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Robust High-Gain Observer for Nonlinear Closed-Loop Stochastic Systems

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In this paper, the problem of robust state observation is tackled. A high-gain observer is employed to carry out the state estimation of a continuous time uncertain nonlinear system subject to external perturbations of stochastic nature. Unmodelled dynamics is assumed to be deterministic and belonging to an *a priori* known class of uncertainties. The control input is constructed based on the state estimates supplied by this observer. An upper bound for the estimation error and the states of this closed-loop system is derived. It is shown to be a linear combination of all *a priori* given uncertainty levels and turns out to be "tight" (reachable). The proposed scheme is applied to a robot manipulator with unknown friction and inaccessible angular velocities.

Keywords: Robustness; High-gain observer; Stochastic systems

AMS Classification Codes: 62G35, 93B52

1 INTRODUCTION

In the presence of any sort of external disturbances (noises or output perturbations) or internal uncertainties (unmodelled dynamics or model uncertainties), the exact and complete knowledge of the current states is impossible and the use of any state estimator (observer) is compulsory to supply any feedback controller with reliable on-line

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information to realize a successful closed-loop control. An important problem to be considered is the problem of the global stability of a nonlinear plant-observer control scheme subject to plant uncertainties as well as to stochastic disturbances.

In the situation where the model of the plant is incomplete or uncertain, the implementation of high-gain observers seems to be convenient [4,5,13,20]. Most of the known results, dealing with this technique, assume that the system neither contains any external disturbances nor has internal unmodelled dynamics [9,10]. In [6] the Lyapunov analysis is performed to prove the stability of the estimation error of a nonlinear system with dynamics perturbed by stochastic noises and whose measured output has no noise. The corresponding output-feedback controller is robust with respect to disturbances (the standard Brownian Motion process), but the output effects of any model uncertainties as well as unmodelled dynamics are not considered. The authors design an output feedback (observer-based) backstepping control law for nonlinear stochastic SISO systems of a very special structure and prove the global asymptotic stability (in probability) for such closed-loop system. Anyway, this work has several drawbacks. One of them is that it considers a very specific structure of the SISO system (as opposite to our case where a broad class of MIMO nonlinear systems is considered). Another disadvantage is that the complete knowledge of the nonlinear plant is assumed and, lastly, the fact that it considers the use of a quartic Lyapunov function to perform the study of the scheme. This makes the analysis very cumbersome due to the size of the resulting mathematical expressions involved.

In *the current paper*, in order to analyze the stability of the closed-loop system containing an observer as a part of the feedback, an extended system is considered, involving the models of the MIMO plant and the observer. The assumptions on this new system are adopted with the aim of assuring the existence of the solution to the plant and observer stochastic differential equations. The stability analysis is based on a Lyapunov approach where satisfaction of two algebraic Riccati equations, constructed artificially for technical reasons, is required to assure boundness (in some probabilistic sense [18]) of the state error, that is, assuring the boundness of both the state and its estimate. The class of systems considered is a broad family of nonlinear stochastic continuous time models with and the class of feedback nonlinear controls employed. The family of controllers may include

a wide class of commonly used controllers such as PID, Sliding Mode type, HJ (Hamilton-Jacobi) type, Locally Optimal, etc. The control scheme proposed in this paper extends ideas from the previous work by Martínez-Guerra, Poznyak and Gortcheva (see for example [9–11] and uses a high-gain observer. The idea of employing a high-gain observer is to make the dynamics of the observer part of the proposed scheme less dependent on the dynamics of the plant containing uncertainties in it. The Lyapunov analysis for stochastic systems has an antecedent in the works of Poznyak, Taksar, Osorio and Iparraguirre (see [7,14,16, 19]). We believe that the technique, used in these papers, is a novelty for the treatment of such sort of problems. Summarizing, this paper presents a closed-loop plant-observer scheme for a broad class of nonlinear plants subject to stochastic disturbances of the Brownian Motion type as well as to unstructured plant uncertainties. The resulting scheme is shown to be robust against these uncertainties and perturbations.

The paper is organized as follows. In Section 2 the class of plants treated is introduced along with the structure of the high-gain observer employed. The basic assumptions about the system are also presented. The main properties of the robust high-gain observer in the closed-loop scheme are presented in Section 3. Section 4 deals with the applications example of the manipulator robot with an uncertainty in the friction description. Finally, conclusions and bibliography are presented.

2 STOCHASTIC SYSTEMS WITH INCOMPLETE INFORMATION AND HIGH-GAIN OBSERVERS

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ be a given filtered probability space, that is,

- the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is complete,
- the sigma-algebra $\mathcal{F}_0$ contains all the $\mathbf{P}$-null sets in $\mathcal{F}$,
- the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is right continuous: $\mathcal{F}_{t+} := \cap_{s>t} \mathcal{F}_s = \mathcal{F}_t$.

On this probability space an m-dimensional standard Brownian motion is defined, i.e., $(W(t),\ t \geq 0)$ (with $W(0)=0$) is an $\{\mathcal{F}_t\}_{t\geq 0}$-adapted R^q-valued process such that

$$E\{W(t) - W(s) \mid \mathcal{F}_s\} = 0 \quad \mathbf{P}\text{-}a.s.$$

$$E\{[W(t)-W(s)][W(t) - W(s)]^\top \mid \mathcal{F}_s\} = (t-s)I \quad \mathbf{P}\text{-}a.s.$$

$$\mathbf{P}\{\omega \in \Omega : W(0) = 0\} = 1.$$

2.1 Stochastic Process with Incomplete Information

Consider the stochastic process $x_t(\omega)$ $(\omega \in \Omega)$ defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ and given by

$$\begin{aligned} \mathrm{d}x_t &= [F_0(x_t) + F_1(x_t)u_t + \Delta F(t, x_t)]\,\mathrm{d}t + \Theta_{1,t}\,\mathrm{d}W_t \\ &\quad x_0 - \text{is a given random vector} \\ \mathrm{d}y_t &= C\,\mathrm{d}x_t + \Theta_{2,t}\,\mathrm{d}W_t \end{aligned} \tag{1}$$

where

$x_t \in \mathrm{R}^n$ is the state or signal process,
$y_t \in \mathrm{R}^{n_o}$ is the output or observable process,
$u_t \in \mathrm{R}^{n_i}$ is the input process (control function or simply control) subject to

$$u_t \in \mathcal{B}_t$$

$\mathcal{B}_t$ is a set of Borel functions measurable with respect to the σ-algebra $\bar{\sigma}(y_s, s \leq t)$ generated by the process y_s),
$F_0 : \mathrm{R}^n \to \mathrm{R}^n$ is a vector field describing the dynamics of the system,
$F_1 : \mathrm{R}^n \to \mathrm{R}^{n_i \times n}$ is a vector field describing the way the input signal enters the system,
$\Delta F : \mathrm{R}^{n+1} \to \mathrm{R}^n$ is a vector field describing the internal unmodelled dynamic part of the system (or the uncertainty on the model),
$\Theta_{1,t} \in \mathrm{R}^{n\times q}, \Theta_{2,t} \in \mathrm{R}^{n_o \times q}, C \in \mathrm{R}^{n_o \times n}$ are real valued deterministic matrix functions defined for any $t \geq 0$.

2.2 High-Gain Observer

Consider *the high-gain observer* constructed based only on the available information and described as

$$\begin{aligned} \mathrm{d}\hat{x}_t &= (I + KC)^{-1}([F_0(\hat{x}_t) + F_1(\hat{x}_t)u_t]\,\mathrm{d}t + K\mathrm{d}y_t) \\ &\quad \hat{x}_0 - \text{fixed (may be, random)} \end{aligned} \tag{2}$$

where the matrix $K \in \mathrm{R}^{n\times n_o}$ satisfies

$$\det(I + KC) \neq 0 \tag{3}$$

According to [9,10], in the case when $\nabla^{\mathrm{T}} F_0(0) \neq 0$, the gain-matrix K can be selected as

$$K = -S_\theta^{-1} C^{\mathrm{T}}$$

where the matrix S_θ should be the positive solution of the following algebraic equation

$$E^{\mathrm{T}}\left(S_\theta + \frac{\theta}{2} I\right) + \left(S_\theta + \frac{\theta}{2} I\right) E = C^{\mathrm{T}} C$$
$$E := (I + KC)^{-1} \nabla^{\mathrm{T}} F_0(0)$$

The positive parameter θ determines the desired convergence rate for the estimation error. Under some certain technical assumptions (locally Lipschitz condition), it follows that $S_\theta^{-1} \simeq O(1/\theta)$ and this nonlinear observer is shown to provide an arbitrary exponential estimation error decay if no uncertainties and random noises at all.

2.3 Class of Nonlinear Controllers

Throughout this paper the nonlinear stochastic systems described by (1) will be considered. The feedback control policy is constructed based on the information available up to the time t.

DEFINITION 1 *The function* u_τ $(\tau \in [0, t])$ *is said to be an* admissible control policy *if*

(1) *for each time* $t \in \mathsf{R}^+$ *the Borel function* u_t *is* $\mathcal{B}_t$*-measurable* (*does not depend on the future*),
(2) *it guarantees the existence of the solution to the system and observer stochastic differential equations* (1) *and* (2) *within any time interval* $[0, t]$.

DEFINITION 2 *An admissible control policy* u_τ $(\tau \in [0, t])$ *is called* a nonlinear (*may be, nonstationary*) feed-back control *if*

$$u_t = u(t, \hat{x}_t) \tag{4}$$

2.4 Extended System

Let us define the extended random vector

$$z_t^{\mathrm{T}} := (x_t^{\mathrm{T}}, y_t^{\mathrm{T}}, \hat{x}_t^{\mathrm{T}}) \in \mathsf{R}^{2n+n_o}$$

In view of this definition we can rewrite the given stochastic differential equations (1) and (2) as follows

$$\mathrm{d}z_t = H(t, z_t)\,\mathrm{d}t + \Xi_t\,\mathrm{d}W_t \tag{5}$$

where

$$H(t, z_t) = \begin{bmatrix} F_0(x_t) + F_1(x_t)u_t + \Delta F(t, x_t) \\ C[F_0(x_t) + F_1(x_t)u_t + \Delta F(t, x_t)] \\ (I + KC)^{-1}([F_0(\hat{x}_t) + F_1(\hat{x}_t)u_t] + KC[F_0(x_t) + F_1(x_t)u_t + \Delta F(t, x_t)]) \end{bmatrix}$$

$$\Xi_t = \begin{bmatrix} \Theta_{1,t} \\ C\Theta_{1,t} + \Theta_{2,t} \\ (I + KC)^{-1}K[C\Theta_{1,t} + \Theta_{2,t}] \end{bmatrix}$$

In order to assure the existence and the uniqueness of the solution to this system of differential stochastic equations (1) and (2) with a fixed nonlinear feedback control (4) for all $t \in \mathsf{R}^+$, the functional matrices $H(t, z)$ and Ξ_t defined above are required to satisfy the following conditions [7]

(a) $H(t, z)$ and Ξ_t are measurable with respect to t and z for all $t \in \mathsf{R}^+$;
(b) there exist constants K_0 and K_H such that for all $t \in \mathsf{R}^+$ and $z, z' \in \mathsf{R}^{2n\times n_o}$ the following properties hold:
 (i)

$$\|H(t, z) - H(t, z')\| \le K_0\|z - z'\|$$

 (ii)

$$\|H(t, z)\| \le K_H(1 + \|z\|)$$

$$\sup_{t\in[0,\infty)} \|\Xi_t\| < \infty$$

Under these conditions, it can be proved [3,7] that there is the solution z_t to (5) defined on $[0, \infty)$ which is continuous with probability 1 and pathwise unique, that is, if z and z' are two solutions to (5), then

$$P\Big(\omega \in \Omega\colon \sup_{t\in[0,\infty)} \|z_t(\omega) - z_t'(\omega)\| = 0\Big) = 1$$

and such that for any finite $T < \infty$

$$\sup_{t\in[0,T]} E\{\|z_t\|^2\} < \infty.$$

2.5 Basic Assumptions

For technical convenience and taking into account the necessary and sufficient conditions for the existence and uniqueness of the solution to the differential equation (5) just presented, the following assumptions should be done.

A.1: there exists a stable matrix A and a strictly positive definite matrix Λ_0 such that

$$\|F_0(x+\Delta) - F_0(x) - A\Delta\|_{\Lambda_0}^2 \le L_0\|\Delta\|_{\Lambda_0}^2$$
$$\forall\Delta,\ x \in \mathsf{R}^n, \quad L_0 \in (0, \infty)$$

with the matrix

$$\bar{A} := [I + KC]^{-1}A$$

also being stable and the matrix K satisfying (3).

A.2: for any $x \in \mathsf{R}^n$ the vector field $F_0(x)$ satisfies the following condition

$$\|F_0(x)\|^2 \le k_0 + k_1\|x\|^2, \quad k_0, k_1 \ge 0$$

A.3: the vector field $F_1(x)$ is bounded, i.e., for any $x \in \mathsf{R}^n$

$$\|F_1(x)\| \le F_1^+ < \infty$$

A.4: the unmodelled dynamics ΔF due to parametric uncertainties, is bounded as

$$\|\Delta F(t,x)\|^2 \leq \mu_0 + \mu_1 \|x\|^2$$
$$\forall t \geq 0, \quad \forall x \in \mathsf{R}^n, \quad \mu_0, \mu_1 \in [0, \infty)$$

A.5: the feedback control law is of the nonstationary feedback type, nonlinear dependent on the current state estimate $\hat{x}_t$ and satisfies

$$\|u(t,\hat{x}_t)\|^2 \leq C_0 + C_1 \|\hat{x}_t\|^2 \quad \forall t \geq 0$$

A.6: the noise matrices $\Theta_{1,t}$ and $\Theta_{2,t}$ are bounded, that is,

$$\theta_1 := \sup_{t \in [0,T]} \|\Theta_{1,t}\| < \infty$$
$$\theta_2 := \sup_{t \in [0,T]} \|\Theta_{2,t}\| < \infty.$$

3 BASIC PROPERTIES OF ROBUST HIGH-GAIN OBSERVERS IN CLOSED-LOOP SYSTEMS

Define the estimation error as

$$\Delta_t \triangleq \hat{x}_t - x_t \tag{6}$$

According to (1) and (2), the dynamic equation describing the behavior of the estimation error is given by

$$\begin{aligned} \mathrm{d}\Delta_t = \mathrm{d}\hat{x}_t - \mathrm{d}x_t &= [F_0(\hat{x}_t(\omega)) + F_1(\hat{x}_t(\omega))u_t]\,\mathrm{d}t + K[\mathrm{d}y_t - C(t)\hat{x}_t(\omega)] \\ &\quad - [F_0(x_t(\omega)) + F_1(x_t(\omega))u_t]\,\mathrm{d}t + \Delta F\,\mathrm{d}t + \Theta_{1,t}\,\mathrm{d}\omega_t \end{aligned}$$

Substituting $\hat{x}_t(\omega) = x_t(\omega) + \Delta_t$ in the last equation and rearranging terms, we finally get

$$\begin{aligned} \mathrm{d}\Delta_t = {} & [I + KC]^{-1}[F_0(x_t(\omega) + \Delta_t) - F_0(x_t(\omega)) \\ & + [F_1(x_t(\omega) + \Delta_t) - F_1(x_t(\omega))]u_t + \Delta F]\,\mathrm{d}t \\ & + [I + KC]^{-1}[K\Theta_{2,t} + \Theta_{1,t}]\,\mathrm{d}\omega_t \end{aligned} \tag{7}$$

Define the Lyapunov function as

$$V_0(\Delta) \triangleq \Delta^{\mathrm{T}} P \Delta$$
$$\Delta \in \mathrm{R}^n, \quad 0 < P = P^{\mathrm{T}} \in \mathrm{R}^{n \times n}$$

Differentiating it over the trajectories of (7) and taking into account the Itô's formula [7], it follows that

$$\mathrm{d}V_0(\Delta_t) \triangleq (2P\Delta_t, \mathrm{d}\Delta_t) + \mathcal{I}(t)\,\mathrm{d}t \tag{8}$$

where the operator $(\cdot, \cdot)$ denotes the scalar product and $\mathcal{I}(t)$ is the Itô's term given by

$$\mathcal{I}(t) = \mathrm{tr}\{\Theta_t^K P (\Theta_t^K)^{\mathrm{T}}\}$$

and

$$\Theta_t^K \triangleq [I + KC]^{-1}[K\Theta_{2,t} + \Theta_{1,t}]$$
$$\bar{\Theta}_t := KC[I + KC]^{-1}[K\Theta_{2,t} + \Theta_{1,t}] + K\Theta_{2,t}$$

that, in view of (8), implies

$$\mathrm{d}V_0(\Delta_t) \triangleq (2P\Delta_t, \mathrm{d}\Delta_t) + \mathrm{tr}\{\Theta_t^K P (\Theta_t^K)^{\mathrm{T}}\}\,\mathrm{d}t \tag{9}$$

An extra technical assumption is needed.

A.7: There exist the positive solutions P and $\hat{P}$ to the matrix Riccati equations [21]

$$L_1 \triangleq P\bar{A} + \bar{A}^{\mathrm{T}} P + P R_1 P + Q_1 = 0$$

$$R_1 \triangleq 2\Lambda_0^{-1} + \Lambda_\Delta^{-1}$$

$$\begin{aligned} Q_1 \triangleq\ & L_0 \|(I + KC)^{-1}\|_{\Lambda_0}^2 \cdot I + \|[I + KC]^{-1}\|_{\Lambda_\Delta}^2 \mu_1 (1 + \eta^{-1}) \cdot I \\ & + L_0 \|KC(I + KC)^{-1}\|^2 \Lambda_0 \\ & + A^{\mathrm{T}} [KC(I + KC)^{-1}]^{\mathrm{T}} \Lambda_2 [KC(I + KC)^{-1}] A + Q_\Delta \end{aligned}$$

and

$$
\begin{aligned}
L_2 &:= \hat{P}A + A^{\mathrm{T}}\hat{P} + \hat{P}R_2\hat{P} + Q_2 = 0 \\
R_2 &= 3I + \Lambda_0^{-1} + \Lambda_\Delta^{-1} + \Lambda_2^{-1} \\
Q_2 &\triangleq \Big[k_1 + F_1^{+2}(C_1 + 4\|KC\|^2 C_1) + \|KC\|^2_{\Lambda_\Delta}\|[I+KC]^{-1}\|^2_{\Lambda_\Delta}\mu_1 \\
&\quad + C_1 + \|[I+KC]^{-1}\|^2_{\Lambda_\Delta}\mu_1(1+\eta^{-1}) \\
&\quad + 4(F_1^+)^2\|(I+KC)^{-1}\|^2_{\Lambda_0}\Big]I + Q_{\hat{x}}
\end{aligned}
$$

The following three lemmas present the basic properties of the considered stochastic processes.

3.1 Property 1

LEMMA 3 *The assumptions* A.1–A.7 *imply that with probability one the following inequality holds:*

$$
\begin{aligned}
&t^{-1}\int_{\tau=0}^{t}\Big[\|\Delta_\tau(\omega)\|^2_{Q_\Delta} + \|\hat{x}_\tau(\omega)\|^2_{Q_{\hat{x}}}\Big]\,\mathrm{d}\tau \\
&\quad \overset{a.s.}{\leq} S^+ + t^{-1}V_{t=0} + t^{-1}\int_{\tau=0}^{t} g_\tau^{\mathrm{T}}(\omega)\,\mathrm{d}W_\tau(\omega)
\end{aligned}
\tag{10}
$$

where

$$
\begin{aligned}
S^+ &:= 4(F_1^+)^2\|(I+KC)^{-1}\|^2_{\Lambda_0}C_0 + \mu_0 \\
&\quad + (\|K\|\theta_2 + \theta_1^2)^2\|P\|\mathrm{tr}\{[I+KC]^{-1}[I+KC]^{-\mathrm{T}}\} \\
&\quad + (\|KC[I+KC]^{-1}\|[\|K\|\theta_2 + \theta_1] + \|K\|\theta_2)^2\mathrm{tr}\{\hat{P}\} \\
&\quad + k_0 + F_1^{+2}(C_0 + 4\|KC\|^2C_0) + \|KC\|^2_{\Lambda_1}\mu_0 \\
g(\hat{x}_t, \Delta, t) &:= 2(\Theta_t^K)^{\mathrm{T}}P\Delta + [\Theta_t^K - \Theta_{2,t}]^{\mathrm{T}}(KC)^{\mathrm{T}}\hat{P}\hat{x}_t \\
V_0(\Delta) &\triangleq \Delta^{\mathrm{T}}P\Delta, \quad V_1(\hat{x}) \triangleq \hat{x}^{\mathrm{T}}\hat{P}\hat{x}, \quad V_t = V_0(\Delta_t)V_1(\hat{x}_t)
\end{aligned}
$$

and Q_Δ, $Q_{\hat{x}}$, P, $\hat{P}$ are positive defined $(n \times n)$-matrices.

Proof What we will do now is to look for the upper bound for the regular term $(2P\Delta_t, \mathrm{d}\Delta_t)$ in the last Eq. (9). The use of (7) leads to the

following:

(1) For the term

$$2(P\Delta_t, [I+KC]^{-1}[F_0(x_t+\Delta_t) - F_0(x_t(\omega))])$$

it follows that

$$\begin{aligned}
&2(P\Delta_t, [I+KC]^{-1}[F_0(x_t+\Delta_t) - F_0(x_t(\omega))]) \\
&\quad = 2(P\Delta_t, \bar{A}\Delta_t) + 2(P\Delta_t, [I+KC]^{-1} \\
&\qquad \times [F_0(x_t+\Delta_t) - F_0(x_t(\omega)) - A\Delta_t]) \\
&\quad \le \Delta_t^{\mathrm{T}}(\bar{A}^{\mathrm{T}}P + P\bar{A})\Delta_t + \Delta_t^{\mathrm{T}}P\Lambda_0^{-1}P\Delta_t \\
&\qquad + \|[I+KC]^{-1}\|_{\Lambda_0}^2 \|F_0(x_t+\Delta_t) - F_0(x_t(\omega)) - A\Delta_t\|_{\Lambda_0}^2 \\
&\quad \le \Delta_t^{\mathrm{T}}[(\bar{A}^{\mathrm{T}}P + P\bar{A}) + P\Lambda_0^{-1}P + L_0\|[I+KC]^{-1}\|_{\Lambda_0}^2 I]\Delta_t \qquad (11)
\end{aligned}$$

(2) For the term

$$2(P\Delta_t, [I+KC]^{-1}[F_1(x_t+\Delta_t) - F_1(x_t(\omega))]u_t)$$

it follows that

$$\begin{aligned}
&2(P\Delta_t, [I+KC]^{-1}[F_1(x_t+\Delta_t) - F_1(x_t(\omega))]u_t) \\
&\quad \le \Delta_t^{\mathrm{T}}P\Lambda_0^{-1}P\Delta_t + \|[I+KC]^{-1}[F_1(x_t+\Delta_t) - F_1(x_t(\omega))]u\|_{\Lambda_0}^2 \\
&\quad \le \Delta_t^{\mathrm{T}}P\Lambda_0^{-1}P\Delta_t + 4(F_1^+)^2\|[I+KC]^{-1}\|_{\Lambda_0}^2 C_0 \\
&\qquad + 4(F_1^+)^2\|[I+KC]^{-1}\|_{\Lambda_0}^2 C_1\|\hat{x}_t\|^2 \qquad (12)
\end{aligned}$$

(3) The term $(2P\Delta_t, \Delta F)$ can be estimated as

$$\begin{aligned}
&(2P\Delta_t, [I+KC]^{-1}\Delta F) \\
&\quad \le \Delta_t^{\mathrm{T}}P\Lambda_\Delta^{-1}P\Delta_t + \Delta^{\mathrm{T}}F([I+KC]^{-1})^{\mathrm{T}}\Lambda_\Delta[I+KC]^{-1}\Delta F \\
&\quad \le \Delta_t^{\mathrm{T}}P\Lambda_\Delta^{-1}P\Delta_t + \|[I+KC]^{-1}\|_{\Lambda_\Delta}^2(\mu_0 + \mu_1\|x_t\|^2) \\
&\quad \le \Delta_t^{\mathrm{T}}P\Lambda_\Delta^{-1}P\Delta_t + \|[I+KC]^{-1}\|_{\Lambda_\Delta}^2(\mu_0 + \mu_1\|\hat{x}_t - \Delta_t\|^2)
\end{aligned}$$

Applying the inequality

$$\|a+b\|^2 \le (1+\eta)\|a\|^2 + (1+\eta^{-1})\|b\|^2$$

valid for any $\eta > 0$, we get

$$\begin{aligned}
&(2P\Delta_t, [I+KC]^{-1}\Delta F) \\
&\quad \le \Delta_t^{\mathrm{T}} P\Lambda_\Delta^{-1} P\Delta_t + \|[I+KC]^{-1}\|^2_{\Lambda_\Delta} \\
&\qquad \times (\mu_0 + \mu_1[(1+\eta)\|\hat{x}_t\|^2 + (1+\eta^{-1})\|\Delta_t\|^2]) \\
&\quad \le \Delta_t^{\mathrm{T}}[P\Lambda_\Delta^{-1}P + \|[I+KC]^{-1}\|^2_{\Lambda_\Delta}\mu_1(1+\eta^{-1})I]\Delta_t \\
&\qquad + \|[I+KC]^{-1}\|^2_{\Lambda_\Delta}(\mu_0 + \mu_1[(1+\eta)\|\hat{x}_t\|^2]) \qquad (13)
\end{aligned}$$

Substituting (11)–(13) into (9), we conclude that

$$\begin{aligned}
\mathrm{d}V_0(\Delta_t) \le \{&\Delta_t^{\mathrm{T}}[(\bar{A}^{\mathrm{T}}P + P\bar{A}) + P(2\Lambda_0^{-1} + \Lambda_\Delta^{-1})P \\
&+ (L_0\|[I+KC]^{-1}\|^2_{\Lambda_0} I + \|[I+KC]^{-1}\|^2_{\Lambda_\Delta}\mu_1(1+\eta^{-1})I)]\Delta_t \\
&+ 4(F_1^+)^2\|[I+KC]^{-1}\|^2_{\Lambda_0} C_0 + \|[I+KC]^{-1}\|^2_{\Lambda_\Delta}\mu_0 \\
&+ [4(F_1^+)^2\|[I+KC]^{-1}\|^2_{\Lambda_0} C_1 \\
&+ \|[I+KC]^{-1}\|^2_{\Lambda_\Delta}\mu_1(1+\eta^{-1})]\|\hat{x}_t\|^2\}\,\mathrm{d}t \\
&+ \mathcal{I}(t)\,\mathrm{d}t + 2(P\Delta, \Theta_t^K\,\mathrm{d}w_t)
\end{aligned}$$

Consider also the other quadratic form defined as

$$V_1(\hat{x}_t) \triangleq \hat{x}_t^{\mathrm{T}}\hat{P}\hat{x}_t$$

Differentiating it in a similar way to the function $V_0(\Delta_t)$, we get

$$\mathrm{d}V_1(\hat{x}_t) \triangleq 2\hat{x}_t^{\mathrm{T}}\hat{P}\,\mathrm{d}\hat{x}_t + \mathrm{tr}\{\bar{\Theta}_t\hat{P}\bar{\Theta}_t^{\mathrm{T}}\}\,\mathrm{d}t$$

The substitution $\mathrm{d}\hat{x}_t$ from the differential observer equation in the expression above implies

$$\begin{aligned}
\mathrm{d}V_1(\hat{x}_t) = {}& 2\hat{x}_t^{\mathrm{T}}\hat{P}(F_0(\hat{x}_t) + F_1(\hat{x}_t)u_t)\,\mathrm{d}t \\
&- 2\hat{x}_t^{\mathrm{T}}\hat{P}KC[I+KC]^{-1}(F_0(x_t+\Delta_t) - F_0(x_t)u_t)\,\mathrm{d}t \\
&- 2\hat{x}_t^{\mathrm{T}}\hat{P}KC(F_1(x_t+\Delta_t) - F_1(x_t))u_t\,\mathrm{d}t - 2\hat{x}_t^{\mathrm{T}}\hat{P}KC\Delta F\,\mathrm{d}t \\
&- 2\hat{x}_t^{\mathrm{T}}\hat{P}\bar{\Theta}_t\,\mathrm{d}w_t - 2\hat{x}_t^{\mathrm{T}}\hat{P}KC\,K\Theta_{2,t}\,\mathrm{d}w_t + \mathrm{tr}\{\bar{\Theta}_t\hat{P}\bar{\Theta}_t^{\mathrm{T}}\}\,\mathrm{d}t
\end{aligned}$$

Next, we bound from above each of the terms of the previous expression:

(1)

$$2\hat{x}_t^{\mathrm{T}}\hat{P}F_0(\hat{x}_t) \le \hat{x}_t^{\mathrm{T}}\hat{P}\hat{P}\hat{x}_t + \|F_0(\hat{x}_t)\|^2 \le \hat{x}_t^{\mathrm{T}}(\hat{P}\hat{P} + k_1 I)\hat{x}_t + k_0 \quad (14)$$

(2)

$$\begin{aligned} 2\hat{x}_t^{\mathrm{T}}\hat{P}F_1(\hat{x}_t)u_t &\le \hat{x}_t^{\mathrm{T}}\hat{P}\hat{P}\,\hat{x}_t + \|F_1(\hat{x}_t)u_t\|^2 \\ &\le \hat{x}_t^{\mathrm{T}}(\hat{P}\hat{P} + (F_1^+)^2 C_1)\hat{x}_t + (F_1^+)^2 C_0 \end{aligned} \quad (15)$$

(3)

$$\begin{aligned} &-2\hat{x}_t^{\mathrm{T}}\hat{P}KC[F_1(\Delta + x_t) - F_1(x_t)u_t] \\ &\quad \le \hat{x}_t^{\mathrm{T}}\hat{P}\hat{P}\hat{x}_t + \|KC(F_1(x_t + \Delta_t) - F_1(x_t))u_t\| \\ &\quad \le \hat{x}_t^{\mathrm{T}}(\hat{P}P\|KC\|^2 C_1 I + 4(F_1^+)^2)\hat{x} + 4(F_1^+)^2\|KC\|^2 C_0 \end{aligned} \quad (16)$$

(4)

$$\begin{aligned} &-2\hat{x}_t^{\mathrm{T}}\hat{P}KC[I + KC]^{-1}(F_0(\Delta + x_t) - F_0(x_t) - A\Delta) \\ &\quad \le \hat{x}_t^{\mathrm{T}}\hat{P}\hat{P}\hat{x}_t + \|KC[I + KC]^{-1}(F_0(\Delta + x_t) - F_0(x_t) - A\Delta)\|_{\Lambda_0}^2 \\ &\quad \le \hat{x}_t^{\mathrm{T}}\hat{P}\Lambda_0^{-1}\hat{P}\hat{x}_t + \|KC[I + KC]^{-1}\|^2 L_0\|\Delta\|_{\Lambda_0}^2 \end{aligned} \quad (17)$$

(5)

$$\begin{aligned} &-2(\hat{x}_t^{\mathrm{T}}, \hat{P}KC[I + KC]^{-1}A\Delta) \\ &\quad \le \hat{x}_t^{\mathrm{T}}\hat{P}\Lambda_2^{-1}\hat{P}\hat{x}_t + \Delta^{\mathrm{T}}A^{\mathrm{T}}[KC[I + KC]^{-1}]^{\mathrm{T}} \\ &\qquad \times \Lambda_2[KC[I + KC]^{-1}]A\Delta \end{aligned} \quad (18)$$

(6)

$$\begin{aligned} -2\hat{x}_t^{\mathrm{T}}\hat{P}KC\Delta F &\le \hat{x}_t^{\mathrm{T}}\hat{P}\Lambda_\Delta^{-1}\hat{P}\hat{x}_t + \Delta F^{\mathrm{T}}(KC)^{\mathrm{T}}\Lambda_\Delta KC\Delta F \\ &\le \hat{x}_t^{\mathrm{T}}\hat{P}\Lambda_\Delta^{-1}\hat{P}\hat{x}_t + \|KC\|_{\Lambda_\Delta}^2(\mu_0 + \mu_1(\|\hat{x}_t\|^2)) \end{aligned}$$

For the joint quadratic form V defined by

$$V_t := V_0(\Delta) + V_1(\hat{x}_t)$$

the following expression holds:

$$\begin{aligned} \mathrm{d}V_t &= \mathrm{d}V_0(\Delta) + \mathrm{d}V_1(\hat{x}_t) \\ &\le \Delta^{\mathrm{T}} L_1 \Delta\,\mathrm{d}t + \hat{x}_t^{\mathrm{T}} L_2 \hat{x}_t\,\mathrm{d}t + \phi_t\,\mathrm{d}t + g(\hat{x}_t, \Delta, t)\,\mathrm{d}w_t \\ &\quad - \Delta^{\mathrm{T}} Q_\Delta \Delta\,\mathrm{d}t - \hat{x}_t^{\mathrm{T}} Q_{\hat{x}} \hat{x}_t\,\mathrm{d}t \end{aligned} \tag{19}$$

with

$$\begin{aligned} \phi_t := {} & 4(F_1^+)^2 \|(I+KC)^{-1}\|^2_{\Lambda_0} C_0 + \|[I+KC]^{-1}\|^2_{\Lambda_\Delta} \mu_0 \\ & + \mathrm{tr}\{[I+KC]^{-1}[K\Theta_{2,t} + \Theta_{1,t}] P [K\Theta_{2,t} + \Theta_{1,t}]^{\mathrm{T}} [I+KC]^{-\mathrm{T}}\} \\ & + \mathrm{tr}\{\bar{\Theta}_t^{\mathrm{T}} \hat{P} \bar{\Theta}_t\} + k_0 + F_1^{+2}(C_0 + 4\|KC\|^2 C_0) \\ & + \|KC\|^2_{\Lambda_1} \|[I+KC]^{-1}\|^2_{\Lambda_\Delta} \mu_0 \le S^+ = \mathrm{const} < \infty \end{aligned}$$

In view of (19) and taking into account the assumptions of this theorem, it follows

$$L_1 = L_2 = 0$$

and

$$\mathrm{d}V_t \overset{a.s.}{\le} S^+\,\mathrm{d}t + g^{\mathrm{T}}(\hat{x}_t, \Delta, t)\,\mathrm{d}Wt - [\Delta_t^{\mathrm{T}} Q_\Delta \Delta_t + \hat{x}_t^{\mathrm{T}} Q_{\hat{x}} \hat{x}_t]\,\mathrm{d}t$$

The integration on t both sides of this inequality leads to

$$V_t - V_{t=0} \overset{a.s.}{\le} S^+ \cdot t + \int_{\tau=0}^{t} g_\tau^{\mathrm{T}}\,\mathrm{d}W_\tau - \int_{\tau=0}^{t} [\|\Delta_\tau\|^2_{Q_\Delta} + \|\hat{x}_\tau\|^2_{Q_{\hat{x}}}]\,\mathrm{d}\tau$$

where

$$\begin{aligned} & t^{-1} \int_{\tau=0}^{t} [\|\Delta_\tau(\omega)\|^2_{Q_\Delta} + \|\hat{x}_\tau(\omega)\|^2_{Q_{\hat{x}}}]\,\mathrm{d}t \\ & \overset{a.s.}{\le} S^+ - t^{-1}(V_t - V_{t=0}) + t^{-1} \int_{\tau=0}^{t} g_\tau^{\mathrm{T}}(\omega)\,\mathrm{d}W_\tau(\omega) \\ & \le S^+ + t^{-1} V_{t=0} + t^{-1} \int_{\tau=0}^{t} g_\tau^{\mathrm{T}}(\omega)\,\mathrm{d}W_\tau(\omega) \end{aligned}$$

lemma is proved.

3.2 Property 2

LEMMA 4 *If*

$$\int_{t=0}^{\infty} \frac{1}{(t+1)^2} E\{\|gt\|^2\}\,\mathrm{d}t < \infty \tag{20}$$

then

$$S_t \triangleq t^{-1} \int_{\tau=0}^{t} g_\tau^{\mathrm{T}}\,\mathrm{d}W_\tau \xrightarrow[n\to\infty]{a.s.} 0$$

Proof This proof follows the ideas given in [15]. Let us introduce the following random sequence $\{S_n\}$ defined as

$$S_n := \frac{1}{n}\int_t^n g_\tau^{\mathrm{T}}\,\mathrm{d}W_\tau \tag{21}$$

This can be represented in an iterative form as follows:

$$\begin{aligned} S_n &:= \frac{n-1}{n(n-1)}\left[\int_{t=0}^{n-1} g_\tau^{\mathrm{T}}\,\mathrm{d}W_\tau + \int_{t=n-1}^{n} g_\tau^{\mathrm{T}}\,\mathrm{d}W_\tau\right] \\ &= (1-n^{-1})S_{n-1} + n^{-1}\int_{n-1}^{n} g_\tau^{\mathrm{T}}\,\mathrm{d}W_\tau \end{aligned}$$

Calculating the conditional expectation of its square, it follows

$$\begin{aligned} &E\{S_n^2/\mathcal{F}_{n-1}\} \\ &\overset{a.s.}{=} (1-n^{-1})^2 S_{n-1}^2 + n^{-2}E\left\{\int_{n-1}^{n}\int_{n-1}^{n}(g_\tau^{\mathrm{T}}\,\mathrm{d}W_\tau)(g_\tau^{\mathrm{T}}\,\mathrm{d}W_\tau)/\mathcal{F}_{n-1}\right\} \\ &\quad + 2(1-n^{-1})n^{-1}S_{n-1}E\left\{\int_{n-1}^{n} g_t^{\mathrm{T}}\,\mathrm{d}W_t/\mathcal{F}_{n-1}\right\} \\ &\overset{a.s.}{\leq} (1+n^{-2})S_{n-1}^2 - 2n^{-1}S_{n-1}^2 + n^{-2}\int_{n-1}^{n}\|g_t\|^2\,\mathrm{d}t \end{aligned} \tag{22}$$

Here we take into account that

$$\begin{aligned} &E\left\{\int_{n-1}^{n} g_t^{\mathrm{T}}\,\mathrm{d}W_t/\mathcal{F}_{n-1}\right\} \overset{a.s.}{=} 0 \\ &E\left\{\int_{n-1}^{n}\int_{n-1}^{n}(g_\tau^{\mathrm{T}}\,\mathrm{d}W_\tau)(g_\tau^{\mathrm{T}}\,\mathrm{d}W_\tau)/\mathcal{F}_{n-1}\right\} \overset{a.s.}{=} \int_{n-1}^{n}\|g_t\|^2\,\mathrm{d}t \end{aligned} \tag{23}$$

In view of the law estimate

$$\int_{t=0}^{\infty}(t+1)^{-2}E\{\|g_t\|^2\}\,\mathrm{d}t \geq \sum_{n=0}^{\infty}\int_{n}^{n-1}(t+1)^{-2}E\{\|g_t\|^2\}\,\mathrm{d}t$$

$$\geq \sum_{n=0}^{\infty}(n+1)^{-2}\int_{n}^{n-1}E\{\|g_t\|^2\}\,\mathrm{d}t$$

from the assumption (20) of this lemma it follows that

$$\sum_{n=0}^{\infty}\frac{1}{(n+1)^2}\int_{n}^{n-1}E\{\|g_t\|^2\}\,\mathrm{d}t < \infty$$

The Skorohod lemma [15] leads to

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\int_{t=n-1}^{n}\|g_t\|^2\,\mathrm{d}t \overset{a.s.}{<} \infty$$

and the Robins–Siegmund lemma [12,17] with

$$x_n := S_n^2, \quad \xi_n := 2n^{-1}S_{n-1}^2$$

$$\alpha_n := n^{-2}, \quad \beta_n := n^{-2}\int_{n-1}^{n}\|g_t\|^2\,\mathrm{d}t$$

implies

$$S_n \overset{a.s.}{\to} S^*(\omega) \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{1}{n}S_{n-1}^2 \overset{a.s.}{<} \infty$$

From the last property, taking into account that

$$\sum_{n=1}^{\infty}\frac{1}{n} = \infty$$

it follows that there exists a subsequence $\{n_k\}$ satisfying

$$\exists n_k : S_{n_k} \underset{k\to\infty}{\overset{a.s.}{\to}} 0$$

and, hence,

$$S^*(\omega) \overset{a.s.}{=} 0 \tag{24}$$

Below, the behavior of the Borel function S_t within the interval $[n, n+1]$ will be studied and its boundness will be proved. To do that consider the random sequence $\{\eta_n\}$ defined as

$$\eta_n := \sup_{t\in[n,n+1]} |S_t - S_n|$$

From above it follows:

$$\begin{aligned}\eta_n &= \sup_{t\in[n,n+1]}\left|\frac{1}{n}\int_{t=n}^{n+1} g_t^{\mathrm{T}}\,\mathrm{d}W_t + \left(\frac{1}{t}-\frac{1}{n}\right)\int_{\tau=0}^{t} g_\tau^{\mathrm{T}}\,\mathrm{d}W_\tau\right| \\ &\le \sup_{t\in[n,n+1]}\left|\frac{1}{n}\int_{t=n}^{n+1} g_t\right| + \sup_{t\in[n,n+1]}\left|\frac{1}{n+1}-\frac{1}{n}\right|\left|\int_{\tau=0}^{t} g_\tau^{\mathrm{T}}\,\mathrm{d}W_\tau\right|\end{aligned}$$

So,

$$\eta_n \le \sup_{t\in[n,n+1]}\left|\frac{1}{n}\int_{n}^{t} g_t^{\mathrm{T}}\,\mathrm{d}W_t\right| + \frac{1}{n^2}\sup_{t\in[n,n+1]}\left|\int_{\tau=0}^{t} g_\tau^{\mathrm{T}}\,\mathrm{d}W_\tau\right|$$

The Chebyshev's inequality (for any $\varepsilon > 0$) implies

$$\begin{aligned}P\{\eta_n \ge \varepsilon\} &\le \frac{1}{\varepsilon^2}E\{\eta_n^2\} \\ &\le \frac{2}{\varepsilon^2}\left(\frac{1}{n^2}E\left\{\left[\sup_{t\in[n,n+1]}\left|\int_n^t g_t^{\mathrm{T}}\,\mathrm{d}W_t\right|\right]^2\right\}\right. \\ &\quad \left. + \frac{1}{n^4}E\left\{\left[\sup_{t\in[n,n+1]}\left|\int_{\tau=0}^{t} g_\tau^{\mathrm{T}}\,\mathrm{d}W_\tau\right|\right]^2\right\}\right) \\ &\le \frac{\text{const}}{\varepsilon^2}\left[\int_n^{n+1}\frac{1}{(t+1)^2}E\{\|g_\tau\|^2\}\,\mathrm{d}t + \frac{1}{n^3}\right] := \psi_n\end{aligned}$$

Since

$$\sum_{n=1}^{\infty} P\{\xi_n \ge \varepsilon\} \le \sum_{n=1}^{\infty}\psi_n < \infty$$

then by the Borel–Cantelli lemma it follows

$$\sum_{n=1}^{\infty}\chi\{\eta_n \ge \varepsilon\} \overset{a.s.}{<} \infty$$

and we can conclude that for almost all random realization $\omega \in \Omega$ there exists a random (but finite) number $n_0(\omega) < \infty$ such that $\eta_n < \varepsilon$ for any consecutive numbers $n > n_0(\omega)$. This means that $\eta_n \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} 0$. And finally, from this fact and in view of (24), it follows that

$$S_t \underset{t\to\infty}{\overset{a.s.}{\longrightarrow}} 0$$

lemma is proved.

3.3 Property 3

LEMMA 5 *If the initial states of the considered stochastic process x_0 as well as the initial states of the suggested high-gain observer $\hat{x}_0$ are quadratically integrable, then for any $T \in \mathrm{R}^+$ the following estimate holds:*

$$\int_{t=0}^{T} (t+1)^{-2} E\{\|g_\tau\|^2\}\,\mathrm{d}t$$
$$\leq \text{const} \cdot T^{-1} + 2\,\text{const} \int_0^T (t+1)^{-2}\,\mathrm{d}t \leq \text{const} < \infty$$

Proof From (10) and from the properties (23), it follows:

$$\int_{\tau=0}^{t} E\{\|\Delta_\tau(\omega)\|^2_{Q_\Delta} + \|\hat{x}_\tau(\omega)\|^2_{\tilde{Q}}\}\,\mathrm{d}t$$
$$\leq S^+ \cdot t + E\{V_{t=0}\} + E\left\{\int_{\tau=0}^{t} g_\tau^{\mathrm{T}}(\omega)\,\mathrm{d}W_\tau(\omega)\right\}$$
$$= S^+ \cdot t + E\{V_{t=0}\}$$

The definition of $g_t(\Delta_t, \hat{x}_t, t)$ provides the property

$$\|g_t\|^2 \leq C(\|\Delta_t\|^2 + \|\hat{x}_t\|^2) \tag{25}$$

Hence,

$$F(T) := \int_{t=0}^{\mathrm{T}} E\{\|g_t\|^2\}\,\mathrm{d}t$$
$$\leq C \int_{t=0}^{\mathrm{T}} E\{\|\Delta_t\|^2 + \|\hat{x}_t\|^2\}\,\mathrm{d}t \leq aT + b$$

where a, b are positive constants. Integrating by parts, we get:

$$\int_{t=0}^{\mathrm{T}} (t+1)^{-2} E\{\|g_\tau\|^2\}\,\mathrm{d}t$$
$$= \int_{t=0}^{\mathrm{T}} (t+1)^{-2}\,\mathrm{d}F(t) = F(T)(T+1)^{-2} + 2\int_0^{\mathrm{T}} F(t)(t+1)^{-3}\,\mathrm{d}t$$

The application of the expectation operator $E\{\cdot\}$ to the Eq. (25) implies

$$\int_0^{\mathrm{T}} (t+1)^{-2} E\{\|g_\tau\|^2\}\,\mathrm{d}t$$
$$\leq \frac{aT+b}{(T+1)^2} + 2\int_0^{\mathrm{T}} \frac{a\cdot t+b}{(t+1)^3}\,\mathrm{d}t$$
$$\leq \max\{a;b\}\left[(T+1)^{-1} + 2\int_0^{\mathrm{T}} (t+1)^{-2}\,\mathrm{d}t\right]$$
$$\leq \max\{a;b\}\left[1 + 2\int_0^{\infty} (t+1)^{-2}\,\mathrm{d}t\right] \leq \text{const} < \infty$$

lemma is proved.

COROLLARY 6 *In view of the Lemmas* 1, 2 *and* 3 *it follows that*

$$\frac{1}{t}\int_{\tau=0}^{t} g_\tau^{\mathrm{T}}\,\mathrm{d}W_\tau \xrightarrow[n\to\infty]{a.s.} 0$$

3.4 Main Result

Now we are ready to formulate our main contribution of this paper.

THEOREM 7 (Main Result) *The assumptions* A.1–A.7 *imply that for any quadratically integrable initial states of the process and the observer the performance index* $\mathcal{L}_t$ *defined as*

$$\mathcal{L}_t := \frac{1}{t}\int_{\tau=0}^{t} (\|\Delta_\tau(\omega)\|_{Q_\Delta}^2 + \|\hat{x}_\tau(\omega)\|_{Q_{\hat{x}}}^2)\,\mathrm{d}\tau$$

possesses the following properties:

(1) *for any random trajectory* $\omega \in \Omega$

$$\limsup_{t\mapsto\infty} \mathcal{L}_t \overset{a.s}{\leq} S^+$$

(2) *speaking in the average sense,*

$$\limsup_{t\mapsto\infty} E\{\mathcal{L}_t\} \leq S^+$$

Proof It follows directly from the statements of Lemmas 1, 2 and 3.

3.5 Upper Bound Tightness and Stochastic Stability

We show next that the upper bound for the performance index obtained above is reachable.

In the case of no internal uncertainties and no stochastic nonmeasurable perturbations in the given system, that is, when

$$\mu_0 = \mu_1 = 0$$
$$\theta_0 = \theta_1 = 0$$

and for the nonlinear fields satisfying the "cone" condition

$$k_0 = 0 \Rightarrow F_0(x = 0) = 0$$
$$C_0 = 0 \Rightarrow u(t, \hat{x}_t = 0) = 0, \quad \forall t \geq 0$$

from the main theorem it follows that

$$S^+ := 0$$

In other words, the obtained upper bound S^+ is "*tight*" (reachable). This leads to the *stochastic stability* property

– in the "with probability one" sense:

$$\|\Delta_\tau(\omega)\|^2_{Q_\Delta} + \|\hat{x}_\tau(\omega)\|^2_{Q_{\hat{x}}} \xrightarrow[n\to\infty]{a.s.} 0$$

– and in the "mean square" sense:

$$E\{\|\Delta_\tau(\omega)\|^2_{Q_\Delta}\} + E\{\|\hat{x}_\tau(\omega)\|^2_{Q_{\hat{x}}}\} \xrightarrow[n\to\infty]{} 0$$

4 APPLICATION TO A ROBOT MANIPULATOR WITH THE FRICTION UNCERTAINTY

4.1 Robotic Model

In this section the dynamic model for a Robot Manipulator with two degrees of freedom containing an internal uncertainty connected with an unknown friction parameter is considered.

4.2 Dynamic Equations

The corresponding Lagrange dynamic equation can be expressed as follows [8]:

$$M(\theta)\ddot{\theta} + W(\theta,\dot{\theta}) = u(\theta, u \in R^2) \tag{26}$$

where $M(\theta)$ represents the positive definite inertia matrix

$$M(\theta) = M^{\mathrm{T}}(\theta) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} > 0 \tag{27}$$

with the elements

$$\begin{aligned} M_{11} &= (m_1 + m_2)a_1^2 + m_2 a_2^2 + 2m_2 a_1 a_2 c_2 \\ M_{12} &= m_2 a_2^2 + m_2 a_1 a_2 c_2, \quad M_{22} = m_2 a_2^2 \\ M_{21} &= M_{12}, \quad a_i = l_i, \quad c_i = \cos\theta_i, \quad s_i = \sin\theta_i \\ c_{12} &= \cos(\theta_1 + \theta_2) \end{aligned}$$

Here m_i, l_i $(i = 1, 2)$ are the masses and lengths of the corresponding links and $W(\theta,\dot{\theta})$ is the Coriolis matrix representing the centrifugal and frictional effects (with the uncertain parameters). It can be described as follows:

$$W(\theta,\dot{\theta}) = W_1(\theta,\dot{\theta}) + W_2(\dot{\theta}) \tag{28}$$

where $W_1(\theta, \dot{\theta})$ corresponds to the Coriolis and centrifugal components:

$$W_1(\theta, \dot{\theta}) = \begin{pmatrix} W_{1a} \\ W_{1b} \end{pmatrix}$$

$$W_{1a} = -m_1 a_1 a_2 (2\dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_2^2) s_2 + (m_1 + m_2) g a_1 c_1 + m_2 g a_2 c_{12}$$
$$W_{1b} = m_2 a_1 a_2 \dot{\theta}_1^2 s_2 + m_2 g a_2 c_{12}$$

and $W_2(\dot{\theta})$ corresponds to the friction component:

$$W_2(\dot{\theta}) = \kappa v$$

$$\kappa := \begin{pmatrix} \nu_1 & \kappa_1 & 0 & 0 \\ 0 & 0 & \nu_2 & \kappa_2 \end{pmatrix}, \quad v^{\mathrm{T}} = \begin{pmatrix} \dot{\theta}_1 & \operatorname{sign} \dot{\theta}_1 & \dot{\theta}_2 & \operatorname{sign} \dot{\theta}_2 \end{pmatrix}.$$

To represent this system in the standard form, introduce the extended vector

$$x^{\mathrm{T}} = (\theta_1, \theta_2; \dot{\theta}_1, \dot{\theta}_2)$$

and, in view of this definition, the dynamic equation (26) can be rewritten as follows:

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ (-M^{-1}(x) W_1(x) - M^{-1}(x) \kappa v(x) + M^{-1}(x) u)_1 \\ (-M^{-1}(x) W_1(x) - M^{-1}(x) \kappa v(x) + M^{-1}(x) u)_2 \end{pmatrix} \tag{29}$$

4.3 Uncertainty Description

Assume that the matrix κ is presented in the following form

$$\kappa := \kappa_0 + \Delta \kappa_t \tag{30}$$

where the internal uncertainty $\Delta\kappa$ satisfies

$$\forall t: \quad \Delta \kappa_t^{\mathrm{T}} \Delta \kappa_t \leq \Delta \tag{31}$$

Here the matrix Δ is *a priori* known.

In view of the notations accepted above, the system (29) can be represented in the following standard form:

$$\mathrm{d}x_t = [F_0(x_t, t) + \Delta F(x_t, t) + F_1(x_t, t)u_t]\,\mathrm{d}t + \Theta_{1,t}\,\mathrm{d}W_t \tag{32}$$

where

$$F_0(x_t, t) = \begin{pmatrix} x_3 \\ x_4 \\ \mathbf{f}_0(x_t) \end{pmatrix} = Ex_t + \overline{F_0(x_t, t)}, \quad \Delta F(x_t, t) = \begin{pmatrix} 0 \\ 0 \\ \Delta\mathbf{f}(x_t) \end{pmatrix}$$

$$F_1(x_t, t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ B(x_t) \end{pmatrix}$$

and

$$\mathbf{f}_0(x_t) := -M^{-1}(x_t)[W_1(x_t) + \kappa_0 v(x_t)] \in R^2$$

$$\Delta\mathbf{f}(x_t) := -M^{-1}(x_t)\Delta\kappa_t \cdot v(x_t) \in R^2, \quad B(x_t) = M^{-1}(x_t) \in R^{2\times 2}$$

$$E = \begin{pmatrix} 0_{2x2} & I_{2x2} \\ 0_{2x2} & 0_{2x2} \end{pmatrix}, \quad \overline{F_0(x_t, t)} = \begin{pmatrix} 0 \\ 0 \\ \mathbf{f}_0(x_t) \end{pmatrix} \tag{33}$$

Taking into account the restrictions (31), the corresponding nonlinear term, containing the uncertainty mentioned above, can be estimated as follows:

$$\begin{aligned} \|\Delta F(x_t, t)\|^2_{\Lambda_0} &= \Delta F^{\mathrm{T}}(x_t, t)\Lambda_0\Delta F(x_t, t) = \Delta\mathbf{f}^{\mathrm{T}}(x_t)\Lambda_{02}\Delta\mathbf{f}(x_t) \\ &= v^{\mathrm{T}}(x_t)\Delta\kappa_t^{\mathrm{T}} M^{-1}(x_t)\Lambda_{02}M^{-1}(x_t)\Delta\kappa_t v(x_t) \\ &\le \lambda_{\max}(S(x_t))v^{\mathrm{T}}(x_t)\Delta\kappa_t^{\mathrm{T}}\Delta\kappa_t v(x_t) \le \mu_t \end{aligned} \tag{34}$$

where

$$\mu_t := \lambda_{\max}(S(x_t))v^{\mathrm{T}}(x_t)\Delta v(x_t) \tag{35}$$

$$S(x_t) := M^{-1}(x_t)\Lambda_{02}M^{-1}(x_t) \tag{36}$$

and Λ_0 is the weight matrix selected for the simplicity in the block-diagonal form:

$$\Lambda_0 := \begin{bmatrix} \Lambda_{01} & 0 \\ 0 & \Lambda_{02} \end{bmatrix} \in R^{4\times 4}$$

Sure, such friction structure covers only Coulomb and Viscous frictions. Any other phenomena in friction such as Stribeck's effect and hysteresis [2] are outlined in.

Consider the robotic system with noise components, that is, we have for any $t \geq 0$:

$$\Theta_{1,t} = \varepsilon_{W_1} \cdot I \in \mathrm{R}^{4\times 4}, \quad \varepsilon_{W_1} = \text{const} > 0 \tag{37}$$

4.4 Measurable Output

In this context it is assumed that only the angular positions can be measured:

$$\mathrm{d}y_t = C\,\mathrm{d}x_t + \Theta_{2,t}\,\mathrm{d}W_t \in \mathrm{R}^2 \tag{38}$$

where

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and

$$\Theta_{2,t} = \varepsilon_{W_2} \cdot I \in \mathrm{R}^{2\times 2}, \quad \varepsilon_{W_2} = \text{const} > 0 \tag{39}$$

In other words, the angular positions are measurable with observation noises.

4.5 Simulation Results

A number of simulations have been performed for a two link manipulator with the friction parameter k given by (30) with

$$k_0 = \begin{pmatrix} 0.35 & 1.25 & 0 & 0 \\ 0 & 0 & 0.35 & 1.25 \end{pmatrix}$$

and the time-varying uncertainty satisfying

$$\Delta k_t = \begin{pmatrix} 0.5\omega\sin(\omega t) & 0.9\omega\cos(\omega t) & 0 & 0 \\ 0 & 0 & 0.2\omega\sin(\omega t) & 0.6\omega\cos(\omega t) \end{pmatrix} \tag{40}$$

with $\omega = 5$.

The control input u_t was selected as in [8]:

$$u_t(\hat{x}_t) = -F_1^+(\hat{x}_t)\left[F_0(\hat{x}_t) - f_0^*(x_t^*, t) + \sqrt{2}\Lambda_0^{-1/2}(\hat{x}_t - x_t^*)\right]$$

$$\dot{x}_t^* = f_0^*(x_t^*, t) = (x_2^*, -x_1^*, -x_1^*, -x_2^*)^{\mathrm{T}}, \quad x_0^* = (1, 1, 0, 0)^{\mathrm{T}}$$

where $F_1^+(\hat{x}_t)$ is the pseudoinverse (in Moore–Penrose sense, see [1]) matrix of $F_1(\hat{x}_t)$.

The tracking trajectory x_t^* is equivalent to the following harmonic oscillations

$$x_t^* = (\sin(t + \phi_1), \cos(t + \phi_2), \cos(t + \phi_1), -\sin(t + \phi_2))^{\mathrm{T}}$$

$$\phi_1 = \text{arc}\sin(1), \quad \phi_2 = \text{arc}\cos(1)$$

The performance index of the corresponding estimation process was calculated as

$$\mathcal{L}_t = \frac{1}{t + 0.001}\int_0^t \|\Delta_\tau\|_{Q_0}^2 \, \mathrm{d}\tau, \quad Q_0 = I \tag{41}$$

or, in the differential form,

$$\mathcal{L}_t = -\frac{1}{t + 0.001}\left[J_t - \|\Delta_t\|_{Q_0}^2\right], \quad J_0 = 0 \tag{42}$$

The simulation results, presented in Figs. 1–5, have been carried out with the help of MATLAB 5.3 Software with SIMULINK 3.0 as toolbox.

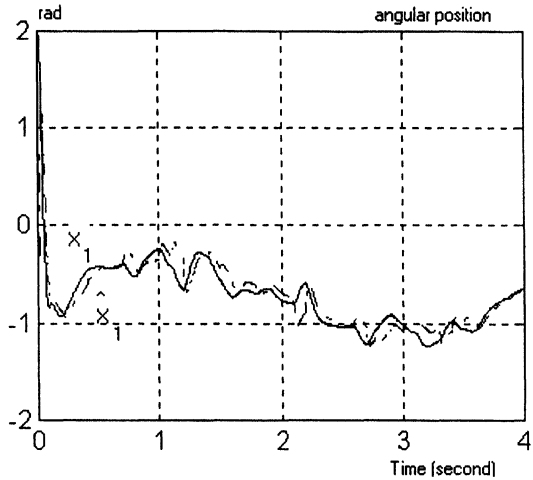

FIGURE 1 The state x_1 and its estimate $\hat{x}_1$.

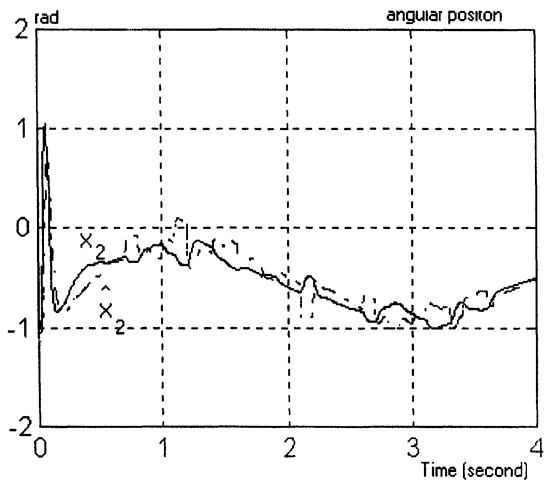

FIGURE 2 The state x_2 and its estimate $\hat{x}_2$.

The Figs. 1–4 show the convergence of the state estimates to the real states in the presence of parametric uncertainties given by (40) and the external noise perturbations satisfying (37) and (39) with $\varepsilon_{W_1} = 1$, $\varepsilon_{W_2} = 0.02$ (the noise in the system output) and $\theta = 0.01$ (the small parameter in the high-gain observer). The initial conditions of the

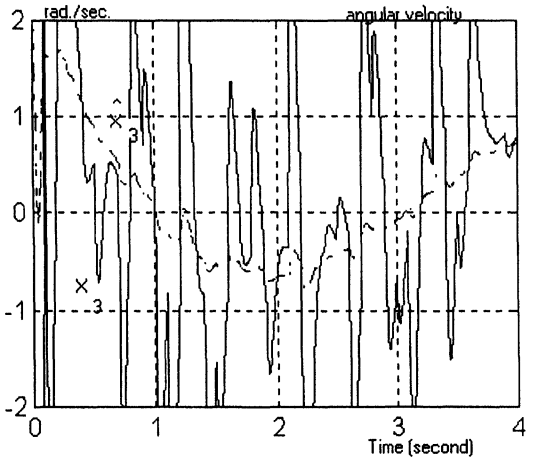

FIGURE 3 The state x_3 and its estimate $\hat{x}_3$.

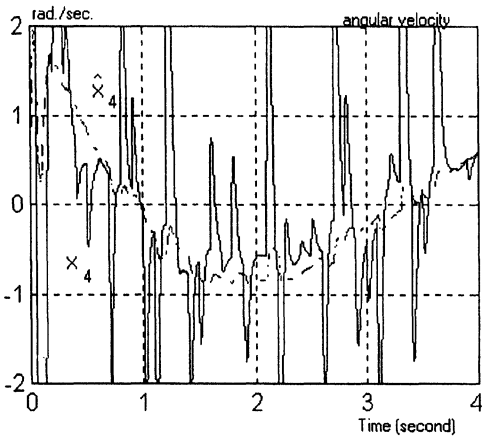

FIGURE 4 The state x_4 and its estimate $\hat{x}_4$.

system and the observer were taken as

$$x_1 = 2, \quad x_2 = -1, \quad x_3 = 1, \quad x_4 = 1$$

and

$$\hat{x}_1 = -2, \quad \hat{x}_2 = 1, \quad \hat{x}_3 = -1, \quad \hat{x}_4 = -1$$

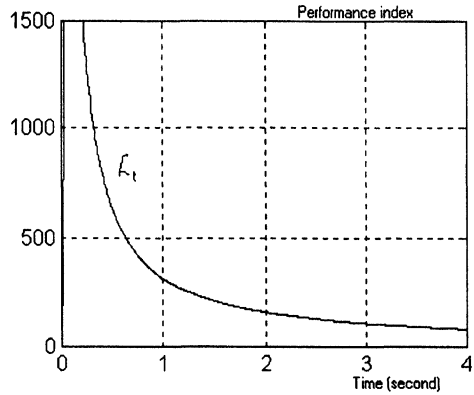

FIGURE 5 Performance index $\mathcal{L}_t$.

Finally, Fig. 5 shows that the quadratic estimation error is bounded on average (the performance index) in t.

5 CONCLUSION

In this paper we present an observer-plant scheme where a nonlinear, continuous time, uncertain model for the plant is considered. External perturbations of the Brownian Motion type were also considered. The analysis of the proposed scheme is based on the Lyapunov approach which allows to assure the boundness of the estimation error of the states of the plant in a closed-loop configuration. As the main result, it is shown that this closed-loop uncertain stochastic system is stable and the performance index attains the value, shown in Theorem 3.5, both in the trajectory sense and in an average sense.

Among the advantages, mentioned about, the suggested scheme serves for the broad class of nonlinear plants closed by the feedback controllers. The considered feedback control laws may be of the nonstationary feedback type, dependent in a nonlinear way on the current state estimate and with the property satisfying the "sector condition". The robustness properties of this scheme with respect to plant uncertainties and with respect to stochastic perturbations in the dynamics of the plant as well as in its output, are reinforced by the use of a high-gain observer that makes the scheme less dependent on the

dynamics of the uncertain plant. The high-gain observer provides the state estimates bounded on average.

Finally, the performance of the closed-loop system in the applications example is shown. The simulation results, concerning the robot manipulator, illustrate the effectiveness of the technique in the case of the incomplete information about the friction and the noises presence at the output of the system.

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