VIBRATION SUPPRESSION IN A FLEXIBLE
GYROSCOPIC SYSTEM USING MODEAL COUPLING
STRATEGIES

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Several recent studies have shown that vibrations in a two-degree-of-freedom system can be suppressed by using modal coupling based control techniques. This involves making the first two natural frequencies commensurable (e.g., in a ratio of 1:1 or 1:2) to establish a state of Internal Resonance (IR). When the system exhibits IR, vibrations in the two directions are strongly coupled resulting in a beat phenomenon. Upon introducing damping in one direction, oscillations in both directions can be quickly suppressed. In this paper we consider vibration suppression of a flexible two-degree-of-freedom gyroscopic system using 1:1 and 1:2 IR. The possibility of using 1:1 and 1:2 IR to enhance the coupling in the system is established analytically using the perturbation method of multiple scales. The results of IR based control strategy are compared with a new method, which is based on tuning the system parameters to make the mode shapes identical. Results indicate that this new technique is more efficient and easy to implement than IR based control strategies. Another advantage of this method is that there is no restriction on the frequencies as in the case of IR. Finally, a control torque is obtained which on application automatically tunes the system parameters to establish modal coupling.

KEYWORDS: Vibration suppression; modal coupling; internal resonance

1. INTRODUCTION

In this paper, we address vibration suppression of a flexible two-degree-of-freedom gyroscopic system with an uncontrollable mode. Using conventional control techniques it is generally not possible to regulate vibrations in such cases using one actuator because one mode is uncontrollable. However if the oscillations are coupled, the coupling may be utilized to regulate the oscillations indirectly. The vibration suppression strategy presented in this work is based on using a simple control scheme like a PD controller to regulate the vibrations in one direction and using the coupling to control the vibrations in the other direction. To effectively employ this strategy, the coupling has to be enhanced. Several recent studies [1–11] have suggested the use of the phenomenon of Internal Resonance (IR) as a means to enhance the coupling effect.

A system is said to be in a state of IR if the natural frequencies ($\Omega$) are commensurable (i.e., $a_1 \Omega_1 + a_2 \Omega_2 + \ldots = 0$, where $a$'s are positive or negative integers). When IR is established, the existing coupling between the two coordinate directions is enhanced. At this state, energy is continually exchanged between the modes resulting in a beat
phenomenon. Control is achieved upon introducing damping into one of the directions, from where the energy is dissipated leading to vibration suppression in both directions.

One of the first authors to investigate IR was Sethna [12]. Later several authors, Stupnicka [13], Van Dooren [14], Haddow et al. [15], and Mook et al. [16] studied resonant response of a system under harmonic excitation forces. Nayfeh and Mook [17] in their book give a comprehensive treatment on this subject. These studies reveal that the choice of IR ratio depends on the type of coupling in the system. When quadratic coupling is present, 1:2 IR ratio gives amplitude modulated response, and for cubic nonlinearities both 1:1 and 1:3 IR ratios can be used [17].

The research in Construct group at University of Waterloo has focused on applications of linear and nonlinear coupling terms to suppress vibrations. The first study in this area is the work by Golnaraghi [1,2], where he used a sliding mass to control the vibrations of a flexible cantilever beam. The controller in this work introduced an additional degree of freedom and kinematic nonlinearities to the system. In this case, control was achieved when the sliding mass motion was slightly damped at the state of 2:1 IR. Subsequently, Tuer et al. [3] and Duquette et al. [4] experimentally studied a flexible cantilever beam. The coupling in their system was introduced through a rigid beam that was attached to the tip of the flexible beam via a DC motor. Tuer [5] and Duquette [6] also discussed the use of linear coupling in vibration suppression of a cantilever beam. In more recent studies, enroute to generalize these control schemes, Tuer et al. [7,8] utilized coordinate coupling (linear coupling through position coordinates) and IR with quadratic coupling methods in active configurations. In these studies, the secondary degree of freedom was introduced in computer software and the coupling effect was introduced via an actuator connected to the plant. The example used by Tuer [7,8] was a flexible arm manipulator, and was tested experimentally for the case of coordinate coupling. The IR case was later tested by Oueini and Golnaraghi [9], experimentally. Later on, Khajepour et al. [10,11] used the center manifold and normal form methods to address the deficiencies of earlier studies and derived design criteria for the generalized coupling control law using quadratic nonlinearities.

In the above mentioned references, 1:2 IR ratio was predominantly used because quadratic nonlinearities were introduced in the system through the controller. However, in gyroscopic systems, the required coupling between the two degrees of freedom exists naturally, due to the gyroscopic forces. This makes gyroscopic type systems a natural candidate for the application of modal coupling based control strategies.

Gyroscopic systems in general can exhibit both linear and nonlinear couplings. For the system under consideration the equations of motion are coupled through linear, quadratic, and cubic terms. Therefore, we can investigate modal coupling through 1:1 and 1:2 frequency ratios. The existence of modal coupling for 1:1 and 1:2 IR is established analytically using the perturbation method of multiple scales and the results are verified numerically. In addition we also present a new vibration suppression strategy which uses the linear velocity coupling that arises through the gyroscopic terms and tunes the linear mode shapes directly to strengthen the link between the two degrees of freedom. It can be shown that 1:1 IR ratio forms a special case of this method. Numerical simulations show that, for gyroscopic systems, this method is more effective in suppressing vibrations than the method of IR. Investigations in the parameter space show that the system cannot be always tuned to exhibit a desired modal coupling. To overcome this, we present a control
law, which will make the system exhibit modal coupling, regardless of the values of the system parameters. This approach opens a new horizon for vibration suppression in other types of systems, on which we will be focusing in the early future.

2. SYSTEM DESCRIPTION AND MATHEMATICAL MODELING

The system studied is shown in Figure 1. A discrete model of a flexible beam of mass $m$ and length $l$ is assumed. The flexibility is modeled as two linear rotational springs of stiffness $k_1$ and $k_2$ in the vertical and horizontal planes, respectively. The beam is assumed to rotate at a constant angular velocity $\omega$ about the vertical axis. The rotation of the beam contributes to the centrifugal and gyroscopic forces on the system. The motion is described by the angular coordinates $\theta_1$ and $\theta_2$. Further, the $\theta_1$ direction is assumed as the control direction to which the controller is applied and the oscillations in the $\theta_2$ direction are controlled indirectly through the coupling. The model retains many fundamental characteristics of physical gyroscopic systems like helicopter rotors, but at the same time it is simple enough to conduct analytical investigations. A detailed study of the gyroscopic characteristics of helicopter rotor blades can be found in [18,19].

Using the Lagrangian method the following equations of motion are obtained [20]:

\[
\begin{bmatrix}
1 & 0 \\
0 & \cos^2(\theta_1)
\end{bmatrix}
\begin{bmatrix}
\dot{\theta}_1^* \\
\dot{\theta}_2^*
\end{bmatrix}
+ \begin{bmatrix}
0 & \frac{1}{2} \dot{\theta}_2^* \sin(2\theta_1) \\
-\dot{\theta}_2^* \sin(2\theta_1) & 0
\end{bmatrix}
\begin{bmatrix}
\dot{\theta}_1^* \\
\dot{\theta}_2^*
\end{bmatrix}
\]

\[
- \begin{bmatrix}
0 & \omega^* \sin(2\theta_1) \\
-\omega^* \sin(2\theta_1) & 0
\end{bmatrix}
\begin{bmatrix}
\dot{\theta}_1^* \\
\dot{\theta}_2^*
\end{bmatrix}
+ \begin{bmatrix}
\omega_1^* & 0 \\
0 & \omega_2^*
\end{bmatrix}
\begin{bmatrix}
\theta_1 \\
\theta_2
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
\frac{1}{2} \omega^* \sin(2\theta_1) - \cos(\theta_1) + T^* \\
0
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

(1)

\[\text{Figure 1 System model.}\]
Similar equations are obtained in [18] in connection with a rigid body model of a helicopter rotor blade. In (1), nondimensional parameters are used which are defined as follows:

\[
t^* = \sqrt{\frac{g}{l^2} t}, \quad \frac{d}{dt^*} = \sqrt{\frac{l}{g}} \frac{d}{dt^*} = \sqrt{\frac{l}{g}} \frac{d}{dt^*^2} = \frac{l}{g} \frac{d^2}{dt^*^2}
\]

\[
\omega^* = \sqrt{\frac{l}{g} - \omega}, \quad \omega^*_1 = \frac{k_1}{mgl}, \quad \omega^*_2 = \frac{k_2}{mgl}
\]

(2)

and \(\dot{\theta}^*_1, \dot{\theta}^*_2, \ddot{\theta}_1^*, \text{and} \ \ddot{\theta}_2^*\) denote the nondimensionalized rates. In (1) the term \(T^*\) represents the controller torque that is applied along \(\theta_1\). To illustrate the modal coupling strategies, we assume Proportional and Derivative (PD) feedback control. Using PD control, \(T^*\) is defined as follows:

\[
T^* = K(\dot{\theta}_{1r} - \dot{\theta}_1) + C\ddot{\theta}_1^*
\]

(3)

where \(K\) is the controller position gain, \(C\) is the controller velocity gain and \(\dot{\theta}_{1r}\) is a constant reference input. Another term \(T^*_c\), which represents the constant element in (3) is defined as follows:

\[
T^*_c = K\dot{\theta}_{1r}
\]

(4)

Using (3) and (4), the equations of motion (1) are written as follows:

\[
\begin{bmatrix}
1 & 0 \\
0 & \cos^2(\theta_1)
\end{bmatrix}
\begin{bmatrix}
\dot{\theta}_1^* \\
\dot{\theta}_2^*
\end{bmatrix}
+ \begin{bmatrix}
0 & \frac{1}{2}\dot{\theta}_2^* \sin(2\theta_1) \\
-\dot{\theta}_2^* \sin(2\theta_1) & 0
\end{bmatrix}
\begin{bmatrix}
\dot{\theta}_1^* \\
\dot{\theta}_2^*
\end{bmatrix}
+ \begin{bmatrix}
C & \omega^* \sin(2\theta_1) \\
-\omega^* \sin(2\theta_1) & 0
\end{bmatrix}
\begin{bmatrix}
\dot{\theta}_1^* \\
\dot{\theta}_2^*
\end{bmatrix}
+ \begin{bmatrix}
\frac{1}{2}\omega^2 \sin(2\theta_1) - \cos(\theta_1) + T^*_c \\
0
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

(5)

where \(\omega^*_{n1}\) and \(\omega^*_{n2}\) (for notational consistency) are defined as:

\[
\omega^*_{n1} = \omega_{1}^* - K \\
\omega^*_{n2} = \omega_{2}^*
\]

(6)
The controller is used to tune the system parameters to establish modal coupling and to introduce damping in the $\theta_1$ direction. From (6) it can be seen that by adjusting the controller position gain $K$, the value of $\omega_{n1}^*$ can be tuned to establish modal coupling in the system.

3. EQUILIBRIUM POSITIONS

The constant solution for (5) is defined as an equilibrium position. The equilibrium positions are obtained by setting velocity and accelerations to zero in (5) ($\dot{\theta}_1^* = \dot{\theta}_2^* = \ddot{\theta}_1^* = \ddot{\theta}_2^* = 0$), resulting in the following equations:

$$\omega_{n1}^* \theta_{1e} + \frac{1}{2} \omega^* \omega^2 \sin(\theta_{1e}) - \cos(\theta_{1e}) + T_c^* = 0$$

$$\omega_{n2}^* \theta_{2e} = 0$$

(7)

where $\theta_{1e}$ and $\theta_{2e}$ are the equilibrium values. From (7) it can be seen that $\theta_{2e}$ is always zero, but the equilibrium values for $\theta_{1e}$ depend on the system parameters $\omega^*$, $\omega_{n1}^*$, and $T_c^*$. It can be seen that the first equation in (7) is nonlinear, so multiple equilibrium values could exist for $\theta_1$. When the system parameters are varied the equilibrium positions change, and by adjusting the controller reference input $\theta_{1r}$, the value of $T_c^*$ and, hence, the equilibrium position can be tuned to a desired value.

The stability of the equilibrium positions can be established using the eigenvalues of the linearized equations of motion or the Lyapunov's direct method. Figure 2 shows the equilibrium values for $\theta_1$ and also points a stable equilibrium position. A detailed investigation of the stability of the equilibrium positions for this system was conducted in [20]. In this paper we consider motion only about the stable equilibria.

![Figure 2](image)

Figure 2 Equilibrium function; $\omega^* = 8.0$, $\omega_{n1}^* = 3.0$, $T_c^* = 0$. 
4. MOTION IN THE NEIGHBORHOOD OF EQUILIBRIUM POINTS

To investigate the motion in the neighborhood of equilibrium points, the equations of motion (5) are expanded about the equilibrium position. In the analysis which follows, the controller velocity gain $C$ is assumed to be small and neglected for simplicity. Using Taylor series expansion about an equilibrium point and collecting terms up to third order we obtain the following approximate equations of motion:

\[
\begin{bmatrix}
1 & 0 & \ddot{u}_1 \\
0 & 1 & \ddot{u}_2
\end{bmatrix}
+ \begin{bmatrix}
0 & s_1 & \dot{u}_1 \\
0 & s_6 & \dot{u}_2
\end{bmatrix}
+ \begin{bmatrix}
0 & s_7 & u_1 \\
0 & 0 & u_2
\end{bmatrix}
= \begin{bmatrix}
s_3 s_1^2 + s_4 u_2^2 + s_5 u_1 u_2 + s_6 u_1^3 + s_7 u_2^2 + s_8 u_1 u_2 + s_9 u_1^2 + s_{10} u_2^2 + s_{11} u_1 u_2 + s_{12} u_2^2 + s_{13} u_1 u_2 + s_{14} u_2^2 + s_{15} u_1 + s_{16} u_2
\end{bmatrix}
\]

(8)

where $u_1 = \theta_1 - \theta_{1e}$, and $u_2 = \theta_2 - \theta_{2e}$ and the $s_1$ terms are given by:

\[
s_1 = \omega^* \sin(2\theta_{1e}), \quad s_2 = \omega^* \cos(2\theta_{1e}) + \omega_{n1}^* + \sin(\theta_{1e})
\]

\[
s_3 = -\frac{1}{2} \cos(\theta_{1e}) + \omega^* \sin(2\theta_{1e}), \quad s_4 = -\frac{1}{2} \sin(2\theta_{1e})
\]

\[
s_5 = -2\omega^* \cos(2\theta_{1e}), \quad s_6 = -2\omega^* \tan(\theta_{1e})
\]

\[
s_7 = \omega_{n2} \sec^2(\theta_{1e}), \quad s_8 = \sin(2\theta_{1e}) \sec^2(\theta_{1e})
\]

\[
s_9 = 2\omega^* \sec^2(\theta_{1e}), \quad s_{10} = -2\omega_{n2} \tan(\theta_{1e}) \sec^2(\theta_{1e})
\]

\[
s_{11} = 2\omega^* \cos(2\theta_{1e}) + \frac{1}{6} \sin(\theta_{1e}), \quad s_{12} = 2\omega^* \sin(2\theta_{1e})
\]

\[
s_{13} = -\cos(2\theta_{1e}), \quad s_{14} = 2 \sec^2(\theta_{1e})
\]

\[
s_{15} = 2\omega^* \sin(\theta_{1e}) \sec^3(\theta_{1e}), \quad s_{16} = \omega_{n2} (2 \cos^2(\theta_{1e}) - 3) \sec^4(\theta_{1e})
\]

(9)

The equations of motion (8) are used later in the perturbation analysis to establish modal coupling under resonant conditions.

5. CHARACTERISTIC ROOTS

Assuming $u_i = a_i e^{i\Omega t}$ as the solution of the linearized equations associated with (8), we obtain the following characteristic equation:

\[
\Omega^4 - (s_2 + s_7 - s_1 s_6) \Omega^2 + s_2 s_7 = 0
\]

(10)
where the roots $\Omega_i$ are given by:

$$\Omega_{1,2}^2 = \frac{(s_2 + s_7 - s_1s_6) \pm \sqrt{(s_2 + s_7 - s_1s_6)^2 - 4s_2s_7}}{2}$$

$$= \frac{(s_2 + s_7 - s_1s_6) \pm \sqrt{(s_2 - s_7 - s_1s_6)^2 - 4s_1s_6s_7}}{2}$$  \hspace{1cm} (11)$$

Note that

$$-4s_1s_6s_7 = 4\omega_s^2\omega_{n_2}^2\tan^2(2\theta_{1e}) \geq 0$$  \hspace{1cm} (12)$$

therefore $\Omega_1^2$ and $\Omega_2^2$, corresponding to the eigenvalues of the linear portion of (8) are always real, and the natural frequencies $\Omega_1$ and $\Omega_2$ are associated with the positive roots of $\Omega_1^2$ and $\Omega_2^2$, respectively.

6. PERTURBATION EXPANSIONS

The perturbation analysis in the case of 1:1 or 1:2 IR is performed using the method of multiple scales. Details of this method can be found in [17] and [21]. Following the procedure of the multiple scales method the dependent variables $u_1$ and $u_2$ are assumed as follows:

$$u_1 = \epsilon u_{11}(T_0,T_1,T_2) + \epsilon^2 u_{12}(T_0,T_1,T_2) + \epsilon^3 u_{13}(T_0,T_1,T_2)$$

$$u_2 = \epsilon u_{21}(T_0,T_1,T_2) + \epsilon^2 u_{22}(T_0,T_1,T_2) + \epsilon^3 u_{23}(T_0,T_1,T_2)$$  \hspace{1cm} (13)$$

where $\epsilon$ represents a nondimensional scaling parameter, and $u_{1i}$ and $u_{2i}$ ($i = 1...3$) represent the solution corresponding to the order $\epsilon^i$. Since we use three terms in the expansion for $u_1$ and $u_2$, multiple scales method requires that the nondimensional time $t^*$ is also measured on three different time scales $T_0$, $T_1$, and $T_2$, which are defined as follows:

$$T_i = \epsilon^it^*; \ i = 0, 1, \text{ and } 2$$  \hspace{1cm} (14)$$

The scale $T_0$ is a fast scale and $T_1$, $T_2$ represent slower time scales. Using the chain rule of differentiation and (14), the derivatives with respect to the nondimensional time $t^*$ are written as follows:

$$\frac{d}{dt^*} = D_0 + \epsilon D_1 + \epsilon^2 D_2$$

$$\frac{d}{dt^*} = D_0^2 + 2\epsilon D_0D_1 + \epsilon^2(2D_0D_2 + D_1^2)$$  \hspace{1cm} (15)$$
where $D_i$ represents the partial derivatives with respect to $T_i$.

Substituting (13) and (15) in the equations of motion (8), and collecting like powers of $\epsilon$, gives the following equations:

Order $\epsilon$:

\[
D_0^2u_{11} + s_1D_0u_{21} + s_2u_{11} = 0
\]
\[
D_0^2u_{21} + s_6D_0u_{11} + s_7u_{21} = 0
\]  
(16)

Order $\epsilon^2$:

\[
D_0^2u_{12} + s_1D_0u_{22} + s_2u_{12} = -2D_0D_1u_{11} - s_1D_1u_{21} + s_3u_{11}^2 + s_4(D_0u_{21})^2
\]
\[
+ s_5u_{11}D_0u_{21}
\]
\[
D_0^2u_{22} + s_6D_0u_{12} + s_7u_{22} = -2D_0D_1u_{21} - s_6D_1u_{11} + s_8D_0u_{11}D_0u_{21} + s_9u_{11}D_0u_{11}
\]
\[
+ s_{10}u_{11}u_{21}
\]  
(17)

Order $\epsilon^3$:

\[
D_0^2u_{13} + s_1D_0u_{23} + s_2u_{13} = -2D_0D_1u_{12} - 2D_0D_2u_{11} - D_1^2u_{11} - s_1(D_1u_{22} + D_2u_{21})
\]
\[
+ 2s_3u_{11}u_{12} + 2s_4(D_0u_{21}D_1u_{21} + D_0u_{22}D_0u_{22})
\]
\[
+ s_5(u_{11}D_0u_{22} + u_{12}D_0u_{21} + u_{11}D_1u_{21})
\]
\[
+ s_{11}u_{11}^3 + s_{12}D_0u_{21}u_{11}^2 + s_{13}(D_0u_{21})^2u_{11}
\]
\[
D_0^2u_{23} + s_6D_0u_{13} + s_7u_{23} = -2D_0D_1u_{22} - 2D_0D_2u_{21} - D_1^2u_{21} - s_6(D_1u_{12} + D_2u_{11})
\]
\[
+ s_8(D_1u_{11}D_0u_{21} + D_0u_{11}D_0u_{22} + D_0u_{11}D_1u_{21} + D_0u_{12}D_0u_{21})
\]
\[
+ s_9(u_{12}D_0u_{11} + u_{11}D_0u_{12} + u_{11}D_1u_{11}) + s_{10}(u_{12}u_{21} + u_{11}u_{22})
\]
\[
+ s_{14}D_0u_{11}D_0u_{21}u_{11} + s_{15}D_0u_{11}u_{11}^2 + s_{16}u_{11}^2u_{21}
\]  
(18)

The perturbation technique involves solving (16), (17), and (18) sequentially. The general solution for (16) can be considered as:

\[
u_{11} = A_1(T_1,T_2)e^{\Omega_1T_0} + A_2(T_1,T_2)e^{\Omega_2T_0} + cc
\]
\[
u_{21} = r_1A_1(T_1,T_2)e^{\Omega_1T_0} + r_2A_2(T_1,T_2)e^{\Omega_2T_0} + cc
\]  
(19)

where $cc$ denotes complex conjugate of the preceding terms, and $A_1$ and $A_2$ are, in general, functions of the slower time scales $T_1$ and $T_2$, indicating that amplitudes vary slowly with time. In (19), $r_1$ and $r_2$ are the modal amplitude ratios corresponding to the first and second natural frequencies, respectively, and are given by:
\[ r_1 = -\frac{s_2 - \Omega_1^2}{is_1\Omega_1} = -\frac{is_6\Omega_1}{s_7 - \Omega_1^2} \]
\[ r_2 = -\frac{s_2 - \Omega_2^2}{is_1\Omega_2} = -\frac{is_6\Omega_2}{s_7 - \Omega_2^2} \]  
(20)

The equations developed in this section are used to show the existence of modal coupling in the 1:1 and 1:2 IR cases, next.

7. VIBRATION SUPPRESSION USING 1:1 IR

The first vibration suppression strategy discussed is 1:1 IR. It requires tuning the value of \( \omega_{n1}^* \) to make \( \Omega_1 = \Omega_2 \). For a given set of parameters it may not be always possible to obtain a real value for the tuning parameter \( \omega_{n1}^* \) to establish a desired ratio of \( \Omega_1 \) and \( \Omega_2 \). Therefore we first consider the problem of tuning the system parameters to establish IR.

7.1 System Parameters

From (11) it follows that the difference between the frequencies \( \Omega_1 \) and \( \Omega_2 \) is given by:

\[ \Omega_1 - \Omega_2 = \sqrt{s_2 + s_7 - s_1s_6} - 2\sqrt{s_2s_7} \]  
(21)

If the difference between the frequencies is set to zero, from (21) the following equation is obtained:

\[ s_2 = \sqrt{s_7} \pm \sqrt{s_1s_6} \]  
(22)

It can be seen from (9) that \( s_1s_6 \) is a negative quantity, and from (22) it follows that \( s_2 \) and hence \( \omega_{n1}^* \) are complex quantities implying that we cannot have an exact ratio of \( \Omega_1 \) and \( \Omega_2 \). Since the system cannot be tuned to establish an exact 1:1 ratio of the natural frequencies, we try to minimize the difference between the frequencies with respect to the tuning parameter \( \omega_{n1}^* \). After simplification it results in the following equations:

\[ s_2 = s_7 \]
\[ \omega_{n1}^* = -\omega^*\cos(2\theta_{1e}) - \sin(\theta_{1e}) + \omega_{n2}^{**}\sec^2(\theta_{1e}) \]  
(23)

When the system parameters are tuned according to (23) the difference between the natural frequencies, given by (21) is reduced to the following equation:

\[ \delta = \Omega_1 - \Omega_2 = \sqrt{-s_1s_6} \]
\[ = 2\omega^* \sin(\theta_{1e}) \]  
(24)
where $\delta$ is called the detuning parameter. It can be seen from (24) that for small values of $\theta_{1e}$, $\delta$ is small and the frequencies can be considered to be approximately equal. For convenience we refer to this approximate ratio as 1:1 IR. For larger values of $\theta_{1e}$, the value of $\delta$ is large and the system cannot be considered as 1:1 resonant. Figure 3(a) shows a set of values of $\omega_{n1}^*$ required for tuning the system to 1:1 IR when $\omega^*$ is varied. Figure 3(b) shows the natural frequencies along the same path. From this figure it can be seen that the range of the tuning parameter $\omega_{n1}^*$ is limited. To overcome this we present a method to modify the control torque in Section (9.1).

7.2 Perturbation Analysis

The perturbation analysis involves extensive algebraic manipulation, which is carried out using the symbolic manipulation program MAPLE. In carrying out the perturbation analysis the approach presented in [21] is adopted. The solution (19) of the order $\epsilon$ equations is substituted into the order $\epsilon^2$ equations (17), and after simplification gives the

![Figure 3](image)

**Figure 3** Tuned parameters and natural frequencies—1:1 IR $\omega_{n2}^* = 5.0$, $\theta_{1e} = 0.1$. (a) Tuned parameter values; (b) Natural frequencies along tuned path.
following expressions for the right-hand sides, denoted by rhs3 and rhs4:

\[
\text{rhs3} = x_{11}A_1'e^{i\Omega_1T_0} + x_{22}A_2'e^{i\Omega_2T_0} + x_{33}A_1'e^{2i\Omega_1T_0} + x_{44}A_2'e^{2i\Omega_2T_0} \\
+ x_{55}A_1A_2'e^{i(\Omega_1+\Omega_2)T_0} + x_{66}A_1A_2'e^{i(\Omega_1-\Omega_2)T_0} + cc + x_{77}A_1A_2 + x_{88}A_2A_2
\]

\[
\text{rhs4} = x_{99}A_1'e^{i\Omega_1T_0} + x_{10}A_2'e^{i\Omega_2T_0} + x_{11}A_1'e^{2i\Omega_1T_0} + x_{12}A_2'e^{2i\Omega_2T_0} \\
+ x_{13}A_1A_2'e^{i(\Omega_1+\Omega_2)T_0} + x_{14}A_1A_2'e^{i(\Omega_1-\Omega_2)T_0} + cc
\]  

(25)

where \( x_i, i = 1 \ldots 14 \), are constants which depend on the system parameters and (') indicates differentiation with respect to \( T_1 \). These constants are not presented because of the length of these terms. Since an exact 1:1 ratio of \( \Omega_1 \) and \( \Omega_2 \) cannot be established we use the detuning parameter \( \delta \). However we first scale \( \delta \) by using \( \delta = \varepsilon^2 \sigma \) to give the correct time scale, and write (24) as follows:

\[
\Omega_2 = \Omega_1 + \varepsilon^2 \sigma
\]  

(26)

The terms \( e^{\pm i\Omega_1T_0} \) and \( e^{\pm i\Omega_2T_0} \) appear in both the homogeneous and the particular solution of (17), this gives rise to secular terms which makes the solution of (17) nonuniform. Eliminating the secular terms results in solvability conditions which relate the modal amplitudes \( A_1 \) and \( A_2 \). To eliminate the secular terms from \( u_{12} \) and \( u_{22} \), the following particular solution is assumed, which neglects all the terms except the ones which lead to secular terms:

\[
u_{12} = P_{11}e^{i\Omega_1T_0} + P_{12}e^{i\Omega_2T_0} \\
u_{22} = P_{21}e^{i\Omega_1T_0} + P_{22}e^{i\Omega_2T_0}
\]

(27)

Note in (27) that the conjugate terms are also neglected because they would lead to four incompatible equations relating \( A_1 \) and \( A_2 \). Substituting (27) in (17) and equating the coefficients of \( e^{i\Omega_1T_0} \) and \( e^{i\Omega_2T_0} \) on both sides results in the following solvability conditions:

\[
(k_2 - \Omega_n^2)P_{1n} + ik_1\Omega_nP_{2n} = R_{1n} \\
k_1\Omega_nP_{1n} + (k_2 - \Omega_n^2)P_{2n} = R_{2n}
\]

(28)

where \( n = 1, 2 \) and \( R_{1n} \) and \( R_{2n} \) are given by:

\[
R_{11} = x_1A_1', R_{12} = x_2A_2' \\
R_{21} = x_9A_1', R_{22} = x_{10}A_2'
\]

(29)

For a nontrivial solution to (28) it requires that

\[
\begin{vmatrix}
\delta_2 - \Omega_n^2 & R_{1n} \\
isl_1\Omega_n & R_{2n}
\end{vmatrix} = 0
\]

(30)
where $s_2$ and $s_6$ are defined in (9). Solving (30) for $n = 1, 2$ gives the following conditions:

\[ A'_1 = 0, \ A'_2 = 0 \]  
(31)

which imply that $A_1 = A_1(T_2), A_2 = A_2(T_2)$ and the coupling between the modes cannot be established using $\varepsilon^2$ equations (17). Therefore, we have to consider the higher order equations (18). The first step involves substituting (31) in (25) and obtain the particular solution for (17) by considering the remaining terms in the right-hand sides. Substituting the homogeneous solution for $u_{11}$ and $u_{21}$ (19), and the particular solution for $u_{21}$ and $u_{22}$ in (18) results in the following terms for the right-hand sides of (18), denoted by $rhs5$ and $rhs6$:

\[
\begin{align*}
\text{rhs5} &= \{x_{15}A'_1(T_2) + x_{16}A'_1\overline{A}_1 + x_{17}A_1A_2\overline{A}_2\}e^{i\Omega_1T_0} \\
&\quad + \{x_{18}A'_1(T_2) + x_{19}A_1\overline{A}_1A_2 + x_{20}A'_2\overline{A}_2\}e^{i\Omega_2T_0} \\
&\quad + x_{21}A'_1e^{i3\Omega_1T_0} + x_{22}A'_1e^{i3\Omega_2T_0} + x_{23}A_1A_2e^{i(2\Omega_2 + \Omega_1)T_0} + x_{24}A_1A_2e^{i(2\Omega_2 + \Omega_1)T_0} \\
&\quad + x_{25}A_1A_2e^{i(\Omega_2 - 2\Omega_1)T_0} + x_{26}A_1A_2e^{i(\Omega_2 - 2\Omega_1)T_0} + cc \\
\text{rhs6} &= \{x_{27}A'_1(T_2) + x_{28}A'_1\overline{A}_1 + x_{29}A_1A_2\overline{A}_2\}e^{i\Omega_1T_0} \\
&\quad + \{x_{30}A'_1(T_2) + x_{31}A_1\overline{A}_1A_2 + x_{32}A'_2\overline{A}_2\}e^{i\Omega_2T_0} \\
&\quad + x_{33}A_1e^{i3\Omega_1T_0} + x_{34}A'_1e^{i3\Omega_2T_0} + x_{35}A_1A_2e^{i(2\Omega_2 + \Omega_1)T_0} + x_{36}A_1A_2e^{i(2\Omega_2 + \Omega_1)T_0} \\
&\quad + x_{37}A_1A_2e^{i(\Omega_2 - 2\Omega_1)T_0} + x_{38}A_1A_2e^{i(\Omega_2 - 2\Omega_1)T_0} + cc
\end{align*}
\]  
(32)

where $(\cdot)$ indicates differentiation with respect to $T_2$, and $x_i$ are constants. Using the resonant condition (26) the following relationships are obtained:

\[
\begin{align*}
e^{i(-\Omega_2 + 2\Omega_1)T_0} &= e^{i\Omega_1T_0}e^{-i2\sigma T_2} \\
e^{i(\Omega_2 - 2\Omega_1)T_0} &= e^{i\Omega_2T_0}e^{i2\sigma T_2}
\end{align*}
\]  
(33)

Substituting (33) in (32) and assuming a solution for $u_{13}$ and $u_{23}$ similar to (27), and equating the coefficients of similar terms on either side of (18), we obtain the following solvability conditions which establish the modal coupling:

\[
\begin{align*}
x_{39}A'_1(T_2) + x_{40}A_1^2\overline{A}_1 + x_{41}A_1A_2\overline{A}_2 + x_{42}A_2^2\overline{A}_1e^{2i\sigma T_2} &= 0 \\
x_{43}A'_2(T_2) + x_{44}A_2^2\overline{A}_2 + x_{45}A_1A_2\overline{A}_1 + x_{46}A_1^2\overline{A}_2e^{-2i\sigma T_2} &= 0
\end{align*}
\]  
(34)
The complex constants $A_1$ and $A_2$ are converted to polar form using:

$$A_1 = \frac{1}{2} a_1 e^{i\alpha_1}$$

$$A_2 = \frac{1}{2} a_2 e^{i\alpha_2}$$

(35)

where $a_1(T_2)$ and $a_2(T_2)$ are modal amplitudes and $\alpha_1(T_2)$ and $\alpha_2(T_2)$ are phases of the response.

Using (35) the solvability conditions (34) are reduced to the following differential equations:

$$\dot{a}_1 = \Gamma_1 a_2^2 a_1 \sin(\gamma)$$

$$\dot{a}_2 = -\Gamma_2 a_1^2 a_2 \sin(\gamma)$$

$$\dot{\alpha}_1 = -2\Gamma_1 a_2^2 \cos(\gamma) + \Gamma_3 a_1^2 + \Gamma_4 a_2^2$$

$$\dot{\alpha}_2 = -2\Gamma_2 a_1^2 \cos(\gamma) + \Gamma_5 a_1^2 + \Gamma_6 a_2^2$$

(36)

where

$$\gamma = 2\sigma T_2 - 2\alpha_1 + 2\alpha_2$$

(37)

and $\Gamma_i, i = 1...6$ are constants which depend on system parameters and are too long to be included here, the reader is referred to [20] for their actual values. Using (37) the last two equations in (36) can be combined reducing the number of equations to three:

$$\gamma' = 2\sigma + 4(\Gamma_1 a_2^2 - \Gamma_2 a_2^2) \cos(\gamma) + 2(\Gamma_5 - \Gamma_3) a_1^2 + 2(\Gamma_6 - \Gamma_4) a_2^2$$

(38)

Eliminating $\gamma$ from the first two equations in (36) and integrating we get the following equation, which shows exchange of energy between the modes:

$$a_1^2 + \nu a_2^2 = E$$

(39)

In the above equation, $\nu = \Gamma_1 / \Gamma_2$ and $E$ is an integration constant which depends on the initial conditions and represents the modal energy. Equation (39) shows that the modal amplitudes are coupled and energy is exchanged between the modes. The system of equations (36) are solved numerically to illustrate the coupling between the modes. Figure 4 shows such a response for 1:1 IR tuned parameters. It illustrates that the amplitude modes $a_1$ and $a_2$ are strongly coupled and 1:1 IR can be successfully used to control the vibrations.
Figure 4 Modal amplitude response—1:1 IR. \( \omega^* = 3.0, \omega_{a1}^* = 4.04, \omega_{a2}^* = 5.0, \theta_{1e} = 0.1, \sigma = -0.599. \)

7.3 Numerical Simulations

To illustrate numerically the vibration suppression strategy, the nonlinear system of equations (5) are used. Figure 5 shows the time response for a set of 1:1 IR tuned parameters for the undamped case. The response shows distinctive beats which are typical for Internal Resonance. The response on the slow time scale \( T_2 \), shown in Figure 4, represents the envelopes of the beats. When damping is introduced in the \( \theta_1 \) direction using the controller velocity gain \( C \), energy can be quickly dissipated from the system. This is illustrated in Figure 6 for different damping coefficients. These figures indicate that when \( C \) is increased we get an optimal response, Figure 6(b), and further increase in \( C \) does not improve the response because the oscillations in the \( \theta_1 \) mode decay quickly and limit the interaction between the modes. To further analyze the role of damping, center manifold and normal form methods can be used as shown in [10] and [11].

8. VIBRATION SUPPRESSION USING 1:2 IR

In this section we consider establishing modal coupling using 1:2 IR. This type of resonance occurs in the system due to the presence of quadratic nonlinearities. Similar to 1:1 IR, we first consider the problem of tuning the system parameters to establish a 1:2 ratio of the natural frequencies.

8.1 System Parameters

If we set \( \Omega_1/\Omega_2 = 2 \), using (9) the following equations for \( s_2 \) are obtained:

\[
s_2 = \frac{(17s_7 + 8s_1s_6) \pm \sqrt{225s_7^2 + 400s_1s_6s_7}}{8}
\]

(40)
Unlike 1:1 IR where the frequencies cannot be tuned to an exact ratio, using (40) it is possible to tune $\Omega_1$ and $\Omega_2$ to an exact 1:2 ratio. In (40), we choose the value of $s_2$ which gives a real value for $\omega_{n1}^*$. Figure 7 shows a set of tuned parameter values obtained using (40). It can be seen from this figure, that for certain values of $\omega^*$ the system can be tuned for 1:2 IR at two different values of $\omega_{n1}^*$, which is not possible in the 1:1 IR case.

### 8.2 Perturbation Analysis

Although the system can be tuned for an exact 1:2 ratio of $\Omega_1$ and $\Omega_2$, we still consider a detuning parameter $\delta$ because it may not be practically possible to tune the system to an exact ratio. By defining $\delta = \varepsilon \sigma$, the frequencies are related through the following equations:

$$
\Omega_1 = 2\Omega_2 + \varepsilon \sigma
$$

(41)
Figure 6 Time response—1:1 IR. \( \omega^* = 3.0, \omega_{n1}^* = 4.04, \omega_{n2}^* = 5.0, \theta_{1e} = 0.1 \). (a) \( C = 0.5 \); (b) \( C = 1.0 \); (c) \( C = 2.0 \).

Using (41) the following relationships are obtained:

\[
\begin{align*}
  e^{2\Omega_2 T_0} &= e^{i\Omega_2 T_0} e^{-i\sigma T_1}, \\
  e^{i(\Omega_1 - \Omega_2) T_0} &= e^{i\Omega_2 T_0} e^{i\sigma T_1}
\end{align*}
\]  

(42)

Substituting (42) in \( rhs3 \) and \( rhs4 \) and following a procedure similar to the 1:1 IR case, the following solvability conditions are obtained for 1:2 IR:
Figure 7 Tuned parameters—1:2 IR. $\omega_{2\ast} = 5.0, \theta_{1e} = 0.1$.

\[
x_{47} \dot{A}_1 + x_{48} A_2^2 e^{-i\sigma T_1} = 0
\]
\[
x_{49} \dot{A}_2 + x_{50} A_1 A_2 e^{i\sigma T_1} = 0
\]

(43)

where ('') indicates differentiation with respect to $T_1$, and $x_i$ are constants. Using (35) the complex constants $A_1$ and $A_2$ are converted to polar form resulting in the following simplification:

\[
a_1' = \Gamma_1 a_2^2 \sin(\gamma)
\]
\[
a_2' = -\Gamma_2 a_1 a_2 \sin(\gamma)
\]
\[
a_1 a_1' = \Gamma_1 a_2^2 \cos(\gamma)
\]
\[
a_2 a_2' = \Gamma_2 a_1 \cos(\gamma)
\]

(44)

where

\[
\gamma = \sigma T_1 + \alpha_1 - 2\alpha_2
\]

(45)

Combining the last two equations in (44) gives:

\[
a_1 \gamma' = \Gamma_1 a_2^2 \cos(\gamma) - 2\Gamma_2 a_1^2 \cos(\gamma) + a_1 \sigma
\]

(46)

and combining the first two equations in (44) gives the following conservation of energy relationship for the modal amplitudes:

\[
a_1^2 + v a_2^2 = E
\]

(47)
Note that to show modal coupling for 1:2 IR we had to use terms only up to $e^2$ order. This is typical for 1:2 IR because only quadratic terms contribute to the modal coupling. Figure 8 shows the modal amplitude response on the slow time scale $T_1$ obtained using (44). The response shows that the amplitudes are coupled but the coupling is not as strong as in 1:1 IR case.

### 8.3 Numerical Simulations

Figure 9 shows the time response for a set of 1:2 IR tuned parameters. Note that the beat is present but it is not as strong as in 1:1 IR case. Figure 10 shows the damped response for the same set of parameters. Adding damping to the system shows that small oscillations remain in the system which decay very slowly. Although using 1:2 IR the vibrations cannot be suppressed as quickly as 1:1 IR, nevertheless it is still a useful technique when only quadratic couplings are present.

### 9. VIBRATION SUPPRESSION USING $r_1 = r_2$

In this section we present a new technique for vibration suppression. This method involves tuning the system parameters to make the linear mode shapes equal. These mode shapes are represented by the modal amplitude ratios $r_1$ and $r_2$ given by (20). When the mode shapes are made equal the response for $\theta_1$ and $\theta_2$ becomes similar, and upon introducing damping in one direction the oscillations in the other direction are also suppressed. To make the mode shapes equal it requires setting $r_1 = r_2$, which gives the following equation:

\[
\begin{align*}
\sigma_2 &= \sigma_7 \\
\omega_n^1 + \omega^*^2 \cos(2\theta_1e) + \sin(\theta_1e) - \omega_n^2 \sec^2(\theta_1e) &= 0
\end{align*}
\]

(48)

![Figure 8](image)

**Figure 8** Modal amplitude response—1:2 IR. $\omega^* = 2.0$, $\omega_n^1 = 1.54915$, $\omega_n^2 = 5.0$, $\theta_1e = 0.1$, $\sigma = 0.0$. 
Note that the above equations are the same as (23) which were obtained by minimizing the difference between the frequencies with respect to $\omega_{n1}^*$. This implies that the difference between the frequencies $\delta$ can be obtained from (24), and for small values of $\theta_{te}$ the difference is small and both $r_1 = r_2$ and 1:1 IR give the same results. When $\theta_{te}$ is not small the difference between the frequencies $\Omega_1$ and $\Omega_2$ is large and the system does not exhibit 1:1 IR, however the mode shapes are identical ($i.e., r_1 = r_2$). Figure 11 shows the numerical simulation results for this case for damped and undamped conditions. The response is nonresonant but still gives very good vibration suppression results. Some applications may not allow tuning the equilibrium positions to large values; in such cases we have to rely on tuning $\omega_{n1}^*$ to establish $r_1 = r_2$ for which the range is limited. To overcome this problem, we present a method to obtain a control torque which will make the system satisfy $r_1 = r_2$, regardless of the values of the system parameters.
9.1 Control Torque for $r_1 = r_2$

To make the system exhibit $r_1 = r_2$, the condition $s_2 = s_7$ has to be satisfied. To achieve this we modify the control torque $T^*$ so that in the linearized equations of motion (8) the $s_2$ term becomes equal to $s_7$. It can be seen that if choose $T^*$ as:

$$T^* = -(s_2 - s_7)\theta_1 + T_c + C\theta_1^*$$

$$= -(\omega_{n1}^* + \omega_{n2}^* \cos(\theta_{te}) + \sin(\theta_{te}) - \omega_{n2}^* \sec^2(\theta_{te}))\theta_1 + T_c + C\theta_1^*$$

(49)

after the differentiation involved in the linearization using Taylor series, in (8), $s_2$ becomes equal to $s_7$. In (49), the term $T_c$ is a constant, which is used to establish the equilibrium position. With the modification of $T^*$ as in (49), the equations for the equilibrium positions
(7) become:

\[
\omega_n^2 \theta_1 + \frac{1}{2} \omega^2 \sin(\theta_1) - \cos(\theta_1) - (\omega_1^2 + \omega_2^2 \cos(2\theta_1))
\]

\[
+ \sin(\theta_1) - \omega_n^2 \sec^2(\theta_1) \theta_1 + T_c^* = 0
\]

\[
\omega_n^2 \theta_2 = 0
\]

Figure 11 Time response—\(r_1 = r_2\), \(\omega^* = 7.0\), \(\omega_{n1}^* = 4.0\), \(\omega_{n2}^* = 5.0\), \(\theta_{1e} = 0.58\). (a) \(C = 0.0\); (b) \(C = 6.0\).

Given a set of parameters, the value of \(T_c^*\) required to achieve a desired equilibrium position is obtained from (50).

With the generalizations introduced in this section we can see the effect of establishing the modal coupling for a given set of parameters. Figure 12(a) shows the response of the system when the control torque is not applied (only damping is introduced using velocity feedback), and Figure 12 (b) shows the effect of applying the control torque to establish \(r_1 = r_2\). It can be clearly seen that when modal coupling is established the vibrations are quickly suppressed.
10. CONCLUSION

In this paper we have discussed vibration suppression of a two-degree-of-freedom flexible gyroscopic system using modal coupling. When modal coupling is established a strong energy link is formed between the modes which is used to transfer energy from the uncontrolled mode to the controlled mode from where it is dissipated. The equations of motion show that the system is coupled through linear, quadratic, and cubic terms, indicating that the coupling can be enhanced using 1:1 and 1:2 IR ratios. A study of the system parameters reveal that the system cannot be tuned to establish an exact 1:1 IR ratio. However, it was shown that for small values of the equilibrium position $\theta_{1e}$, the frequencies can be approximated as 1:1 IR. It was also shown the range of the tuning parameter to establish a desired IR ratio was limited. To overcome this a control torque was introduced which upon application makes the system exhibit modal coupling, regardless of the choice of the system parameters.

The modal interaction under Internal Resonance conditions was ascertained analytically using the perturbation method of multiple scales. Results indicate that when 1:1 IR was
established the modal interactions were strong and the vibrations were suppressed quickly. However for 1:2 IR the modal coupling was not as strong as in 1:1 IR case resulting in oscillations which continued to exist for a long period of time. A new method was presented based on the equality of the mode shapes \((r_1 = r_2)\), which is a generalization of 1:1 IR. Using this method a strong modal interaction was established even under nonresonant conditions.

References

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