ON EXISTENCE AND UNIQUENESS OF THE SOLUTION OF THE EQUATION OF MOTION FOR CONSTRAINED MECHANICAL SYSTEMS

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In this paper we provide sufficient conditions for the existence and uniqueness of the solution of the newly obtained equation of motion for constrained mechanical systems.

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Consider an unconstrained mechanical system consisting of n masses $m_1, m_2, \dots, m_n$. Its motion in an inertial Cartesian rectangular coordinate system is governed by the system of differential equations

$$M\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t), \quad (1)$$

where the $3n$ by $3n$ constant diagonal matrix M has the masses m_i in sets of three along its diagonal, and the $3n$ -vector $\mathbf{F}$ is the vector containing the components of the “given” or “impressed” forces in the three coordinate directions. Consider the point $\mathbf{x}(0) = \mathbf{x}_0$, $\dot{\mathbf{x}}(0) = \dot{\mathbf{x}}_0$. We shall assume that $\mathbf{F}$ has continuous partial derivatives in a closed bounded domain G around the point $(\mathbf{x}_0, \dot{\mathbf{x}}_0)$ and for values of t in an interval $-a \leq t \leq a$. Then the solution of equation (1) exists and is unique, locally. Hence we assume that the solution of the unconstrained equation of motion leads, locally, to a unique solution.

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We now impose further, a set of m smooth (actually C^2 is sufficient) constraints of the form

$$\varphi_i(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) = 0, \quad i = 1, 2, \dots, m, \quad (2)$$

which, upon differentiation with respect to t , yield the equation

$$A(\mathbf{x}, \dot{\mathbf{x}}, t) \ddot{\mathbf{x}} = \mathbf{b}(\mathbf{x}, \dot{\mathbf{x}}, t), \quad (3)$$

where the matrix A and the vector $\mathbf{b}$ are again continuous functions of their arguments. As is usual in mechanics, we shall assume that the initial conditions $\mathbf{x}_0, \dot{\mathbf{x}}_0$ at time $t = 0$ for the constrained system are such that equation set (2) is satisfied. Then the equation of motion of the constrained mechanical system was derived in [1] and [2], and can be expressed as

$$M\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t) + M^{1/2} (AM^{-1/2})^+ [\mathbf{b}(\mathbf{x}, \dot{\mathbf{x}}, t) - A(\mathbf{x}, \dot{\mathbf{x}}, t) M^{-1} \mathbf{F}]. \quad (4)$$

We present here two useful results related to the existence and uniqueness of the solutions of equation (4).

THEOREM 1 *Let G be a closed bounded domain in the $6n$ -dimensional space $(\mathbf{x}, \dot{\mathbf{x}})$. Consider now the closed bounded domain G_1 in the $(6n+1)$ -dimensional space determined by G and values of t in the interval $-b \leq t \leq b$. Let the point $(\mathbf{x}_0, \dot{\mathbf{x}}_0, 0)$ be an interior point of G_1 . Furthermore, let*

- (1) *$A, \mathbf{b}$ and $\mathbf{F}$ be defined, and continuous functions of their arguments, in G_1 , and,*
- (2) *the rank of A remain the same throughout G_1 .*

Then a solution $\mathbf{x}(t)$ of (4) passing through $(\mathbf{x}_0, \dot{\mathbf{x}}_0, 0)$ exists and is defined in the interval $(-h, h)$, where,

$$h = \frac{\min(D, b)}{(1 + \mu \sqrt{6n})}. \quad (5)$$

Here D is the minimum Euclidean distance from the point $(\mathbf{x}_0, \dot{\mathbf{x}}_0, 0)$ to the boundary of G_1 , and μ is the maximum absolute value in G_1 among the components of the right-hand-side vector when equation (4) is expressed as a system of first order differential equations.

Proof. Since $AM^{-1/2}$ is a continuous function of $\mathbf{x}, \dot{\mathbf{x}}$, and t , and its rank is constant in G_1 , $(AM^{-1/2})^+$ is a continuous function in the same domain and hence the right-hand side of eq. (4) becomes a continuous function of its arguments. Hence the result (see, e.g., [3]).

THEOREM 2 *If $A, \mathbf{b}$ and $\mathbf{F}$ are continuous and differentiable with respect to their arguments in G_1 and if the rank of A is constant in G_1 then eq. (4) has a unique solution in a neighborhood of the point $(\mathbf{x}_0, \dot{\mathbf{x}}_0, 0)$, and passing through it.*

Proof. Since the rank of $AM^{-1/2}$ is constant in G_1 , it is continuous in G_1 . Hence $(AM^{-1/2})^+$ is differentiable in G_1 [4]. The result then follows, (see [3]).

Changes in the rank of the matrix A occur infrequently in practical, well-modeled problems in mechanics. When they do, they can usually be averted by the use of alternative, yet equivalent, ways of specifying the constraints, and by a reparametrization of the problem through the choice of a different set of Lagrangian coordinates. Most practical problems which arise in the dynamics of mechanical systems with bilateral nonholonomic constraints (such as those illustrated, for example, in [5] and [6]) thus satisfy the conditions of Theorem 2, and therefore yield unique motions (trajectories), at least locally.

If the matrix A is not of a constant rank for all t in $[a, b]$, then there exists a collection of open intervals (a_i, b_i) whose union is dense in $[a, b]$ such that $A^+(t)$ is continuous on each of the subintervals (a_i, b_i) . The situation for generalized inverses of operator-valued functions is more complicated; continuity and perturbation results for generalized inverses of linear operators are given in [7] and the references cited therein. The physical interpretation and implications of the case when the rank of the matrix $A(t)$ is not constant in constrained mechanical systems lead to some interesting questions in the modeling and analysis of such systems.

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