \sin 2\phi_i \right) ,
\]

\[
\left( A^H A' \right)_{22} = \sum_{i=1}^{N} \cos^2 \phi_i , \\
\left( A^H b \right)_{11} = \sum_{i=1}^{N} \left( x_i \sin^2 (\phi_i + \delta \phi_i) - \frac{y_i}{2} \sin 2(\phi_i + \delta \phi_i) \right) \\
= \sum_{i=1}^{N} \left( \left( \frac{y_i}{2} \left( 1 - \cos 2\phi_i \cos 2\phi_i - \sin 2\phi_i \sin 2\phi_i \right) \right) \\
- \frac{y_i}{2} \left( \sin 2\phi_i - 2\delta \phi_i \sin 2\phi_i + 2\delta \phi_i \cos 2\phi_i \right) \right) \\
= \sum_{i=1}^{N} \left( x_i \sin^2 \phi_i - \frac{y_i}{2} \sin 2\phi_i \right) \\
+ \sum_{i=1}^{N} \left( \delta \phi_i^2 (x_i \cos 2\phi_i + y_i \sin 2\phi_i) \right) \\
+ \delta \phi_i (x_i \sin 2\phi_i - y_i \cos 2\phi_i) \\
= \left( A^H b + \delta(A^H b) \right)_{11}^{(k-2)} , \\
\left( A^H b \right)_{22} = \sum_{i=1}^{N} \left( x_i \sin^2 \phi_i + \frac{y_i}{2} \sin 2\phi_i \right) .
\]

\[
\delta(A^H b)_{22} = \sum_{i=1}^{N} \left( \delta \phi_i^2 (x_i \cos 2\phi_i - y_i \cos 2\phi_i) \right) + \delta \phi_i (x_i \sin 2\phi_i + y_i \cos 2\phi_i) \\
= \left( A^H b + \delta(A^H b) \right)_{22}^{(k-2)} , \\
\left( A^H b \right)_{22} = \sum_{i=1}^{N} \left( \delta \phi_i (x_i \cos 2\phi_i + y_i \sin 2\phi_i) \right) , \\
\delta(A^H b)_{22} = \sum_{i=1}^{N} \left( -\delta \phi_i^2 (x_i \sin 2\phi_i - y_i \cos 2\phi_i) \right) .
\]

**B. Third-Order Approximation of A'^H and A'^H b'**

The third-order Taylor expansion-based approximation of the entries of A'^H and A'^H b' is presented.

\[
\left( A'^H A' \right)_{11} = \sum_{i=1}^{N} \sin^2 (\phi_i + \delta \phi_i) = \sum_{i=1}^{N} \left( 1 - \cos (2\phi_i + 2\delta \phi_i) \right) \\
= \sum_{i=1}^{N} \left( 1 - \cos 2\phi_i \cos 2\phi_i + \sin 2\phi_i \sin 2\phi_i \right)
\]
\[
\begin{align*}
\cdot \left( \cos 2 \delta \phi_i &= 1 - \frac{(2 \delta \phi_i)^2}{2}, \sin 2 \delta \phi_i = 2 \delta \phi_i - \frac{(2 \delta \phi_i)^3}{6} \right) \\
= \sum_{i=1}^{N} \frac{1}{2} \left( 1 - \cos 2 \phi_i + 2 \delta \phi_i^2 \cos 2 \phi_i + 2 \delta \phi_i \sin 2 \phi_i \right) \\
- \frac{4}{3} \delta \phi_i^3 \sin 2 \phi_i) &= \sum_{i=1}^{N} \sin^2 \phi_i + \sum_{i=1}^{N} \left( -\frac{2}{3} \delta \phi_i^3 \sin 2 \phi_i \right) \\
+ \delta \phi_i^2 \cos 2 \phi_i + \delta \phi_i \sin 2 \phi_i) \equiv \left( \mathbf{A}^{H} \mathbf{A} + \delta (\mathbf{A}^{H} \mathbf{A})^{(k=2)} \right)_{11}, \\
(\mathbf{A}^{H} \mathbf{A})_{11} &= \sum_{i=1}^{N} (\sin^2 \phi_i), \\
\delta (\mathbf{A}^{H} \mathbf{A})^{(k=3)}_{11} &= \sum_{i=1}^{N} \left( -\frac{2}{3} \delta \phi_i^3 \sin 2 \phi_i + \delta \phi_i^2 \cos 2 \phi_i \right) \\
+ \delta \phi_i \sin 2 \phi_i) \right), \\
(\mathbf{A}^{H} \mathbf{A}^{(k=3)})_{12} &= \sum_{i=1}^{N} \left( \frac{x_i \sin^2 \phi_i + \frac{1}{2} y_i \sin 2 \phi_i \right), \\
\delta (\mathbf{A}^{H} \mathbf{b})_{1} &= \sum_{i=1}^{N} \left( x_i \sin^2 \phi_i + \frac{1}{2} y_i \sin 2 \phi_i \right), \\
\delta (\mathbf{A}^{H} \mathbf{b})^{(k=3)}_{1} &= -\sum_{i=1}^{N} \left( \frac{2}{3} \delta \phi_i^3 \sin 2 \phi_i - y_i \cos 2 \phi_i \right) - \delta \phi_i^2 \left( x_i \sin 2 \phi_i - y_i \cos 2 \phi_i \right) \\
&- \delta \phi_i^3 \left( x_i \sin 2 \phi_i - y_i \cos 2 \phi_i \right) \right), \\
(\mathbf{A}^{H} \mathbf{A})_{22} &= \sum_{i=1}^{N} \cos^2 (\phi_i + \delta \phi_i) \\
= \sum_{i=1}^{N} \left( 1 + \cos 2 \phi_i \cos 2 \delta \phi_i - \sin 2 \phi_i \sin 2 \delta \phi_i \right) \\
\cdot \left( \cos 2 \delta \phi_i = 1 - \frac{(2 \delta \phi_i)^2}{2}, \sin 2 \delta \phi_i = 2 \delta \phi_i - \frac{(2 \delta \phi_i)^3}{6} \right) \\
= \sum_{i=1}^{N} \left( 1 + \cos 2 \phi_i \cos 2 \delta \phi_i - \sin 2 \phi_i \sin 2 \delta \phi_i \right) \\
+ \delta \phi_i^2 \cos 2 \phi_i + \delta \phi_i \sin 2 \phi_i) \equiv \left( \mathbf{A}^{H} \mathbf{A} + \delta (\mathbf{A}^{H} \mathbf{A})^{(k=3)} \right)_{22}, \\
(\mathbf{A}^{H} \mathbf{A})_{22} &= \sum_{i=1}^{N} (\cos^2 \phi_i), \\
\delta (\mathbf{A}^{H} \mathbf{A})^{(k=3)}_{22} &= \sum_{i=1}^{N} \left( \frac{2}{3} \delta \phi_i^3 \sin 2 \phi_i - \delta \phi_i^2 \cos 2 \phi_i - \delta \phi_i \sin 2 \phi_i \right) \right), \\
(\mathbf{A}^{H} \mathbf{A}^{(k=3)})_{21} &= \sum_{i=1}^{N} \left( x_i \sin^2 \phi_i + \frac{1}{2} y_i \sin 2 \phi_i \right), \\
(\mathbf{A}^{H} \mathbf{A}^{(k=3)})_{12} &= \sum_{i=1}^{N} \left( \frac{x_i \sin^2 \phi_i + \frac{1}{2} y_i \sin 2 \phi_i \right), \\
\delta (\mathbf{A}^{H} \mathbf{b})^{(k=3)}_{1} &= \sum_{i=1}^{N} \left( \frac{2}{3} \delta \phi_i^3 \sin 2 \phi_i - y_i \cos 2 \phi_i \right) - \delta \phi_i^2 \left( x_i \sin 2 \phi_i + y_i \sin 2 \phi_i \right) \\
&- \delta \phi_i \left( x_i \sin 2 \phi_i + y_i \sin 2 \phi_i \right) \right). \\
\end{align*}
\]
Comparing (C.1) and (C.2), it is clear that \( x(\varepsilon) = 0 \) in (C.4) is equal to \( x \) in (C.1):

\[
x \equiv x(\varepsilon) = 0.
\]

Using (C.5) in (C.4) results in

\[
F x + C x(\varepsilon) = 0 = f.
\]

Solving for \( x(\varepsilon) = 0 \) yields

\[
\dot{x}(\varepsilon) = 0 = C^{-1}(f - F x).
\]

First-order approximation of \( x(\varepsilon) \) based on the Taylor expansion can be written as

\[
x(\varepsilon) = x(\varepsilon) = 0 + \varepsilon x(\varepsilon) = x + C^{-1}(\varepsilon F - e F x).
\]

Comparing (C.2) and (16), \( e F \) and \( e f \) in (C.2) correspond to \( \delta(A^H A)_{11}^{(k=1)} \) and \( \delta(A^H b)_{11}^{(k=1)} \) in (16), respectively. Therefore, the approximation of \( x_{11}^{(k=1)} \) and \( x_{r(k,1)} \), is obtained from (C.3) by substituting \( e f = \delta(A^H b)^{(k)} \) and \( e F = \delta(A^H A)^{(k)} \).

**D. Mean of Entries of \( \delta(A^H A') \) and \( \delta(A^H b') \) Based on the First-Order Taylor Series**

Explicit expression of many expectation terms in (28), (29) and (30) is derived. The expectation of \( \delta \phi_j \delta \phi_j \) can be written as

\[
E(\delta \phi_j \delta \phi_j) = \begin{cases} 
\sigma_j^2, & i = j, \\
0, & \text{otherwise}. 
\end{cases}
\]

Explicit expression of the expectation of \( \delta(A^H A)_{11}^{(k=1)} \) is derived:

\[
E\left(\delta(A^H A)_{11}^{(k=1)} \right)^2 = E\left(\delta(A^H A)_{11}^{(k=1)} \right)^2 = E\left(\delta(A^H A)_{11}^{(k=1)} \delta(A^H A)_{11}^{(k=1)} \right) = E\left(\sum_{i=1}^{N} \delta \phi_i \sin 2 \phi_i \left(\sum_{j=1}^{N} \delta \phi_j \sin 2 \phi_j \right)\right) = E\left(\sum_{i=1}^{N} \delta \phi_i^2 \right) \sum_{j=1}^{N} \sin^2 2 \phi_j = \sigma_\phi^2 \sum_{i=1}^{N} \cos^2 2 \phi_i.
\]

Since the other entries can be derived similarly, we list the final expressions for the other entries:

\[
E\left(\delta(A^H A)_{12}^{(k=1)} \right)^2 = \sigma_\phi^2 \sum_{i=1}^{N} \cos^2 2 \phi_i.
\]
\[ E \left( \left( \delta (A^H A)_{22}^{(k=1)} \right)^2 \right) = \sigma^2_\delta \sum_{i=1}^{N} (x_i^2 \sin^2 2\phi_i) \]  
\[ (D.4) \]

\[ E \left( \left( \delta (A^H b)_{11}^{(k=1)} \right)^2 \right) = \sigma^2_\delta \left( \frac{1}{2} \sum_{i=1}^{N} \sin 4\phi_i \right) \]  
\[ \sigma^2_\delta \sum_{i=1}^{N} \left( \frac{1}{2} y_i \sin 4\phi_i \right) \]  
\[ (D.5) \]

\[ E \left( \left( \delta (A^H A)_{22}^{(k=1)} \right)^2 \right) = \sigma^2_\delta \sum_{i=1}^{N} \sin^2 2\phi_i \]  
\[ \sigma^2_\delta \sum_{i=1}^{N} (x_i^2 \sin^2 2\phi_i + y_i^2 \cos^2 2\phi_i - x_i y_i \sin 4\phi_i) \]  
\[ (D.6) \]

\[ E \left( \left( \delta (A^H b)_{1}^{(k=1)} \right)^2 \right) \]  
\[ \sigma^2_\delta \sum_{i=1}^{N} \left( -x_i^2 \sin^2 2\phi_i + \frac{1}{2} y_i \sin 4\phi_i \right) \]  
\[ (D.7) \]

\[ E \left( \left( \delta (A^H A)_{22}^{(k=1)} \right)^2 \right) \]  
\[ \sigma^2_\delta \sum_{i=1}^{N} \left( \frac{1}{2} x_i \sin 4\phi_i + y_i \sin^2 2\phi_i \right) \]  
\[ (D.8) \]

\[ E \left( \left( \delta (A^H A)_{11}^{(k=1)} \right) \delta (A^H b)_{1}^{(k=1)} \right) \]  
\[ -\sigma^2_\delta \sum_{i=1}^{N} \left( -x_i^2 \sin^2 2\phi_i + \frac{1}{2} y_i \sin 4\phi_i \right) \]  
\[ (D.9) \]

\[ E \left( \left( \delta (A^H A)_{11}^{(k=1)} \right) \delta (A^H b)_{2}^{(k=1)} \right) \]  
\[ -\sigma^2_\delta \sum_{i=1}^{N} \left( \frac{1}{2} x_i \sin 4\phi_i + y_i \sin^2 2\phi_i \right) \]  
\[ (D.10) \]

\[ E \left( \left( \delta (A^H A)_{22}^{(k=1)} \right) \delta (A^H b)_{1}^{(k=1)} \right) \]  
\[ -\sigma^2_\delta \sum_{i=1}^{N} \left( x_i \sin 2\phi_i - \frac{1}{2} y_i \sin 4\phi_i \right) \]  
\[ (D.11) \]

\[ E \left( \left( \delta (A^H A)_{22}^{(k=1)} \right) \delta (A^H b)_{2}^{(k=1)} \right) \]  
\[ \sigma^2_\delta \sum_{i=1}^{N} \left( \frac{1}{2} x_i \sin 4\phi_i + y_i \sin^2 2\phi_i \right) \]  
\[ (D.12) \]

\[ E \left( \left( \delta (A^H b)_{1}^{(k=1)} \right) \delta (A^H A)_{12}^{(k=1)} \right) \]  
\[ E \left( \left( \delta (A^H b)_{1}^{(k=1)} \right) \delta (A^H A)_{21}^{(k=1)} \right) \]  
\[ -\sigma^2_\delta \sum_{i=1}^{N} \left( \frac{1}{2} x_i \sin 4\phi_i - y_i \cos^2 2\phi_i \right) \]  
\[ (D.13) \]

\[ E \left( \left( \delta (A^H b)_{2}^{(k=1)} \right) \delta (A^H A)_{12}^{(k=1)} \right) \]  
\[ E \left( \left( \delta (A^H b)_{2}^{(k=1)} \right) \delta (A^H A)_{21}^{(k=1)} \right) \]  
\[ \sigma^2_\delta \sum_{i=1}^{N} \left( x_i \cos^2 2\phi_i - \frac{1}{2} y_i \sin 4\phi_i \right) \]  
\[ (D.14) \]

\[ E \left( \left( \delta (A^H A)_{1}^{(k=2)} \right)^2 \right) \]  
\[ E \left( \left( \delta (A^H A)_{1}^{(k=2)} \right)^2 \right) \]  
\[ \sigma^2_\delta \sum_{i=1}^{N} \sin^2 2\phi_i + 3\sigma^4_\delta \sum_{i=1}^{N} \cos^2 2\phi_i \]  
\[ (E.1) \]

\[ E \left( \left( \delta (A^H A)_{12}^{(k=2)} \right)^2 \right) \]  
\[ E \left( \left( \delta (A^H A)_{12}^{(k=2)} \right)^2 \right) \]  
\[ \sigma^2_\delta \sum_{i=1}^{N} \sin^2 2\phi_i + 3\sigma^4_\delta \sum_{i=1}^{N} \cos^2 2\phi_i \]  
\[ (E.2) \]

\[ E \left( \left( \delta (A^H A)_{22}^{(k=2)} \right)^2 \right) \]  
\[ E \left( \left( \delta (A^H A)_{22}^{(k=2)} \right)^2 \right) \]  
\[ \sigma^2_\delta \sum_{i=1}^{N} \sin^2 2\phi_i + 3\sigma^4_\delta \sum_{i=1}^{N} \cos^2 2\phi_i \]  
\[ (E.3) \]

\[ E \left( \left( \delta (A^H b)_{1}^{(k=2)} \right)^2 \right) \]  
\[ E \left( \left( \delta (A^H b)_{1}^{(k=2)} \right)^2 \right) \]  
\[ \sigma^2_\delta \sum_{i=1}^{N} \left( x_i \sin 2\phi_i - y_i \cos 2\phi_i \right)^2 \]  
\[ + 3\sigma^4_\delta \sum_{i=1}^{N} \left( x_i \cos 2\phi_i - y_i \sin 2\phi_i \right)^2 \]  
\[ (E.4) \]
\[ E \left( \left( \delta (A^H \mathbf{b})^{(k=2)} \right)_2 \right) \]
\[ = \sigma^2_\\delta \sum_{i=1}^{N} (-x_i \cos 2\phi_i - y_i \sin 2\phi_i)^2 \]
\[ + 3\sigma^4_\\delta \sum_{i=1}^{N} (x_i \sin 2\phi_i - y_i \cos 2\phi_i)^2 \]
\[ + \sigma^4_\\delta \sum_{i=1}^{N} \sum_{j=1}^{N} (x_i \sin 2\phi_i - y_j \cos 2\phi_j) \cdot (x_j \sin 2\phi_j - y_j \cos 2\phi_j) \]  
\[ \cdot (x_j \sin 2\phi_j - y_j \cos 2\phi_j) \] (E.5)

\[ E \left( \left( \delta (A^H \mathbf{A})^{(k=2)} \right)_{11} \left( \delta (A^H \mathbf{A})^{(k=2)} \right)_{12} \right) \]
\[ = \sigma^2_\\delta \sum_{i=1}^{N} \sin 2\phi_i \cdot (-\sin 2\phi_i) + 3\sigma^4_\\delta \sum_{i=1}^{N} \cos 2\phi_i \cdot (-\cos 2\phi_i) \]
\[ + \sigma^4_\\delta \sum_{i=1}^{N} \sum_{j=1}^{N} \cos 2\phi_i \cdot (-\cos 2\phi_j) \] (E.6)

\[ E \left( \left( \delta (A^H \mathbf{A})^{(k=2)} \right)_{12} \left( \delta (A^H \mathbf{A})^{(k=2)} \right)_{22} \right) \]
\[ = \sigma^2_\\delta \sum_{i=1}^{N} (-\cos 2\phi_i) \cdot (-\sin 2\phi_i) + 3\sigma^4_\\delta \sum_{i=1}^{N} \sin 2\phi_i \cdot (-\cos 2\phi_i) \]
\[ + \sigma^4_\\delta \sum_{i=1}^{N} \sum_{j=1}^{N} \sin 2\phi_i \cdot (-\cos 2\phi_j) \] (E.7)

\[ E \left( \left( \delta (A^H \mathbf{A})^{(k=2)} \right)_{11} \left( \delta (A^H \mathbf{b})^{(k=2)} \right) \right) \]
\[ = \sigma^2_\\delta \sum_{i=1}^{N} (\sin 2\phi_i) \cdot (x_i \sin 2\phi_i - y_i \cos 2\phi_i) \]
\[ + 3\sigma^4_\\delta \sum_{i=1}^{N} (\cos 2\phi_i) \cdot (x_i \cos 2\phi_i - y_i \sin 2\phi_i) \]
\[ + \sigma^4_\\delta \sum_{i=1}^{N} \sum_{j=1}^{N} (\cos 2\phi_i) \cdot (x_i \cos 2\phi_i - y_j \sin 2\phi_j) \] (E.9)

\[ E \left( \left( \delta (A^H \mathbf{A})^{(k=2)} \right)_{12} \left( \delta (A^H \mathbf{b})^{(k=2)} \right) \right) \]
\[ = \sigma^2_\\delta \sum_{i=1}^{N} (-\cos 2\phi_i) \cdot (x_i \sin 2\phi_i - y_i \cos 2\phi_i) \]
\[ + 3\sigma^4_\\delta \sum_{i=1}^{N} (\sin 2\phi_i) \cdot (x_i \cos 2\phi_i - y_i \sin 2\phi_i) \]
\[ + \sigma^4_\\delta \sum_{i=1}^{N} \sum_{j=1}^{N} (-\sin 2\phi_i) \cdot (x_i \sin 2\phi_i - y_j \sin 2\phi_j) \] (E.10)

\[ E \left( \left( \delta (A^H \mathbf{A})^{(k=2)} \right)_{22} \left( \delta (A^H \mathbf{b})^{(k=2)} \right) \right) \]
\[ = \sigma^2_\\delta \sum_{i=1}^{N} (-\sin 2\phi_i) \cdot (x_i \sin 2\phi_i - y_i \cos 2\phi_i) \]
\[ - 3\sigma^4_\\delta \sum_{i=1}^{N} (\cos 2\phi_i) \cdot (x_i \cos 2\phi_i - y_i \sin 2\phi_i) \]
\[ - \sigma^4_\\delta \sum_{i=1}^{N} \sum_{j=1}^{N} \cos 2\phi_i \cdot (x_i \cos 2\phi_i - y_j \sin 2\phi_i) \] (E.11)

\[ E \left( \left( \delta (A^H \mathbf{A})^{(k=2)} \right)_{12} \left( \delta (A^H \mathbf{b})^{(k=2)} \right) \right) \]
\[ = \sigma^2_\\delta \sum_{i=1}^{N} (-\cos 2\phi_i) \cdot (x_i \sin 2\phi_i - y_i \cos 2\phi_i) \]
\[ + 3\sigma^4_\\delta \sum_{i=1}^{N} (\sin 2\phi_i) \cdot (x_i \cos 2\phi_i - y_i \sin 2\phi_i) \]
\[ + \sigma^4_\\delta \sum_{i=1}^{N} \sum_{j=1}^{N} (-\sin 2\phi_i) \cdot (x_i \sin 2\phi_i - y_j \sin 2\phi_j) \] (E.12)

\[ E \left( \left( \delta (A^H \mathbf{A})^{(k=2)} \right)_{22} \left( \delta (A^H \mathbf{b})^{(k=2)} \right) \right) \]
\[ = \sigma^2_\\delta \sum_{i=1}^{N} (-\sin 2\phi_i) \cdot (x_i \sin 2\phi_i - y_i \cos 2\phi_i) \]
\[ + 3\sigma^4_\\delta \sum_{i=1}^{N} (\cos 2\phi_i) \cdot (x_i \cos 2\phi_i - y_i \sin 2\phi_i) \]
\[ + \sigma^4_\\delta \sum_{i=1}^{N} \sum_{j=1}^{N} \cos 2\phi_i \cdot (x_i \cos 2\phi_i - y_j \sin 2\phi_i) \] (E.13)

\[ E \left( \left( \delta (A^H \mathbf{A})^{(k=2)} \right)_{11} \left( \delta (A^H \mathbf{b})^{(k=2)} \right) \right) \]
\[ = \sigma^2_\\delta \sum_{i=1}^{N} (\sin 2\phi_i) \cdot (x_i \sin 2\phi_i - y_i \cos 2\phi_i) \]
\[ + 3\sigma^4_\\delta \sum_{i=1}^{N} (\cos 2\phi_i) \cdot (x_i \cos 2\phi_i - y_i \sin 2\phi_i) \]
\[ + \sigma^4_\\delta \sum_{i=1}^{N} \sum_{j=1}^{N} (\cos 2\phi_i) \cdot (x_i \cos 2\phi_i - y_j \sin 2\phi_j) \] (E.14)
\[
E\left(\left(\delta (A^H \mathbf{A})^{(k=2)}\right)\left(\delta (A^H \mathbf{b})^{(k=2)}\right)\right)
= \sigma_\phi^2 \sum_{i=1}^{N} (-\sin 2\phi_i)(-x_i \cos 2\phi_i - y_i \sin 2\phi_i)
+ 3\sigma_\phi^4 \sum_{i=1}^{N} (-\cos 2\phi_i)(x_i \sin 2\phi_i - y_i \cos 2\phi_i)
+ \sigma_\phi^4 \sum_{i=1}^{N} \sum_{j=1}^{N} (-\cos 2\phi_i)(x_i \sin 2\phi_i - y_i \cos 2\phi_i),
\]

(E.15)

\section*{F. Mean of Entries of $\delta(A^H \mathbf{A}')$ and $\delta(A^H \mathbf{b}')$ Based on the Third-Order Taylor Series}

\[
E\left(\left(\delta (A^H \mathbf{A})^{(k=3)}\right)^2\right)
= \sigma_\phi^2 \sum_{i=1}^{N} \sin^2 2\phi_i + 3\sigma_\phi^4 \sum_{i=1}^{N} \cos^2 2\phi_i
+ \sigma_\phi^4 \sum_{i=1}^{N} \sum_{j=1}^{N} \sin 2\phi_i \sin 2\phi_j + 6\sigma_\phi^4 \sum_{i=1}^{N}
- (\cos 2\phi_i) \left(\frac{2}{3} \cos 2\phi_i\right) + 15\sigma_\phi^6 \sum_{i=1}^{N} \left(\frac{2}{3} \cos 2\phi_i\right)
\left(\frac{2}{3} \cos 2\phi_i\right),
\]

(F.1)

\[
E\left(\left(\delta (A^H \mathbf{b})^{(k=3)}\right)^2\right)
= \sigma_\phi^2 \sum_{i=1}^{N} (x_i \sin 2\phi_i - y_i \cos 2\phi_i)^2
+ 3\sigma_\phi^4 \sum_{i=1}^{N} (x_i \sin 2\phi_i - y_i \cos 2\phi_i)^2
+ \sigma_\phi^4 \sum_{i=1}^{N} \sum_{j=1}^{N} (x_i \sin 2\phi_i - y_i \cos 2\phi_i)
- (x_i \cos 2\phi_i - y_i \sin 2\phi_i)
\left(\frac{2}{3} \cos 2\phi_i\right) + y_i \left(\frac{2}{3} \sin 2\phi_i\right)
\left(\frac{2}{3} \cos 2\phi_i\right).
\]

(F.2)

\[
E\left(\left(\delta (A^H \mathbf{A})^{(k=3)}\right)\left(\delta (A^H \mathbf{A})^{(k=3)}\right)\right)
= \sigma_\phi^2 \sum_{i=1}^{N} \sin 2\phi_i (-\cos 2\phi_i) + 3\sigma_\phi^4 \sum_{i=1}^{N} \cos 2\phi_i \sin 2\phi_i
+ \sigma_\phi^4 \sum_{i=1}^{N} \sum_{j=1}^{N} \cos 2\phi_i \cos 2\phi_j + 3\sigma_\phi^4 \sum_{i=1}^{N}
- (\sin 2\phi_i) \left(\frac{2}{3} \sin 2\phi_i\right) + 3\sigma_\phi^6 \sum_{i=1}^{N} \left(\frac{2}{3} \sin 2\phi_i\right)
\left(\frac{2}{3} \sin 2\phi_i\right),
\]

(F.3)
\[
E \left( \left( \delta(A^H A)^{(k-3)} \right)_{11} \right) \left( \delta(A^H A)^{(k-3)} \right)_{22} \) \\
= \sigma^2 \sum_{i=1}^{N} \sin 2\phi_i (-\sin 2\phi_i) + 3\sigma^2 \sum_{i=1}^{N} \cos 2\phi_i (-\cos 2\phi_i) \\
+ \sigma^2 \sum_{i=1}^{N} \sum_{j=1}^{N} \cos 2\phi_i (-\cos 2\phi_j) + 3\sigma^2 \sum_{i=1}^{N} \sin 2\phi_i \\
\cdot \left( \frac{2}{3} \sin 2\phi_i \right) + 3\sigma^2 \sum_{i=1}^{N} (-\sin 2\phi_i) \left( -\frac{2}{3} \sin 2\phi_i \right) \\
+ 15\sigma^2 \sum_{i=1}^{N} \left( -\frac{2}{3} \sin 2\phi_i \right) \left( \frac{2}{3} \sin 2\phi_i \right). \\
\text{(F.7)}
\]

\[
E \left( \left( \delta(A^H A)^{(k-3)} \right)_{12} \right) \left( \delta(A^H A)^{(k-3)} \right)_{22} \) \\
= \sigma^2 \sum_{i=1}^{N} (-\cos 2\phi_i) (-\sin 2\phi_i) + 3\sigma^2 \sum_{i=1}^{N} \sin 2\phi_i (-\cos 2\phi_i) \\
+ \sigma^2 \sum_{i=1}^{N} \sum_{j=1}^{N} \sin 2\phi_i (-\cos 2\phi_j) + 3\sigma^2 \sum_{i=1}^{N} (-\cos 2\phi_i) \\
\cdot \left( \frac{2}{3} \sin 2\phi_i \right) + 3\sigma^2 \sum_{i=1}^{N} (-\sin 2\phi_i) \left( \frac{2}{3} \cos 2\phi_i \right) \\
+ 15\sigma^2 \sum_{i=1}^{N} \left( \frac{2}{3} \cos 2\phi_i \right) \left( \frac{2}{3} \sin 2\phi_i \right). \\
\text{(F.8)}
\]

\[
E \left( \left( \delta(A^H A)^{(k-3)} \right)_{11} \right) \left( \delta(A^H b)^{(k-3)} \right) \\
= \sigma^2 \sum_{i=1}^{N} \sin 2\phi_i (x_i \sin 2\phi_i - y_i \cos 2\phi_i) \\
+ 3\sigma^2 \sum_{i=1}^{N} (\cos 2\phi_i) (x_i \cos 2\phi_i - y_i \sin 2\phi_i) \\
+ \sigma^2 \sum_{i=1}^{N} \sum_{j=1}^{N} (\cos 2\phi_i) (x_i \cos 2\phi_j - y_j \sin 2\phi_i) \\
+ 3\sigma^2 \sum_{i=1}^{N} (\sin 2\phi_i) \left( x_i \left( -\frac{2}{3} \sin 2\phi_i \right) + y_i \left( \frac{2}{3} \cos 2\phi_i \right) \right) \\
- 3\sigma^2 \sum_{i=1}^{N} \left( \frac{2}{3} \sin 2\phi_i \right) (x_i \sin 2\phi_i - y_i \cos 2\phi_i) \\
+ 15\sigma^2 \sum_{i=1}^{N} \left( -\frac{2}{3} \sin 2\phi_i \right) \left( x_i \left( -\frac{2}{3} \sin 2\phi_i \right) \right) \\
+ y_i \left( \frac{2}{3} \cos 2\phi_i \right). \\
\text{(F.9)}
\]
\[
E\left(\left(\delta(A^HA)_{11}^{(k-3)}\right)\left(\delta(A^Hb)_{12}^{(k-3)}\right)\right) \\
= \sigma_\delta^2 \sum_{i=1}^{N} \sin 2\phi_i (-x_i \cos 2\phi_i - y_i \sin 2\phi_i) \\
+ 3\sigma_\delta^2 \sum_{i=1}^{N} \cos 2\phi_i (x_i \sin 2\phi_i - y_i \cos 2\phi_i) \\
+ \sigma_\delta^2 \sum_{i=1}^{N} \sum_{j \neq i}^{N} \cos 2\phi_i (x_i \sin 2\phi_i - y_i \cos 2\phi_i) \\
+ 3\sigma_\delta^2 \sum_{i=1}^{N} \left( \sin 2\phi_i \left( \frac{2}{3} \cos 2\phi_i \right) + y_i \left( \frac{2}{3} \sin 2\phi_i \right) \right) \\
+ 3\sigma_\delta^2 \sum_{i=1}^{N} \left( - \frac{2}{3} \sin 2\phi_i \right) (-x_i \cos 2\phi_i - y_i \sin 2\phi_i) \\
+ 15\sigma_\delta^2 \sum_{i=1}^{N} \left( - \frac{2}{3} \sin 2\phi_i \right) \left( x_i \left( \frac{2}{3} \cos 2\phi_i \right) \right) \\
+ y_i \left( \frac{2}{3} \sin 2\phi_i \right) \right),
\]

\[
E\left(\left(\delta(A^HA)_{12}^{(k-3)}\right)\left(\delta(A^Hb)_{22}^{(k-3)}\right)\right) \\
= \sigma_\delta^2 \sum_{i=1}^{N} (-\cos 2\phi_i) (-x_i \cos 2\phi_i - y_i \sin 2\phi_i) \\
+ 3\sigma_\delta^2 \sum_{i=1}^{N} \sin 2\phi_i (x_i \sin 2\phi_i - y_i \cos 2\phi_i) \\
+ \sigma_\delta^2 \sum_{i=1}^{N} \sum_{j \neq i}^{N} \sin 2\phi_i (x_i \sin 2\phi_i - y_i \cos 2\phi_i) \\
+ 3\sigma_\delta^2 \sum_{i=1}^{N} (-\cos 2\phi_i) \left( x_i \left( \frac{2}{3} \cos 2\phi_i \right) + y_i \left( \frac{2}{3} \sin 2\phi_i \right) \right) \\
+ 3\sigma_\delta^2 \sum_{i=1}^{N} \cos 2\phi_i (-x_i \cos 2\phi_i - y_i \sin 2\phi_i) \\
+ 15\sigma_\delta^2 \sum_{i=1}^{N} \left( \frac{2}{3} \cos 2\phi_i \right) \left( x_i \left( \frac{2}{3} \cos 2\phi_i \right) \right) \\
+ y_i \left( \frac{2}{3} \sin 2\phi_i \right) \right),
\]

\[
E\left(\left(\delta(A^HA)_{22}^{(k-3)}\right)\left(\delta(A^Hb)_{12}^{(k-3)}\right)\right) \\
= \sigma_\delta^2 \sum_{i=1}^{N} (-\sin 2\phi_i) (-x_i \cos 2\phi_i - y_i \sin 2\phi_i) \\
+ 3\sigma_\delta^2 \sum_{i=1}^{N} (-\cos 2\phi_i) (x_i \sin 2\phi_i - y_i \cos 2\phi_i) \\
+ \sigma_\delta^2 \sum_{i=1}^{N} \sum_{j \neq i}^{N} (-\cos 2\phi_i) (x_i \sin 2\phi_i - y_i \cos 2\phi_i) \\
+ 3\sigma_\delta^2 \sum_{i=1}^{N} (-\sin 2\phi_i) \left( x_i \left( \frac{2}{3} \cos 2\phi_i \right) + y_i \left( \frac{2}{3} \sin 2\phi_i \right) \right) \\
+ 3\sigma_\delta^2 \sum_{i=1}^{N} \left( \frac{2}{3} \sin 2\phi_i \right) (-x_i \cos 2\phi_i - y_i \sin 2\phi_i) \\
+ 15\sigma_\delta^2 \sum_{i=1}^{N} \left( \frac{2}{3} \sin 2\phi_i \right) \left( x_i \left( \frac{2}{3} \cos 2\phi_i \right) \right) \\
+ y_i \left( \frac{2}{3} \sin 2\phi_i \right) \right) \right). \tag{F.15}
\]

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References


