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Nonnegativity of Uncertain Polynomials

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The purpose of this paper is to derive tests for robust nonnegativity of scalar and matrix polynomials, which are algebraic, recursive, and can be completed in finite number of steps. Polytopic families of polynomials are considered with various characterizations of parameter uncertainty including affine, multilinear, and polynomic structures. The zero exclusion condition for polynomial positivity is also proposed for general parameter dependencies. By reformulating the robust stability problem of complex polynomials as positivity of real polynomials, we obtain new sufficient conditions for robust stability involving multilinear structures, which can be tested using only real arithmetic. The obtained results are applied to robust matrix factorization, strict positive realness, and absolute stability of multivariable systems involving parameter dependent transfer function matrices.

Keywords: Interval polynomials; uncertain parameters; polytope families; matrix polynomials; absolute stability

Classification Categories: 93B35, 26C10, 34D25

I. INTRODUCTION

Nonnegativity of polynomials, being the underlying condition in the general problem of spectral factorization, can be traced back to the classical work of D. Hilbert and N. Wiener. It has appeared since in a wide variety of engineering problems including passivity of electric circuits [1], absolute stability and hyperstability of control systems [2,3], optimal control [4], stability of multi-dimensional [5] and repetitive systems [6], and stability of systems with time-delay [7]. Our objective is to formulate the robustness problem of nonnegativity and positivity of polynomials, and offer solutions that can be useful in engineering applications involving parameter uncertainty.

Robustness of positivity with respect to plant parameters has been considered in the classical work on absolute stability by Aizerman and Gantmacher [8]. The analysis was limited to examples of second and third order systems, and it was left unclear how the obtained inequalities involving plant parameters can be generalized for higher order systems. After the Modified Routh Array [3,9] was proposed for testing positivity of real polynomials, it became obvious that the analysis could not be generalized, because the number of inequalities grew at an unacceptable rate with the dimension of the system [10]. The envelope method [3], which was proposed at the same time to generate the boundaries of the absolute stability regions in the parameter space for high order systems, was limited to two or three parameters. Recent versions [11,12] of the envelope method have resolved the dimensionality problem of the parameter space and offered new possibilities for deciding the robustness issues.

In the late 1960's, it was recognized that the problem of robust absolute stability in the parameter space is one of interpretation [3]. The idea was to embed a geometric figure in the stability region and maximize its size to get the best results. The mathematical programming methods were applied to the optimization problem [13], but no advantage was taken of the known fact [3] that under affine uncertainty assumptions the stability region is convex. Recently, the convexity was exploited using the powerful ellipsoidal method to solve the nominal absolute stability problem [14] without addressing the issue of robustness (see also [15]). The potential of the method of linear matrix inequalities [16] in this context remains unexplored.

The seminal work of Kharitonov [17] introduced a new way to look at the problems of robust stability of linear systems. A hyperrectangle, which was defined in the coefficient space of a real polynomial, was shown by Kharitonov to be fully embedded inside the stability region if and only if *four* of its vertices corresponded to stable polynomials, regardless of the order of the system. This elegant result has been generalized and applied to a wide variety of robust control problems [18,19]. When we apply Kharitonov's approach to positivity of interval polynomials, we find that positivity of the interval family is equivalent to positivity of a polynomial at a *single* vertex of the corresponding hyperrectangle. Furthermore, unlike the stability case, to verify positivity of a polytope family we need only test positivity of polynomials at the vertices of the polytope [20]. This result allows us to estimate absolute stability regions of continuous and discrete

systems by exploiting the fact [3] that with respect to the numerator coefficients of the transfer function the stability region is convex. When the polynomial results are specified to strict positive realness and absolute stability of interval plants, where a polytope becomes a hyperrectangle, the results do not compare well with the transfer function approach [19,21-25]. The reason is that one needs to test only a subset of vertices of the rectangle – not all of them as in the polynomial method [20]. The transfer function results, however, are based upon special properties of scalar rational functions, which cannot be readily extended to multivariable systems involving matrix transfer functions. The main purpose of this work is to derive criteria for nonnegativity of uncertain matrix polynomials, and show how the obtained results apply to robust factorization, absolute stability, and positive realness involving multivariable systems and matrix transfer functions.

The organization of the paper is as follows: In the next section, we define nonnegativity of scalar polynomials on the imaginary axis. Section III is devoted to converting imaginary axis nonnegativity to real axis nonnegativity which can be tested by the algebraic, recursive, and finite algorithm of the Modified Routh Array. In Section IV, we consider uncertain scalar polynomials and establish conditions for positivity of interval, polytope, and multilinear polynomial families. We also use the Zero Exclusion Condition and show that in all these situations, there are interesting distinctions between the positivity and stability of uncertain polynomials. In Section V, the robust stability problem of a complex polynomial with multilinear uncertainty structure is reformulated as problem of robust positivity of a *real* polynomial but with polynomic uncertainty. Examples are provided to indicate when such a reformulation may be useful. Section VI contains the criteria for robust positivity of polynomial matrices. The main result in this context is the reduction of positivity of a matrix polynomial to positivity of its determinant. Both the direct matrix criteria and the scalar tests involving the determinant are derived to conclude robust positivity of matrix polynomials. Finally, in Sections VII and VIII, we apply the obtained positivity criteria to robust factorization, absolute stability, and positive realness of multivariable systems.

II. NONNEGATIVITY

Let us consider a polynomial

$$f(s) = \sum_{k=0}^{n} a_k s^k \tag{2.1}$$

where a_k are complex numbers and $a_n \neq 0$. To avoid trivialities we assume that $f(s) \not\equiv a_0$. Our interest is to study nonnegativity of $f(s)$ on the imaginary axis

$$\mathbf{I} = \{s \in \mathbb{C} : \operatorname{Re} s = 0\} \tag{2.2}$$

of the complex plane $\mathbb{C}$.

(2.3) DEFINITION A polynomial $f(s)$ is **I**-nonnegative if

$$f(s) \geq 0 \quad \forall s \in \mathbf{I}. \tag{2.4}$$

A strict version of this definition is

(2.5) DEFINITION A polynomial $f(s)$ is **I**-positive if

$$f(s) > 0 \quad \forall s \in \mathbf{I}. \tag{2.6}$$

We use the notation

$$f_*(s) = \bar{f}(-s) = \sum_{k=0}^{n} (-1)^k \bar{a}_k s^k, \tag{2.7}$$

for the para-conjugate polynomial of $f(s)$. If

$$f_*(s) = f(s), \tag{2.8}$$

then we say that $f(s)$ is **I**-symmetric. This term is justified by the following theorem, where we provide several characterizations of nonnegativity:

(2.9) THEOREM If $f(s)$ is **I**-nonnegative, then the following statements are true:

i. $f(s)$ is **I**-symmetric and the coefficients a_k are such that

$$a_k = (-1)^k \bar{a}_k, \quad k = 0, 1, \ldots, n \tag{2.10}$$

which is equivalent to

$$\begin{aligned} \operatorname{Im} a_k &= 0, \qquad k \text{ even} \\ \operatorname{Re} a_k &= 0, \qquad k \text{ odd.} \end{aligned} \tag{2.11}$$

ii. The number $\tilde{n}$ of zeros of $f(s)$ is even and they can be grouped in para-conjugate pairs

$$(s_1, \hat{s}_1), (s_2, \hat{s}_2), \ldots, (s_{\tilde{n}}, \hat{s}_{\tilde{n}}), \tag{2.12}$$

where $\hat{s}_k = -\bar{s}_k$, and $\tilde{n} = n/2$.

iii. $f(s)$ admits at least one factorization of the form

$$f(s) = h(s)h_*(s), \tag{2.13}$$

and $h(s)$ is a polynomial defined by

$$h(s) = b(s - s_{i_1})(s - s_{i_2}) \dots (s - s_{i_{\tilde{n}}}). \tag{2.14}$$

The numbers s_{i_k} are arbitrarily chosen, one from each pair in (2.12), and b is a complex number (with arbitrary argument), which depends on the choice of numbers s_{i_k}.

iv. If $f(s)$ has real coefficients, that is, it is a real polynomial, then there exists at least one factorization (2.13), such that $h(s)$ is a real polynomial as well, and (2.13) becomes

$$f(s) = h(s)h(-s)\ . \tag{2.15}$$

(2.16) REMARK Theorem (2.9) is the "axis" version of the "circle" Proposition 1 in Appendix B of [2]. From this result, it is clear that if a polynomial $f(s)$ is **I**-nonnegative its zeros have the axis symmetry with respect to the imaginary axis **I**; it is **I**-symmetric. This property will allow us to formulate in the next section a nonnegativity test that requires only real arithmetic. We should also note that the results can be rephrased in terms of the nonnegativity with respect to the unit circle along the lines of [27,28].

III. MODIFIED ROUTH ARRAY

Now we want to present the algorithm for testing **I**-nonnegativity of a polynomial $f(s)$ with complex coefficients. This requires that we reformulate **I**-nonnegativity in terms of the zeros of $f(s)$.

(3.1) PROPOSITION An **I**-symmetric polynomial $f(s)$ is **I**-nonnegative if and only if it has no zeros on **I** of odd multiplicity, and $f(s_0) > 0$ for some $s_0 \in \mathbf{I}$.

In applications we often need the strict version of the stated proposition, which is

(3.2) PROPOSITION An **I**-symmetric polynomial $f(s)$ is **I**-positive if and only if $f(s)$ has no zeros on **I**, and $f(s_0) > 0$ for some $s_0 \in \mathbf{I}$.

Let us now use the transformation $s \mapsto i\tilde{s}$ to get the new polynomial $\tilde{f}(\tilde{s})$. Since the transformation rotates the zeros of *f(s)* by $\pi/2$, and *f(s)* is **I**-symmetric, the complex (non-real) zeros of $\tilde{f}(\tilde{s})$ appear in conjugate pairs; $\tilde{f}(\tilde{s})$ is a real polynomial. To see this directly in terms of the coefficients $a_k = \alpha_k + i\beta_k$ of *f(s)*, we first note that condition (2.11) implies that *f(s)* has the form

$$f(s) = \alpha_0 + i\beta_1 s + \alpha_2 s^2 + i\beta_3 s^3 + \alpha_4 s^4 + i\beta_5 s^5 + \dots . \tag{3.3}$$

After the announced transformation $s \mapsto i\tilde{s}$, we get

$$\tilde{f}(\tilde{s}) = \alpha_0 - \beta_1 \tilde{s} - \alpha_2 \tilde{s}^2 + \beta_3 \tilde{s}^3 + \alpha_4 \tilde{s}^4 - \beta_5 \tilde{s}^5 - \dots, \tag{3.4}$$

which is a real polynomial and, thus, real on the real line $\mathbb{R}$. Furthermore, complex zeros of $\tilde{f}(\tilde{s})$ appear in conjugate pairs. In other words, $\tilde{f}(\tilde{s})$ is $\mathbb{R}$-symmetric, that is, it is a real polynomial. Most importantly, the existence and multiplicity of zeros of *f(s)* remain intact after transformation, when they become the real zeros of $\tilde{f}(\tilde{s})$.

We start with the $\mathbb{R}_+$ part of the real line $\mathbb{R}$ and state the following:

(3.5) DEFINITION A real polynomial *f(s)* is $\mathbb{R}_+$-nonnegative if

$$f(s) \geq 0 \quad \forall s \in \mathbb{R}_+ . \tag{3.6}$$

The interpretation of $\mathbb{R}_+$-nonnegativity in terms of zeros of *f(s)* is obvious:

(3.7) PROPOSITION A real polynomial *f(s)* is $\mathbb{R}_+$-nonnegative if and only if it has no positive real zeros of odd multiplicity, and $f(s_0) > 0$ for some $s_0 \in \mathbb{R}_+$.

(3.8) REMARK Equally obvious are the strict versions of Definition (3.5) and Proposition (3.7) characterizing the $\mathbb{R}_+$-positivity of *f(s)*.

It has been shown on several occasions [3,9] how the array of Routh can be modified to count the number of positive zeros of a real polynomial, including their multiplicities. The *Modified Routh Array* provides a numerical nonnegativity test, which is algebraic, recursive and finite. We consider a real polynomial *f(s)* defined in (2.1), and state the following [9]:

(3.9) THEOREM The number π of positive zeros of a real polynomial *f(s)* is

$$\pi = n - V(r_0, r_1, \dots, r_{2n}), \tag{3.10}$$

where V is the number of sign variations in the first column of the Modified Routh Array,

$$\begin{array}{lcllll} r_0 & = & (-1)^n a_n & (-1)^{n-1} a_{n-1} & \dots & -a_1 \quad a_0 \\ r_1 & = & (-1)^n n a_n & (-1)^{n-1}(n-1) a_{n-1} & \dots & -a_1 \\ & \vdots & & & & \\ r_{2n} & = & a_0 & & & \end{array} \tag{3.11}$$

and $r_1, r_2, \dots, r_{2n}$ are all different from zero.

If the array (3.11) is *regular,* that is, all numbers r_i are nonzero, then we have from Theorem (3.9):

(3.12) THEOREM A real polynomial $f(s)$ is $\mathbb{R}_+$-positive if and only if $V(r_0, r_1, \dots, r_{2n}) = n$, and $a_0 > 0$.

When the array is *singular,* and a whole row of the array goes to zero, then $f(s)$ may have real positive zeros of odd multiplicity, which is relevant for nonnegativity of $f(s)$. To count the multiplicities of positive zeros by array (3.11), we follow the procedure of Karmarkar [29] and first enumerate the rows of the array by $j = 1, 2, \dots, 2n+1$, and the row preceding the identically zero row by j_m, $m = 1, 2, \dots, M-1$. Let us also define

$$n_m = \frac{1}{2}(j_m - j_{m-1}), \quad m = 1, 2, \dots, M \tag{3.13}$$

with $j_0 = 1$ and $j_M = 2n + 1$. For the number of sign variations in the first column of (3.11), between the two consecutive rows j_{m-1} and j_m, we use the symbol V_m. Finally, we denote by π_m the number of positive zeros of $f(s)$ with multiplicity m. Then, we have

$$\begin{aligned} \pi_m &= (n_m - V_m) - (n_{m+1} - V_{m+1}), \quad m = 1, 2, \dots, M \\ \pi_{m+1} &= n_{m+1} - V_{m+1}, \end{aligned} \tag{3.14}$$

and

(3.15) THEOREM A real polynomial $f(s)$ is $\mathbb{R}_+$-nonnegative if and only if the Modified Routh Array produces $\pi_m = 0$ for all odd $m = 1, 2, \dots, M$, and $f(s_0) > 0$ for some $s_0 \in \mathbb{R}_+$.

(3.16) REMARK To determine $\mathbb{R}_-$-positivity or nonnegativity of a complex $\mathbb{R}$-symmetric polynomial $f(s)$ we apply Theorem (3.9) to $f(-s)$ and use Theorem (3.12) or (3.15) in an obvious way. If we want to test positivity

(nonnegativity) of $f(s)$ on the whole real line $\mathbb{R}$ that is, $\mathbb{R}$-positivity (-nonnegativity), we apply the theorems to both $f(s)$ and $f(-s)$.

(3.17) REMARK The important case in applications [3,30] is when $f(s)$ is real to start with. Then $a_k = \alpha_k$, and from (3.3) we get

$$f(s) = \sum_{k=0}^{n} a_{2k} s^{2k} \tag{3.18}$$

which is an even polynomial. The zeros of $f(s)$ have a "rectangle symmetry," that is, they are symmetrically distributed with respect to both imaginary axis $\mathbf{I}$ and real axis $\mathbb{R}$. To determine $\mathbb{R}_+$ positivity in this case, all we need to do is use the transformation $s^2 \mapsto \tilde{s}$ and apply array (3.11) to the new polynomial $\tilde{f}(\tilde{s})$. Obviously, if $\tilde{f}(\tilde{s})$ is found to be $\mathbb{R}_+$-positive, it is automatically $\mathbb{R}$-positive as well.

When circle nonnegativity is considered, the Marden-Jury algorithm can be used to determine the number and multiplicity of the zeros of a real polynomial on the unit circle [27]. The algorithm can be implemented using well-known Jury's table [10].

IV. UNCERTAIN POLYNOMIALS

We start with positivity considerations of a real scalar polynomial

$$f(s,p) = \sum_{k=0}^{n} a_k(p) s^k, \tag{4.1}$$

where $p \in \mathbb{R}^\ell$ is the uncertain parameter vector which belongs to an uncertainty bounding set $\mathbf{P}$. For a fixed $p^0 \in \mathbf{P}$, the polynomial $f(s, p^0) \equiv f(s)$ has numerical coefficients a_k.

The uncertain polynomial $f(s, p)$ and set $\mathbf{P}$ form a polynomial family

$$\mathcal{F} = \{f(\cdot, p) : p \in \mathbf{P}\}. \tag{4.2}$$

Positivity of family $\mathcal{F}$ is expressed by the following:

(4.3) DEFINITION A family $\mathcal{F}$ is positive if $f(s, p)$ is positive for all $p \in \mathbf{P}$.

Testing positivity of $\mathcal{F}$ depends crucially on the type of coefficient functions $a_k(p)$, as well as the shape of set $\mathbf{P}$. In all cases, it is standard to assume that a_k's are continuous functions of p, $\mathbf{P}$ is a bounded pathwise connected set, and $f(s, p)$ has invariant degree, that is, the degree of $f(s, p)$

is n for all $p \in \mathbf{P}$. Favorite special cases of $\mathbf{P}$ are hyperrectangles and convex polytopes.

Let us start with the special case when the coefficients a_k are independent parameters, that is, $p = (a_0, a_1,..., a_n)$. We also assume that a_k's belong to given intervals,

$$a_k \in [\underline{a}_k, \overline{a}_k], \quad k \in \mathbf{n} \tag{4.4}$$

where $\mathbf{n} = \{0, 1,..., n\}$. Then, the uncertainty set is a hyperrectangle

$$\mathbf{P} = \{p : \mathbb{R}^{n+1} : a_k \in [\underline{a}_k, \overline{a}_k], \quad k \in \mathbf{n}\} \tag{4.5}$$

We define the minorizing polynomial

$$\underline{f}(s) = \sum_{k=0}^{n} \underline{a}_k s^k, \tag{4.6}$$

and prove:

(4.7) THEOREM An interval family $\mathcal{F}$ of polynomials $f(s, p)$ is $\mathbb{R}_+$-positive if and only if the polynomial $\underline{f}(s)$ is $\mathbb{R}_+$-positive.

Proof The fact $\underline{f}(s) \in \mathcal{F}$ implies necessity. Since

$$f(s,p) \geq \underline{f}(s) > 0 \quad \forall s \in \mathbb{R}_+ \quad \forall p \in \mathbf{P} \tag{4.8}$$

sufficiency follows. Q.E.D.

The bounding set $\mathbf{P}$ of (4.5) is a hyperrectangle in $\mathbb{R}^{n+1}$ having 2^{n+1} vertices. Theorem (4.7) says that for $\mathcal{F}$ to be positive, it is necessary and sufficient that a *single* polynomial $\underline{f}(s)$ at a vertex of $\mathbf{P}$ be positive. By contrast, when stability of an interval family $\mathcal{F}$ is considered, the well-known result of Kharitonov [17] states that stability of $\mathcal{F}$ is equivalent to stability of *four* vertices of $\mathbf{P}$.

(4.9) REMARK Theorem (4.7) can be applied to more general cases of coefficients a_k being dependent on parameter vector p. Then, a suitable bounding

$$a_k(p) \geq \underline{a}_k, \quad k \in \mathbf{n} \tag{4.10}$$

by numbers $\underline{a}_k$ can produce the minorizing polynomial $\underline{f}(s)$. Since the resulting polynomial $\underline{f}(s)$ may be outside of the family $\mathcal{F}$ its positivity is generally only sufficient for positivity of $\mathcal{F}$.

A way to generalize Theorem (4.7) and retain necessity is to consider *convex polytopes* bounding the uncertain parameters and assume $a_k(p)$ to be *affine linear functions* of p. This generalization, however, requires test-

ing positivity of corresponding (fixed) polynomials at all vertices of the chosen polytope. The affine assumption is

$$a_k(p) = b_k^T p + c_k, \quad k \in \mathbf{n} \tag{4.11}$$

where $b_k \in \mathbb{R}^\ell$ are constant column vectors, c_k are scalars, p is a column vector, and T denotes transpose.

Let us define a convex polytope $\mathbf{P}$ in $\mathbb{R}^\ell$ as the convex hull of a finite set of points $\{p^1, p^2, \ldots, p^v\}$, that is,

$$\mathbf{P} = \operatorname{conv}\{p^i\}, \quad i \in \mathbf{v} \tag{4.12}$$

and $\mathbf{v} = \{1, 2, \ldots, v\}$. Then, $\mathcal{F}$ is a *polytope of polynomials,*

$$\mathcal{F} = \operatorname{conv}\{f(\cdot, p^i)\}, \quad i \in \mathbf{v} \tag{4.13}$$

where vertex polynomials $f(s, p^i)$ are *generators* for $\mathcal{F}$.

(4.14) THEOREM A polytope family $\mathcal{F}$ is $\mathbb{R}_+$-positive if and only if all generators $f(s, p^i)$ for $\mathcal{F}$ are $\mathbb{R}_+$-positive.

Proof Necessity is obvious. To prove sufficiency, we note that for any point $p \in \mathbf{P}$, there exist numbers $\lambda_i \geq 0$, $i \in \mathbf{v}$, such that

$$p = \sum_{i=1}^{v} \lambda_i p^i, \quad \sum_{i=1}^{v} \lambda_i = 1 \tag{4.15}$$

Then, the corresponding polynomial $f(s, p)$ can be expressed as

$$\begin{aligned} f(s,p) &= f(s, \sum_{i=1}^{v} \lambda_i p^i) \\ &= \sum_{k=0}^{n} (b_k^T \sum_{i=1}^{v} \lambda_i p^i + c_k) s^k \\ &= \sum_{i=1}^{v} \lambda_i \sum_{k=0}^{n} (b_k^T p^i + c_k) s^k \\ &= \sum_{i=1}^{v} \lambda_i f(s, p^i) \end{aligned} \tag{4.16}$$

and $\mathbb{R}_+$-positivity of generators $f(s, p^i)$ implies $\mathbb{R}_+$-positivity of $\mathcal{F}$. Q.E.D.

A more complex uncertainty structure is the case when $a_k(p)$ are *multilinear functions* of p. This case can be reduced to that of affine functions if we define a vector $p^{[j]} = (p_1, p_2, \ldots, p_{j-1}, p_{j+1}, \ldots, p_\ell)^T \in \mathbb{R}^{\ell-1}$ as the vector p with component p_j deleted. Then, each $a_k(p)$ can be expressed as

$$a_k(p) = b_k(p^{[j]}) p_j + c_k(p^{[j]}). \tag{4.17}$$

If we fix $p^{[j]}$, then b_k and c_k are constants, and coefficients $a_k(p)$ become affine functions of the component p_j. Therefore, to apply Theorem (4.14) to the multilinear case, we have to restrict uncertainty bounding set **P** to a hyperrectangle,

$$\mathbf{P} = \{p \in \mathbb{R}^\ell : \ p_k \in [\underline{p}_k, \overline{p}_k], \quad k \in \mathbf{l}\} \tag{4.18}$$

where $\mathbf{l} = \{1,2,...,\ell\}$. We denote 2^ℓ vertices of **P** by p^i, consider the family of polynomials $\mathcal{F} = \{f(\cdot, p) \ : p \in \mathbf{P}\}$, and prove the following:

(4.19) THEOREM A multilinear family $\mathcal{F}$ is $\mathbb{R}_+$-positive if and only if all generators $f(s, p^i)$ for $\mathcal{F}$ are $\mathbb{R}_+$-positive.

Proof We note first that for each fixed $s \in \mathbb{R}_+$, the image of the hyperrectangle **P** under the mapping $f(s, \cdot)$ is a closed and bounded interval $\mathbf{J}(s) = [\alpha(s), \beta(s)]$. Since $f(s, p)$ is a multilinear function of p, the minimum of $f(s, p)$ is attained at an extreme point p^i of the hyperrectangle **P**, (see Lemma 14.5.5 of [18]). This fact implies that for any fixed $s \in \mathbb{R}_+$,

$$\min_{p \in \mathbf{P}} f(s,p) = \min_i f(s,p^i) = \alpha(s), \tag{4.20}$$

and the sufficiency follows. The necessity is obvious. Q.E.D.

(4.21) REMARK Theorem (4.19) is stronger than the corresponding result in the stability context ([18] Remarks 14.8.2). The reason is that for each fixed $s \in \mathbb{R}_+$, the value set $f(s, \mathbf{P})$ is an interval, and $0 \in \text{conv } f(s, \mathbf{P})$ implies $0 \in f(s, \mathbf{P})$. We will exploit this advantage in the next section, where we convert stability of complex uncertain polynomials to positivity of real uncertain polynomials.

(4.22) REMARK Theorem (4.19) can be applied to *polynomic uncertainty* structures via the transformation proposed in [31] (see also Lemma 14.3.9 of [18]). The transformation converts uncertainty hyperrectangle **P** to a polytope, which prevents the use of the vertex result offered by Theorem (4.19). To retain the hyperrectangle uncertainty structure, one can use "overbounding" by a higher dimensional hyperrectangle at the expense of losing the necessity part of the theorem and, thus, incurring conservativeness in the end result. Alternatively, one can stay with the polytopic set and retain necessity by applying the value set concept capitalizing on the fact that the value sets are straight line segments on the real line $\mathbb{R}$. In this context, either the minimization (4.20) is carried out for each $s \in \mathbb{R}_+$, or the Zero Exclusion Condition [18, 19] is applied as explained next.

We show how the Zero Exclusion Condition [18,19] applies to testing positivity of a family $\mathcal{F}$ of polynomials under the standard assumptions: **P** is bounded and pathwise connected, $a_k(p)$ are continuous, and $f(s, p)$ has invariant degree. We again rely on the fact that the value set $f(s, \mathbf{P})$ for each $s \in \mathbb{R}$ is an interval $\mathbf{J}(s)$, and prove the following:

(4.23) THEOREM A family $\mathcal{F}$ is $\mathbb{R}$-positive if and only if $\mathcal{F}$ has at least one $\mathbb{R}$-positive member $f(s, p^0)$ and $0 \notin f(s, \mathbf{P})$ for all $s \in \mathbb{R}$.

Proof To prove the "only if" part, we assume that $\mathcal{F}$ is positive, but $0 \in f(s^*, \mathbf{P})$ for some $s^* \in \mathbb{R}$. Then, $f(s^*, p^*) = 0$ for some $p^* \in \mathbf{P}$, which contradicts positivity of $\mathcal{F}$.

To establish the "if" part, we again proceed by contradiction. Let us assume that $f(s^*, p^1)$ is negative for some $s^* \in \mathbb{R}$ and $p^1 \in \mathbf{P}$, but $0 \notin f(s, \mathbf{P})$ for all $s \in \mathbb{R}$. Since **P** is pathwise connected, there exists a continuous function θ: $[0, 1] \to \mathbf{P}$ such that $\theta(0) = p^0$ and $\theta(1) = p^1$. By continuity of $\theta(\lambda)$, when λ goes from 0 to 1, $f(s^*, \theta(\lambda))$ goes from $f(s^*, p^0) > 0$ to $f(s^*, p^1) < 0$. Due to continuity of $f(s^*, \theta(\lambda))$ with respect to λ, there exists some λ^* such that $f(s^*, \theta(\lambda^*)) = 0$, that is, $f(s^*, p^*) = 0$, where $p^* = \theta(\lambda^*)$. Therefore, $0 \in f(s, \mathbf{P})$, which is a contradiction. Q.E.D.

All results of this section can be reformulated for robust circle positivity of uncertain polynomials along the lines of [20]. The relevant circle positivity test of numerical polynomials has been introduced in [27] (see also [10], [28], and [32]).

V. ROBUST STABILITY

We consider a complex polynomial

$$h(s,p) = \sum_{k=1}^{m} b_k(p) s^k, \tag{5.1}$$

where we assume that $b_k(p)$ are multilinear functions of $p \in \mathbf{P}$, **P** is a hyperrectangle, and $h(s, p)$ has invariant degree. We recall [18, 19] that $h(s, p)$ for a fixed p, is stable if all zeros of $h(s, p)$ lie in the strict left half of plane $\mathbb{C}$. We introduce the polynomial family

$$\mathcal{H} = \{h(\cdot, p) : p \in \mathbf{P}\}, \tag{5.2}$$

and state the following

(5.3) DEFINITION A multilinear family $\mathcal{H}$ is stable if $h(s, p)$ is stable for all $p \in \mathbf{P}$.

To establish stability of O we form the **I**-symmetric polynomial

$$f(s,p) = h(s,p)h_*(s,p), \tag{5.4}$$

which has the polynomic uncertainty structure. The corresponding polynomic family is

$$\mathcal{F} = \{f(\cdot,p) : p \in \mathbf{P}\}, \tag{5.5}$$

and we prove the following:

(5.6) THEOREM A multilinear family $\mathcal{H}$ is stable if and only if there is a stable polynomial $f(s,p^0) \in \mathcal{H}$ and the corresponding polynomic family $\mathcal{F}$ is **I**-positive.

Proof To prove the "only if" part, we assume that $\mathcal{H}$ is stable, but $\mathcal{F}$ is not **I**-positive. Then, $f(s^*, p^*) = 0$ for $p^* \in \mathbf{P}$ and $s^* \in \mathbf{I}$ implying $h(s^*, p^*) = 0$, which is a contradiction. For the "if" part we again proceed by contradiction and assume that $\mathcal{H}$ is not stable, but $\mathcal{F}$ is **I**-positive. Since $\mathcal{H}$ is not stable, there exists an unstable polynomial $f(s, p^1) \in \mathcal{H}$. From the fact that the coefficients $b_k(p)$ are continuous in p, it follows that the root functions $s_1(p), s_2(p),..., s_m(p)$ of $h(s, p)$ are continuous as well. This means that at least one root function $s_i(p)$ in going from $s_i(p^0)$ to $s_i(p^1)$ which is situated in the closed right half plane $\mathbb{C}_+$, must reach the imaginary axis **I**. This implies that $\mathcal{F}$ is not positive, which is the contradiction we want. Q.E.D.

There are several advantages that Theorem (5.6) offers over robust stability analysis by existing methods [18,19,33]. First, the value sets are intervals and, therefore, convex. Second, the positivity test of complex families $\mathcal{F}$ require only real arithmetic. Third, the family $\mathcal{F}$ has polynomic structure with (at most) quadratic terms in p. In deciding positivity of $f(\omega, p)$, these facts open up a possibility to use standard nonlinear programming methods for minimization of $f(\omega, p)$ over **P**.

Alternatively, we can use the transformation of Remark (4.22) to get a vertex test for positivity of $\mathcal{F}$. Unfortunately, the transformation causes the loss of necessity in Theorem (5.6). We illustrate this fact by the following:

(5.7) EXAMPLE Let us consider stability of the complex polynomial

$$h(s,p) = s^2 + (p_1 - p_2 - ip_3)s - p_1p_2 - ip_1p_3 \tag{5.8}$$

and form the real polynomial

$$f(\omega,p) = \omega^4 - 2p_3\omega^3 + (p_1^2+p_2^2+p_3^2)\omega^2 - 2p_1^2p_3\omega + p_1^2(p_2^2+p_3^2) \tag{5.9}$$

using (5.3) and $s = i\omega$. If

$$\mathbf{P} = \{p \in \mathbb{R}^3 : p_1 \in [1,1.3],\ p_2 \in [-1.5,-1.2],\ p_3 \in [3.2,3.5]\} \tag{5.10}$$

we apply the transformation $p_1^2 = \tilde{p}_1\tilde{p}_2$, $p_2^2 = \tilde{p}_3\tilde{p}_4$, $p_3^2 = \tilde{p}_5\tilde{p}_6$, and plot the value set $\tilde{f}(s, \tilde{\mathbf{P}}_1)$ in Figure 1. To plot $\tilde{f}(s, \tilde{\mathbf{P}}_1)$ we considered only the vertices of $\tilde{\mathbf{P}}_1$ obtained from $\mathbf{P}$ using the new parameter vector $\tilde{p} \in \mathbb{R}^6$. Since $s_1 = -1$, $s_2 = -1.5 + i3.2$ are zeros of $h(s, p^0)$ for $p_1^0 = 1$, $p_2^0 = -1.5$, $p_3^0 = 3.2$, the polynomial $h(s, p^0)$ is stable, and Figure 1 implies stability of the corresponding family $\mathcal{H}$.

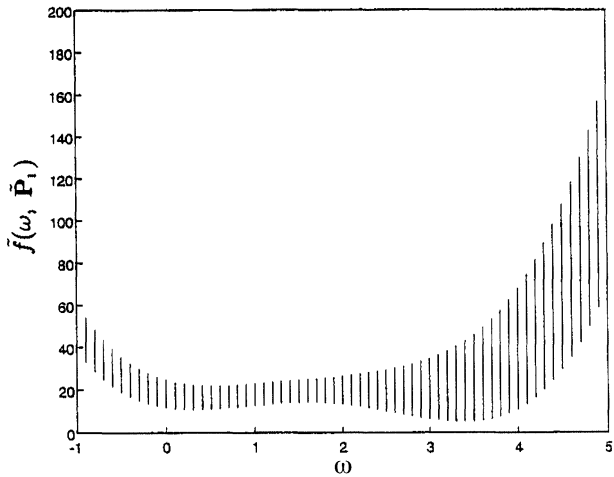

FIGURE. 1. Positive value sets

When we increase the uncertainty interval of p_3 to [3.2,4], we obtain Figure 2, which shows that the corresponding family $\tilde{\mathcal{F}}$ is not positive; the stability test of $\mathcal{H}$ is inconclusive. Yet, zeros of $h(s, p)$ are $s_1 = -p_1$, $s_2 = p_2 + ip_3$, implying stability of $\mathcal{H}$ for the enlarged uncertainty set $\tilde{\mathbf{P}}_2$; therefore, the conservativeness of the overbounding approach.

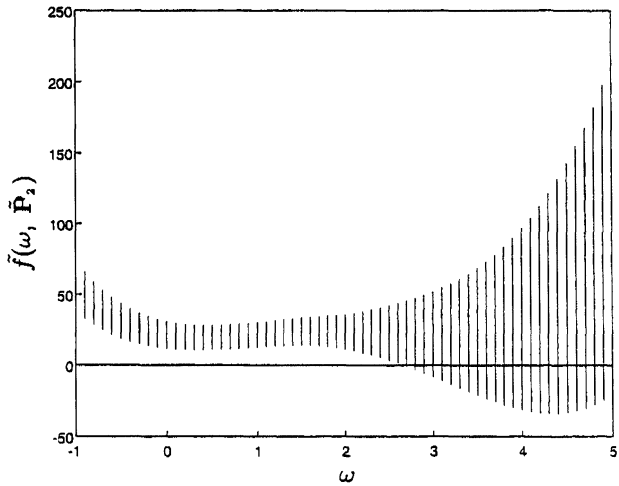

FIGURE 2 Value sets fail positivity

VI. MATRIX POLYNOMIALS

Let us consider a regular matrix polynomial

$$F(s) = \sum_{k=0}^{N} A_k s^k, \tag{6.1}$$

where A_k are constant complex $m \times m$ matrices, and $A_N \neq 0$. We assume that matrix $F(s)$ is paraconjugate Hermitian, that is, $F_*(s) = F(s)$, where $F_*(s) = F^*(\bar{s})$. For $s \in \mathbf{I}$, $F(i\omega)$ is Hermitian in the ordinary sense, since $F^*(i\omega) = F(i\omega)$.

Our immediate objective is to provide the necessary and sufficient conditions for $F(i\omega)$ to be positive definite. By $\mathbb{C}^m$ we denote the m-dimensional unitary space and recall the standard.

(6.2) DEFINITION A Hermitian matrix polynomial $F(s)$ is **I**-positive, written $F(i\omega) > 0$, if

$$z^* F(i\omega) z > 0 \tag{6.3}$$

for all $z \in \mathbb{C}^m - \{0\}$, and all $\omega \in \mathbb{R}$.

Positivity of $F(i\omega)$ can be established by the following [34]:

(6.4) THEOREM A Hermitian matrix polynomial $F(i\omega)$ is positive if and only if

$$F(0) > 0, \text{ and } \det F(i\omega) \neq 0 \quad \forall \omega \in \mathbb{R}. \tag{6.5}$$

Proof Since $F(iw)$ is a Hermitian matrix, there exists a unitary matrix $T(i\omega)$ such that $\Lambda(i\omega) = T^*(i\omega)F(i\omega)T(i\omega)$, where $\Lambda(i\omega) = \mathrm{diag}\{\lambda_1(i\omega), \lambda_2(i\omega),..., \lambda_m(i\omega)\}$. Now, obviously, (6.3) implies (6.5). Conversely, from (6.5) we have $\lambda_j(0) > 0$ and $\lambda_j(i\omega) \neq 0$ for all $\omega \in \mathbb{R}$ and all $j \in \{1, 2,..., m\}$. Since $\lambda_j(i\omega)$ are continuous functions of ω, we have $\lambda_j(i\omega) > 0$ for all $\omega \in \mathbb{R}$, that is, $F(i\omega) > 0$. Q.E.D.

A useful aspect of this result is the fact that positivity of a matrix polynomial $F(i\omega)$ is reduced to checking positivity of a *single* real scalar polynomial which is its determinant. This fact has been exploited in a number of applications, namely testing positive realness of rational matrices [27], stability of two-variable polynomials [5,10,32,34], and absolute stability and optimality of multivariable systems [30]. In each of these cases, all that is needed to do is to compute the Modified Routh Array (3.11) to test positivity of det $F(i\omega)$ and check positivity of a constant symmetric matrix $F(0)$.

We can use the Modified Routh Array to test nonnegativity of $F(i\omega)$ at the price of more elaborate testing. We first assume that $F(i\omega)$ has rank r, that is, there is an r-th order principal minor of $F(i\omega)$, which is not identically zero, and all principal minors of order higher than r vanish identically. We denote by $f^{(r)}(\omega)$ this r-th order minor, and by $f^{(j)}(\omega)$ the leading principal minors of order $j = 1, 2,..., r-1$, which are generated by $f^{(r)}(\omega)$. From [9], we have:

(6.6) THEOREM A Hermitian matrix polynomial $F(s)$ is **I**-nonnegative, written $F(i\omega) \geq 0$, if and only if

$$f^{(j)}(\omega) > 0 \quad \forall\, \omega \in \mathbb{R}, \quad j = 1, 2, \ldots, r. \tag{6.7}$$

Our first and foremost interest is to consider **I**-positivity of a paraconjugate Hermitian matrix polynomial $F(s,p)$ containing uncertain parameter vector $p \in \mathbb{R}^\ell$,

$$F(s,p) = \sum_{k=0}^{N} A_k(p)s^k, \tag{6.8}$$

where p belongs to a convex hull **P** of v points p^i as in (4.12), and $A_N(p) \neq 0$ for all $p \in \mathbf{P}$. We assume that the entries of the coefficient matrices $A_k(p)$ are affine linear functions of p. Then, matrix polynomial $F(s, p)$ and polytope **P** form a polytope family of matrix polynomials,

$$\mathcal{M} = \{F(\cdot, p) : p \in \mathbf{P}\}. \tag{6.9}$$

Robust positivity of uncertain matrix polynomials is captured by the following:

(6.10) DEFINITION A polytope family $\mathcal{M}$ of matrix polynomials $F(s, p)$ is **I**-positive if $F(i\omega, p) > 0$ for all $\omega \in \mathbb{R}$ and all $p \in \mathbf{P}$.

Positivity of $\mathcal{M}$ in terms of its generators, which can be tested for positivity by repeated use of Theorem (5.4), is characterized by

(6.11) THEOREM A polytope family $\mathcal{M}$ is **I**-positive if and only if all generators $F(i\omega, p^i)$ for $\mathcal{M}$ are positive.

Proof As in the proof of Theorem (4.14), we express $p \in \mathbf{P}$ as a convex combination (4.15) and obtain

$$\begin{aligned} F(s,p) &= \sum_{k=0}^{N} A_k \Big(\sum_{i=1}^{m} \lambda_i p^i\Big) s^k \\ &= \sum_{i=1}^{v} \sum_{k=0}^{N} \lambda_k A_k(p^i) s^k \qquad (6.12) \\ &= \sum_{i=1}^{v} \lambda_i F(s,p^i) \end{aligned}$$

which expresses $F(s, p)$ as a convex combination of $F(s, p^i)$ for each $p \in \mathbf{P}$. Therefore, positivity of $F(i\omega, p)$ for all $p \in \mathbf{P}$ is equivalent to positivity of all $F(i\omega, p^i)$. Q.E.D.

An alternative approach to the problem of positivity of the polynomial family $\mathcal{M}$ is to use condition (6.5) directly to consider uncertainty in the polynomial $F(s, p)$.This effectively reduces the robustness analysis of matrix polynomial $F(s, p)$ to that of the scalar polynomial $f(s, p) = \det F(s, p)$. The price for this simplification is the fact that affine uncertainty structure in $F(s, p)$ becomes generally a polynomic structure in $f(s, p)$, which makes Remark (4.22) relevant; see Example (8.21).

As in the case of Theorem (4.19) for scalar polynomials, we can retain the extreme point result of Theorem (6.11) even if we deal with $F(s, p)$ with multilinear uncertainty structure, provided we consider a rectangle **P** as the uncertainty bounding set. We recall the notation $p^{[j]}$ of (4.17) and state the obvious fact that if $F(s, p)$ has multilinear structure, it can be expressed as

$$F(s,p) = G_j(s,p^{[j]})p_j + H_j(s,p^{[j]}), \qquad (6.13)$$

where $G_j(s, p^{[j]})$ and $H_j(s, p^{[j]})$ are matrix polynomials having multilinear uncertainty structure as well. When $p^{[j]}$ is fixed, $F(s, p)$ is affine in p_j. Therefore, for any edge of **P**, which is parallel to p_j axis and connects two vertices $\underline{p}^i$ and $\overline{p}^i$ of **P**, we can substitute the convex combination

$$p_j = \lambda \underline{p}_j + (1-\lambda)\overline{p}_j, \quad \lambda \in [0,1] \tag{6.14}$$

into (6.13) to get

$$\begin{aligned} &F(s, \lambda \underline{p}^i + (1-\lambda)\overline{p}^i) \\ &= G_j(s, p^{[j]})[\lambda \underline{p}_j + (1-\lambda)\overline{p}_j] + H_j(s, p^{[j]}) \\ &= \lambda[G_j(s, p^{[j]})\underline{p}_j + H_j(s, p^{[j]})] + (1-\lambda)[G_j(s, p^{[j]})\overline{p}_j + H_j(s, p^{[j]})] \\ &= \lambda F_j(s, \underline{p}^i) + (1-\lambda)F_j(s, \overline{p}^i) \end{aligned} \tag{6.15}$$

It is obvious that $F(s, p)$ is **I**-positive along the corresponding edge of **P** if and only if the two vertex polynomials $F_j(s, \underline{p}^i)$ and $F_j(s, \overline{p}^i)$ are **I**-positive. It is further obvious that this argument can be extended to all facets of **P**, as well as its interior in pretty much the same way it was done in proving the well known Mapping Theorem in [35]. Thus we arrive at

(6.16) THEOREM A multilinear family $\mathcal{M}$ is **I**-positive if and only if all generators $F(i\omega, p^i)$ for $\mathcal{M}$ are positive.

We immediately note that, while the extreme point result of [35] is only sufficient for stability of a scalar polynomial, the condition of Theorem (6.16) is both necessary and sufficient for positivity of a polynomial matrix. The reason is that the image of the hyperrectangle **P** under the mapping $F(s, \cdot)$ is identical to the convex hull of the generators $F(s, p^i)$ corresponding to the vertices p^i of **P**. This is not the case in the stability investigations in [35] (see also [18, 19]).

VII. SPECTRAL FACTORIZATION

In a number of areas in system theory, notably in linear prediction and filtering [36] and stability [2], it is required to factor a real paraconjugate Hermitian matrix polynomial $F(s)$ as

$$F(s) = H(s)H_*(s) \tag{7.1}$$

so that det $H(s)$ does not have zeros in the open right half of plane $\mathbb{C}$. There are quite a few algorithms (*e.g.*, [37–40]) with varied numerical reli-

ability, which are designed to perform the factorization of $F(s)$. Using a simple example we will show that the spectral factorization problem may be ill-conditioned, because small changes in the coefficients of $F(s)$ can destroy an existing solution $H(s)$.

(7.2) EXAMPLE Let us consider a real scalar polynomial

$$f(s) = s^4 + (p_1 + p_2)s^2 + p_1 p_2 \,, \tag{7.3}$$

where p_1 and p_2 are positive parameters. If $p_1 = p_2$, the polynomial $f(s)$ can be factored as

$$f(s) = h(s)h(-s) \,, \tag{7.4}$$

where

$$h(s) = s^2 + p_1 \,. \tag{7.5}$$

By a slight change of parameters, so that $p_1 \neq p_2$, the factorization is destroyed. To see this, we note that as long as $p_1 = p_2$, $f(s)$ has a pair of complex zeros $s_{1,\,2} = \pm i\sqrt{p_1}$ with multiplicity two, which can be split to produce $h(s)$ in (7.5). When $p_1 \neq p_2$, $f(s)$ has two distinct pairs of imaginary zeros, and factorization (7.4) is impossible.

Alternatively, whenever p_1 and p_2 are nonpositive, that is, the parameter vector $p \in \mathbf{P} = \{p \in \mathbb{R}^2 : p_1 \leq 0,\ p_2 \leq 0\}$, the polynomial $f(s)$ is robustly factorizable. To show this, let $s \mapsto i\omega$ in $f(s)$ and get the new polynomial

$$\tilde{f}(\omega) = \omega^4 - (p_1 + p_2)\omega^2 + p_1 p_2. \tag{7.6}$$

Now, imaginary zeros of $f(s)$, if any, become real zeros of $\tilde{f}(\omega)$, and we see by inspection that $\tilde{f}(\omega)$ is positive; it has no real zeros for all $p \in \mathbf{P}$. Factorizability of $f(s)$ is equivalent to positivity of $\tilde{f}(\omega)$, which now can be shown to be robust by the criterion stated in the preceding sections.

To consider the robust factorization problem, we recall the well-known result [38]:

(7.7) THEOREM Let $F(s)$ be a paraconjugate Hermitian matrix polynomial. Then, there exists a real matrix polynomial $H(s)$ such that $F(s) = H(s)H_*(s)$ if and only if $F(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$.

For a robust version of Theorem (7.7), we need.

(7.8) DEFINITION A polytope family $\mathcal{M}$ is robustly factorizable if for each fixed $p \in \mathbf{P}$, there exists a matrix polynomial $H(s, p)$ such that $F(s, p) = H(s, p)H_*(s, p)$.

Then, from Theorems (6.11) and (7.7), we have

(7.9) THEOREM A polytope family $\mathcal{M}$ is robustly factorizable if all generators $F(i\omega, p^i)$ for $\mathcal{M}$ are positive.

Factorizability of $\mathcal{M}$ can be tested by a repeated use of Theorem (6.4). We could extend Theorem (7.9) to include the "only if" part of Theorem (7.7) by using the nonnegativity results of [37] and Theorem (6.6). The nonnegativity testing of each generator $F(i\omega, p^i)$, however, would require extensive computations.

(7.10) EXAMPLE Let us parameterize a numerical matrix polynomial of reference [39] to get an uncertain matrix polynomial

$$F(s,p) = \begin{bmatrix} -s^2+p_5 & -p_1s+p_3 & 0 \\ p_1s+p_3 & -s^2+p_6 & -p_2s+p_4 \\ 0 & p_2s+p_4 & -s^2+p_7 \end{bmatrix} \tag{7.11}$$

The uncertainty set is a hyperrectangle **P** defined by

$$\begin{aligned} &p_1 \in [0,1],\ p_2 \in [-1,1],\ p_3 \in [-0.25,0],\ p_4 \in [-0.25,0.25] \\ &p_5 \in [0.25,0.5], p_6 \in [2,4], p_7 \in [0.25,0.5]. \end{aligned} \tag{7.12}$$

We want to show that the matrix

$$F(i\omega,p) = \begin{bmatrix} \omega^2+p_5 & -ip_1\omega+p_3 & 0 \\ ip_1\omega+p_3 & \omega^2+p_6 & -ip_2\omega+p_4 \\ 0 & ip_2\omega+p_4 & \omega^2+p_7 \end{bmatrix} \tag{7.13}$$

is positive definite for all $p \in \mathbf{P}$. If we use Theorem (6.11) we need to test $2^7 = 128$ polynomials for positivity. Instead, we can determine $f(\omega, p) = \det F(i\omega, p)$ as

$$\begin{aligned} f(\omega,p) = \omega^6 &+(-p_1^2-p_2^2+p_5+p_6+p_7)\omega^4 \\ &+(-p_1^2p_7-p_2^2p_5-p_3^2-p_4^2+p_5p_6+p_5p_7+p_6p_7)\omega^2 \\ &-p_3^2p_7-p_4^2p_5+p_5p_6p_7 \end{aligned} \tag{7.14}$$

and show that it has no real zeros for all $p \in \mathbf{P}$. We notice that $f(\omega, p)$ has polynomic uncertainty structure, as opposed to $F(i\omega, p)$which has an affine uncertainty form. Before we attempt the extensive testing of $f(\omega, p)$ it is recommended to test for positivity the coefficient-by-coefficient minorizing polynomial $\underline{f}(\omega)$ on **P**, which is

$$\underline{f}(\omega) = \omega^6 + 0.5\omega^4 - 0.0625\omega^2 + 0.0625. \tag{7.15}$$

By computing the corresponding Modified Routh Array

$$\begin{array}{llll} -1 & 0.5 & -0.0625 & 0.0625 \\ -3 & 1 & -0.0625 & \\ 0.1667 & -0.0417 & 0.0625 & \\ 0.2501 & 1.0622 & & \\ -0.7497 & 0.0625 & & \\ 1.0830 & & & \\ 0.0625 & & & \end{array} \tag{7.16}$$

we find three sign variations in the first column of the array and conclude that $\underline{f}(\omega)$ has no real zeros. Next, we use [41] to conclude that the interval matrix

$$A_0 = \begin{bmatrix} p_5 & p_3 & 0 \\ p_3 & p_6 & p_4 \\ 0 & p_4 & p_7 \end{bmatrix} \tag{7.17}$$

is positive definite on **P**, and we are done.

We should note here that nonexistence of real zeros of $\underline{f}(\omega)$ in this case is only a sufficient condition that worked. If $\underline{f}(\omega)$ had failed the test, the testing would have been inconclusive; there is no member $F(s, p)$ of the underlying family $\mathcal{M}$ that corresponds to the polynomial $\underline{f}(\omega)$.

VIII. ROBUST ABSOLUTE STABILITY

Let us consider a multivariable Lur'e-Postnikov system

$$\begin{aligned} \mathcal{S}: \quad \dot{x} &= Ax + Bu \\ y &= Cx \\ u &= -\phi(t, y) \end{aligned} \tag{8.1}$$

where $x \in \mathbb{R}^q$, $u \in \mathbb{R}^r$, and $y \in \mathbb{R}^r$ are state, input, and output of the linear part of $\mathcal{S}$, which is characterized by the triple (A, B, C) of constant matrices having appropriate dimensions. We make the following assumption about $\mathcal{S}$:

(A_1) The triple (A, B, C) is a minimal realization of a strictly proper transfer function $G(s) = C(sI - A)^{-1}B$.

(A_2) The nonlinear time-varying function $\phi(t,y) = [\phi_1(t, y_1), \phi_2(t, y_2), \ldots, \phi_r(t, y_r)]^T$ belongs to the class of sector-bounded continuous functions

$$\Phi_C = \{\phi : \mathbb{R}^{r+1} \to \mathbb{R}^r : \phi^T(t,y)[Ky - \phi(t,y)] \geq 0 \quad \forall y \in \mathbb{R}^r\}, \tag{8.2}$$

where $K \in \mathbb{R}^{m \times m}$ is a constant positive definite matrix.

To establish absolute stability of $\mathcal{S}$, we need the concept of strictly positive real functions. From [9], we recall:

(8.3) THEOREM A real rational function $W(s) = Q(s)/q(s)$, with a real polynomial $m \times m$ matrix $Q(s)$ and a real scalar polynomial $q(s)$ relatively prime to $Q(s)$, is strictly positive real if and only if

i. The polynomial $q(s)$ is stable, and
ii. The polynomial matrix

$$F(i\omega) = | q(i\omega) |^2 [W(i\omega) + W^*(i\omega)] \tag{8.4}$$

is positive.

We recall that absolute stability of $\mathcal{S}$ is defined as global asymptotic stability of the equilibrium $x = 0$ of $\mathcal{S}$ for all $\phi \in \Phi_C$. By using the circle criterion [42,43], absolute stability of $\mathcal{S}$ takes place if the function

$$W(s) = I + KG(s) \tag{8.5}$$

is strictly positive real and the following assumption holds:
(A_3) $W(i\infty) + W^*(i\infty) > 0$.

In terms of part ii. of Theorem (8.3), this means that we require

$$F(i\omega) = | q(i\omega) |^2 [2I + KG(i\omega) + G^*(i\omega)K^T] > 0, \tag{8.6}$$

where $q(s) = \det (sI - A)$.

The standard problem in absolute stability analysis is to determine the effect of the matrix K on stability of $\mathcal{S}$. For this purpose, we consider a set $\{K^l, K^2, ..., K^\upsilon\}$ of matrices with finite elements and define the polytope

$$\mathbf{P} = \operatorname{conv}\{K^i\}, \tag{8.7}$$

where K^i, $i \in \mathbf{v}$, are the generators of $\mathbf{P}$. Using Theorems (6.11) and (8.3), we can establish the following:

(8.8) THEOREM: Under the assumptions (A_1)–(A_3), the system $\mathcal{S}$ is absolutely stable with respect to the polytope $\mathbf{P}$ if

i. The polynomial $q(s)$ is stable, and
ii. All generators $F(i\omega, K^i)$, $i \in \mathbf{v}$, for the matrix polytope $\mathcal{M} = \{F(\cdot, K): K \in \mathbf{P}\}$ are positive.

(8.9) REMARK Obviously, an alternative to this result is provided by Theorem (6.4). To determine the robust absolute stability of $\mathcal{S}$ we can use the condition

$$F(0) > 0 \text{ and } \det\{|\, q(i\omega)\,|^2 [2I + KG(i\omega) + G^*(i\omega)K^T]\} \neq 0 \quad \forall \omega \in \mathbb{R} \tag{8.10}$$

instead of (8.6). We also note that (8.6) is equivalent to the scalar frequency condition

$$\det \operatorname{Re}[\mathrm{I} + \mathrm{KG}(i\omega)] \neq 0 \quad \forall \omega \in \mathbb{R}, \tag{8.11}$$

which may appear attractive in studying robust absolute stability of multivariable systems by frequency techniques proposed in [19,21].

Finally, we want to show how we can determine absolute stability when the uncertain parameters appear in the linear part of $\mathcal{S}$, that is, when we have *G(s, p)* instead of a fixed *G(s)*. For this purpose, the Popov criterion [42,43] is attractive because we can separate matrix K from transfer function matrix *G(s, p)*. First, we need to reformulate assumption (A_2) as

(A'_2) The nonlinear function $\phi(y) = [\phi_1(y_1), \phi_2(y_2),\ldots,\phi_r(y_r)]^T$ is time invariant and belongs to the class of sector-bounded continuous functions

$$\Phi_P = \{\phi : \mathbb{R}^r \to \mathbb{R}^r : \phi^T(y)[y - K^{-1}\phi(y)] \geq 0 \quad \forall y \in \mathbb{R}^r\}, \tag{8.12}$$

where $K \in \mathbb{R}^{m\times m}$ is a constant positive definite matrix.

We also need the following assumption:

(A_4) The parameter vector $p \in \mathbb{R}^\ell$ belongs to an uncertainty bounding set **P**.

(8.13) DEFINITION. The system $\mathcal{S}$ is robustly absolutely stable if it is absolutely stable for all $p \in \mathbf{P}$.

We recall [43] that $\mathcal{S}$ is absolutely stable if $W(s) = K^{-1} + G(s)$ is strictly positive real and $W(i\infty) + W^*(i\infty) > 0$. A parametrized version of this result is the following:

(8.14) THEOREM Under the assumptions (A_1), (A'_2), (A_3) and (A_4), the system $\mathcal{S}$ is robustly absolutely stable if

i. The polynomial *q(s, p)* is stable for all $p \in \mathbf{P}$, and
ii. The polynomial matrix

$$F(i\omega, p) = |\, q(i\omega, p)\,|^2 [2K^{-1} + G(i\omega, p) + G^*(i\omega, p)] \tag{8.15}$$

is positive for all $\omega \in \mathbb{R}$ and all $p \in \mathbf{P}$.

We first note that to test condition i. we can use a number of effective methods which are available in the context of robust stability of linear systems [18,19]. As for condition (8.15), we can rewrite *F(iω, p)* in terms of the triple *(A, B, C)*,

$$\begin{aligned} &F(i\omega,p) = \\ &\det\,(\omega^2 I + A^2)\{2K^{-1} - CA(\omega^2 I + A^2)^{-1}B - [CA(\omega^2 I + A^2)^{-1}B]^T\} \end{aligned} \tag{8.16}$$

to point out the possibility that the elements of the system matrices A, B, and C can be considered as components of the uncertain parameter vector p, and how each matrix enters in *F(iω, p)*. We note that the matrices B and C appear in *F(iω, p)* the same way K appears in *F(iω)* of (8.6). Therefore, Theorem (8.8) can be used when either B or C are uncertain, but not both. However, when the uncertainty set is a rectangle, both B and C can be uncertain since their entries appear multilinearly, as illustrated by the following:

(8.17) EXAMPLE Let us consider the system $\mathcal{S}$ where

$$A = \begin{bmatrix} -2 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & -2 \end{bmatrix},\ B = \mathrm{diag}\{b_1, b_2, b_3\},\ C = \mathrm{diag}\{c_1, c_2, c_3\} \tag{8.18}$$

where the interval uncertainty is

$$\begin{aligned} &b_1 \in [-1.1, 0.1],\ b_2 \in [-1.2, 0.2],\ b_3 \in [-0.9, 0.3] \\ &c_1 \in [-0.1, 1.1],\ c_2 \in [-0.2, 1.2],\ c_3 \in [-0.3, 0.9] \end{aligned} \tag{8.19}$$

The uncertain transfer function is

$$G(s,p) = \frac{1}{(s+2)^3 - 1}\begin{bmatrix} c_1 b_1 (s+2)^2 & c_1 b_2 & c_1 b_3 (s+2) \\ c_2 b_1 (s+2) & c_2 b_2 (s+2)^2 & c_2 b_3 \\ c_3 b_1 & c_3 b_2 (s+2) & c_3 b_3 (s+2)^2 \end{bmatrix} \tag{8.20}$$

where $p = (b_1, b_2, b_3, c_1, c_2, c_3)$ is the parameter vector. We assume that K = diag {3,3,3}, and form the matrix polynomial *F(iω, p)* of (8.15). By testing positivity of *F(iω, p)* at each of the $2^6 = 64$ vertices of the uncertainty rectangle we conclude from Theorem (6.16) that the system S with multilinear uncertainty structure is absolutely stable. Using Theorem (6.4), the formidable task of testing 64 polynomial matrices for positivity, is reduced to testing of 64 numerical matrices and scalar polynomials for positivity.

To illustrate our scalar positivity result of Theorem (6.4) in the context of the Popov criterion, we provide another example.

(8.21) EXAMPLE We consider system $\mathcal{S}$ of [42] with the block diagram of Figure 3, with the uncertain transfer function

$$G(s,p) = \begin{bmatrix} 0 & 0 & G_{13}(s) \\ G_{21}(s,p) & 0 & 0 \\ 0 & G_{32}(s,p) & 0 \end{bmatrix} \tag{8.22}$$

and the bounding matrix $K = \text{diag}\,\{k_1, k_2, k_3\}$. We choose

$$G_{13}(s) = \frac{1}{s+1},\ G_{21}(s,p) = \frac{p_2 s + p_1}{s^2+2s+2},\ G_{32}(s) = \frac{1}{s+3}, \tag{8.23}$$

and compute $f(\omega, p) = \det F(i\omega, p)$ using (8.15) to get

$$\begin{aligned} f(\omega,p) = \ & 8p_3\omega^8 + 80p_3\omega^6 + (2p_1 + 12p_2 + 104p_3)\omega^4 \\ & +(-26p_1 + 28p_2 + 320p_3)\omega^2 + 12p_1 + 288p_3 \end{aligned} \tag{8.24}$$

where $P_3 = k_1^{-1}k_2^{-1}k_3^{-1}$. When $p_3 = 1$ and

$$\mathbf{P} = \{p \in \mathbb{R}^2 : p_1 \in [-1,2], p_2 \in [-5,1]\}, \tag{8.25}$$

all four generators $f(\omega, p^i)$ and $F(0, p)$, which correspond to four vertices of $\mathbf{P}$, are positive. In this case, the minorizing polynomial

$$\underline{f}(\omega) = 8\omega^8 + 80\omega^6 + 90\omega^4 + 128\omega^2 + 276 \tag{8.26}$$

is obviously positive, and testing of the four generators is not necessary.

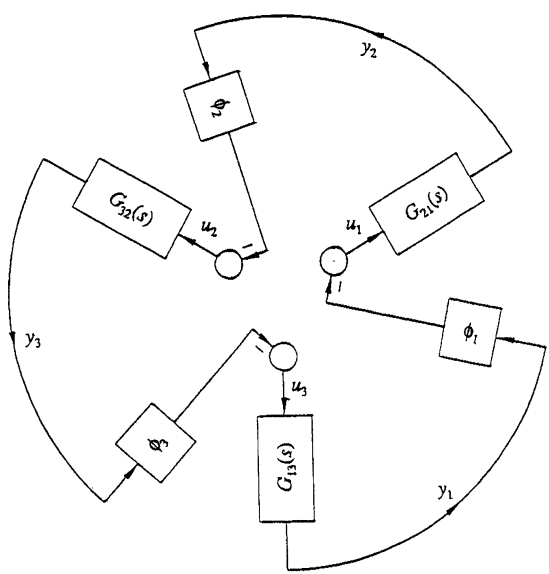

FIGURE 3 Block diagram

When we choose

$$G_{32}(s,p) = \frac{1}{s+p_4} \tag{8.27}$$

we obtain

$$\begin{aligned} f(\omega,p) = \quad & 8p_3\omega^8 + (8p_3 + 8p_3p_4^2)\omega^6 + (2p_1 - 6p_2 + 32p_3 \\ & -2p_2p_4 + 8p_3p_4^2)\omega^4 + (-8p_1 + 4p_2 + 32p_3 - 6p_1p_4 \\ & +8p_2p_4 + 32p_3p_4^2)\omega^2 + 4p_1p_4 + 32p_3p_4^2 \end{aligned} \tag{8.28}$$

which has a polynomic uncertainty structure. Let us assume that $p_3 = 1$ as before, but

$$\mathbf{P} = \{p \in \mathbb{R}^3 : p_1 \in [-2,2], p_2 \in [-3,3], p_4 \in [2,4]\}. \tag{8.29}$$

We first verify that *F(0,p)* is positive at the vertices of **P**. Then, we form the minorizing polynomial

$$\underline{f}(\omega) = 8\omega^8 + 40\omega^6 + 18\omega^4 - 36\omega^2 + 56 \tag{8.30}$$

Since $\underline{f}(\omega)$ is positive, we do not need to "expand" **P** by transformation of Remark (4.22), nor test any of the vertices of the expanded rectangle $\tilde{\mathbf{P}}$. Positivity of matrix *F(0, p)* and polynomial $\underline{f}(\omega)$ implies absolute stability of $\mathcal{S}$.

Besides the references mentioned in the Introduction, there are a considerable number of papers with a wealth of results concerning robust absolute stability and positive realness. With some notable exceptions [44], the results are derived for Single-Input-Single-Output (SISO) systems. Most of these results are surveyed in the paper [21] and the recent book [19]. It would be interesting to explore possibilities of using some of these results in our approach to multivariable systems. New problems in this context are the robustness analysis of absolute stability [45,46], adaptive control [47,48], H_∞ control [49], and parametric stability of nonlinear control systems [50].

IX. CONCLUSION

Nonnegativity and positivity of complex polynomials with uncertain parameters can be established by a variety of techniques. Polytopes of both scalar and matrix polynomials are shown to be positive by testing positivity of a subset of vertex polynomials. The actual testing involves the Mod-

ified Routh Array, which can decide positivity of a vertex polynomial by algebraic, recursive, and finite computations involving only real arithmetic. Another technique, which is shown to be especially attractive in the positivity context, is the Zero Exclusion Condition. Unlike in robust stability analysis of linear systems, the value sets in positivity investigations are intervals that are identical to their convex hulls, which makes the applications of the condition to positivity testing inherently nonconservative. The proposed positivity criteria are of particular interest in the robustness analysis of spectral factorization and absolute stability of multivariable systems, where the existing scalar positivity tests are not readily applicable. Future research should explore the ways in which the polytopes and value sets can grow in a systematic fashion, thus providing improved estimates of the positivity regions at moderate computation costs.

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