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Robust Block Decomposition Sliding Mode Control Design*

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The paper examines the problem of sliding mode manifold design for uncertain nonlinear system with discontinuous control. The original plant first is decomposed such that the problem is divided into a number of simpler sub-problems. Then the block control recursive procedure is presented in which nonlinear sliding manifold is derived. Finally combined high gain and Lyapunov functions techniques are applied to establish hierarchy of the control gains and to estimate the upper bounds of the sliding mode equation solutions.

Key words: Uncertain nonlinear system; Sliding mode control; Block decomposition; High gain; Lyapunov function

1 INTRODUCTION

The problem of decomposition and design robust control for dynamical system to be controlled is one of interesting problem in the control theory. A fruitful and relatively simple approach to solving this problem, especially when dealing with multivariable nonlinear uncertain is based on the use of Variable Structure Control approach with sliding mode, Utkin [1]. First and foremost, this enables high accuracy and robustness to disturbances and system parameter variations to be obtained. Second, the control design problem is conveniently divided into two sub-problems: (a) the design of nonlinear sliding surfaces enforcing motion according to the specified closed-loop performance, and (b) determination of a control law providing stable motion in the sub-state space of the surface.

In order to illustrate the potential of decomposition with the use of the above technique, consider the following system subject to uncertainty:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t) + \boldsymbol{B}(\boldsymbol{x}, t)\boldsymbol{u} + \boldsymbol{g}(\boldsymbol{x}, t) \tag{1}$$

where $\boldsymbol{x} \in \boldsymbol{X} \subset \boldsymbol{R}^n$ is the state vector, $\boldsymbol{u} \in \boldsymbol{U} \subset \boldsymbol{R}^m$ is the control vector to be bounded by

$$|u_i| \leq U_0 \quad \text{con} \quad U_0 > 0, \quad \boldsymbol{u} = (u_1, \ldots, u_m)^T. \tag{2}$$

The unknown mapping $\boldsymbol{g}(\boldsymbol{x}, t)$ characterizes external disturbances and parameter variations which should be not affect the feedback systems. It is assumed the vector fields $\boldsymbol{f}(\boldsymbol{x}, t)$ and

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$g(x, t)$, and the columns of $B(x, t)$ are smooth and bounded mappings of class $C^{\infty}_{[0,\infty)}$, $f(0, t) = 0$, and rank $B(x, t) = m$ for all $x \in X$ and $t \geq 0$. The standard sliding mode design procedure comprises of the two sub-problems.

First, the nonlinear sliding manifold in the state space of the system

$$s(x) = 0, \quad s = (s_1, \ldots, s_m)^T \tag{3}$$

must to chosen such that the matrix $GB(G = \partial s/\partial x)$ has full rank for $x \in X$ and $t \geq 0$, and the sliding mode (SM) equation

$$\dot{x} = f_s(x, t) + g_s(x, t), \quad s(x) = 0 \tag{4}$$

where $f_s = (I_n - B(GB)^{-1}G)f$ and $g_s = (I_n - B(GB)^{-1}G)g$, has the desired properties, including stability as a minimum requirement. Secondly, a discontinuous control

$$u_i(x, t) = \begin{cases} u_i^+(x, t) & \text{if } s_i(x) > 0 \\ u_i^-(x, t) & \text{if } s_i(x) < 0 \end{cases}, \quad i = 1, \ldots, m \tag{5}$$

is introduced to guarantees convergence of a projection of the motion of the closed-loop system in the subspace s, described by

$$\dot{s} = Gf + GBu + Gg$$

where $u_i^+(x, t)$ and $u_i^-(x, t)$, are smooth functions to be selected. If $g(x, t)$ satisfies the so called *matching condition*, Drajevonic [2] $g(x, t) \in$ *span* $B(x, t)$, *i.e.* there exists vector $\mu(x, t)$ such that

$$g(x, t) = B(x, t)\mu(x, t) \quad \forall x \in X \text{ and } t \geq 0 \tag{6}$$

then $g_s = (I_n - B(GB)^{-1}G)B\mu = 0$. In this case, SM equation (4) reduces to simply:

$$\dot{x} = f_s(x, t), \quad s(x) = 0 \tag{7}$$

Note this equation has the reduced order $(n - m)$, however, it is still nonlinear and nonautonomous. One possible approach to ensuring stability of the nominal system (7), is connected with the input–output linearization technique, Isidory [3]. Another approach is the "backstepping" that based on the use step by step of Lyapunov functions, Krstic [4].

In this paper the universal decomposition block control method, Drakunov [5], is adopted to design a nonlinear time-varying sliding manifold (3) which stabilizes the perturbed SM equation (4). Another important aim is to provide robustness of the sliding mode motion with respect to non vanishing perturbation, $g(x, t)$ in cases where it does *not* satisfy to the matching condition (6). A solution for the control of nonlinear, time-varying plants with both matched and unmatched uncertainties is offered here. A solution is achieved by a combination of three techniques:

First, the block control method is applied to decompose the control law synthesis problem into a number of sub-problems of lower order which can be solved independently of one another. For, a special state representation of the system must be used which will be referred to as the *Block Controllable form* (or *BC-form*). This is achieved either by multiple decomposition of the original system under some structural conditions on unmatched uncertainties, employing the integral method, Loukianov [6].

Secondly, the sliding mode technique is used to compensate the matched uncertainty.

Finally, a high gain approach is used to obtain hierarchical fast motions on the sliding manifold, the goals being to achieve stabilization of the sliding the mode equation and compensation of the unmatched uncertainty.

Note that the block control approach has, in fact, successfully been employed for control of linear systems, Dodds [7], Loukianov [8], including linear systems with delay, Escoto-Hernández [9]; for stabilization and regulation of nonlinear (including mechanical) systems, Loukianov [10, 11], Utkin [12, 13]; for robot and automotive control, Loukianov [14, 15]; for electric motors and power systems control, Loukianov [16, 18], Sanchez [17]. Here the possibility of applying the same method for obtaining upper estimations and bounds of uncertain nonlinear system solutions, is investigated.

2 CONTROL METHOD

2.1 Block Representation of a Class of Nonlinear Systems

The essential feature of the proposed method is the conversion of the system (1) to the BC-form consisting of r blocks:

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{f}_1(\boldsymbol{x}_1, t) + \boldsymbol{B}_1(\boldsymbol{x}_1, t)\boldsymbol{x}_2 + \boldsymbol{g}_1(\boldsymbol{x}_1, t) \tag{8a}$$
$$\dot{\boldsymbol{x}}_i = \boldsymbol{f}_i(\bar{\boldsymbol{x}}_i, t) + \boldsymbol{B}_i(\bar{\boldsymbol{x}}_i, t)\boldsymbol{x}_{i+1} + \boldsymbol{g}_i(\bar{\boldsymbol{x}}_i, t), \quad i = 2, \dots, r-1 \tag{8b}$$
$$\dot{\boldsymbol{x}}_r = \boldsymbol{f}_r(\bar{\boldsymbol{x}}_r, t) + \boldsymbol{B}_r(\bar{\boldsymbol{x}}_r, t)\boldsymbol{u} + \boldsymbol{g}_r(\bar{\boldsymbol{x}}_r, t) \tag{8c}$$

where the vector $\boldsymbol{x}$ is decomposed as $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_r, \boldsymbol{x}_{r+1})^T$, $\bar{\boldsymbol{x}}_i = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_i)^T$, $i = 2, \dots, r$, $\boldsymbol{x}_i$ is a $n_i \times 1$ vector, and the indices $(n_1, n_2, \dots, n_r)$ define the structure of the system and satisfy the following relation:

$$n_1 \le n_2 \le \cdots \le n_r \le m \quad \text{and} \quad \sum_{i=1}^{r} n_i = n. \tag{9}$$

The matrix $\boldsymbol{B}_i$, before the fictitious $\boldsymbol{x}_{i+1}$ in each ith block of (8a)–(8c), has full rank, that is

$$\operatorname{rank} \boldsymbol{B}_i = n_i \quad \forall \boldsymbol{x} \in \boldsymbol{X} \subset \boldsymbol{R}^n \quad \text{and} \quad t \in [0, \infty), \quad i = 1, \dots, r. \tag{10}$$

The procedure of reducing the system (1) to the BC-form (8a)–(8b) based on the integral transformation method, Luk'yanov [6], as well as conditions of the BC-form existence, are presented in Appendix A.

The relation (9) means $n_i = n_{i+1}$ or $n_i < n_{i+1}$. Let us first consider the plant with the structure

$$n_1 < n_2 < \cdots < n_r < m. \tag{11}$$

2.2 Block Recursive Transformation

The following assumptions on the bounds on the unknown terms in (8a)–(8c) are stated:
(H1) There exist positive constants $\bar{q}_{i,j}$ and $\bar{d}_i$ such that

$$\|\boldsymbol{g}_1(\boldsymbol{x}_1, t)\| \le \bar{q}_{11}\|\boldsymbol{x}_1\| + \bar{d}_1,$$
$$\|\boldsymbol{g}_2(\bar{\boldsymbol{x}}_2, t)\| \le \bar{q}_{21}\|\boldsymbol{x}_1\| + \bar{q}_{22}\|\boldsymbol{x}_2\| + \bar{d}_2,$$

$$\|g_i(\bar{x}_i, t)\| \le \sum_{j=1}^{i} \bar{q}_{ij}\|x_j\| + \bar{d}_i, \quad i = 3, \ldots, r-1.$$

Taking in the account the structure (11), the following recursive transformation is introduced:

$$z_1 = x_1 := \Phi_1(x_1, t) \tag{12a}$$

$$z_2 = \tilde{B}_2(\bar{x}_1, t)x_2 + \begin{bmatrix} f_1(x_1, t) + k_1\Phi_1(x_1, t) \\ 0 \end{bmatrix} :\equiv \Phi_2(\bar{x}_2, t) \tag{12b}$$

$$z_{i+1} = \tilde{B}_{i+1}(\bar{x}_i, t)x_{i+1} + \begin{bmatrix} \bar{f}_i(\bar{x}_i, t) + k_i\Phi_i(\bar{x}_i, t) \\ 0 \end{bmatrix}$$
$$:\equiv \Phi_{i+1}(\bar{x}_{i+1}, t), \quad i = 3, \ldots, r-1 \tag{12c}$$

where z_i is a $n_i \times 1$ new variables vector, $k_i > 0$, $\tilde{B}_{i+1} = \begin{bmatrix} \bar{B}_i \\ E_{i,2} \end{bmatrix}$, $E_{i,2} = [0 \quad I_{n_{i+1}-n_i}]$, $E_{i,2} \in R^{(n_{i+1}-n_i)\times n_{i+1}}$, $I_{n_{i+1}-n_i}$ is the identity matrix.

PROPOSITION 1 *The transformation* (12*a*)–(12*c*) *reduces the system* (8*a*)–(8*c*) *to the following desired form*:

$$\dot{z}_1 = -k_1 z_1 + E_{11} z_2 + \bar{g}_1(z_1, t) \tag{13a}$$

$$\dot{z}_i = -k_i z_i + E_{i,1} z_{i+1} + \bar{g}_i(\bar{z}_i, t), \quad i = 2, \ldots, r-1 \tag{13b}$$

$$\dot{z}_r = \bar{f}_r(z, t) + \bar{B}_r(z, t)u + \bar{g}_r(z, t) \tag{13c}$$

where $z = (z_1, \ldots, z_r)^T$, $\bar{f}_r(z, x_{r+1}, t)$ *is a bounded function*, $\operatorname{rank}\bar{B}_r = n_1$, $\bar{B}_r = \tilde{B}_{r-1}B_r$.

The proof is given in Appendix B.

2.3 Discontinuous Control

In order to generate sliding mode in (13a)–(13c), a natural choice of the sliding manifold using transformation (12a)–(12c), is

$$z_r = 0, \quad z_r = \Phi_r(\bar{x}_r, t). \tag{14}$$

Then, taking in the account the bound (2), the following discontinuous control strategy, is proposed:

$$u = -U_0 \operatorname{sign}(\bar{B}_r^T z_r). \tag{15}$$

PROPOSITION 2 *The control law* (15) *guaranties the convergence of the closed-loop system motion to manifold* $z_r = 0$ (14) *in a finite time defined as*

$$t_s < t_0 + \frac{1}{\eta}\|z_r(t_0)\|_2, \quad \eta > 0. \tag{16}$$

The proof is given in the Appendix C.

2.4 Robustness to Unmatched Uncertainty

For the system constrained to the sliding surface $z_r = 0$ the system (13a)–(13c) reduces to

$$\dot{z}_1 = -k_1 z_1 + \boldsymbol{E}_{11} z_2 + \bar{\boldsymbol{g}}_1(z_1, t) \tag{17a}$$

$$\dot{z}_i = -k_i z_i + \boldsymbol{E}_{i,1} z_{i+1} + \bar{\boldsymbol{g}}_i(\bar{z}_i, t), \quad i = 2, \ldots, r-1 \tag{17b}$$

$$\dot{z}_{r-1} = -k_{r-1} z_{r-1} + \bar{\boldsymbol{g}}_r(\bar{z}_{r-1}, t). \tag{17c}$$

Thus now, the original stability analysis problem is reduced to analysis of robustness property of a reduced-order sliding mode dynamics (17a)–(17c) which can be considered as linear system with nonlinear perturbation. Note that this perturbation is unmatched with respect to the control $\boldsymbol{u}$ in (8a)–(8c). It will be shown that the convergence rate of the linear part of (17a)–(17c) is defined by values of coefficients $k_1, \ldots, k_{r-1}$. For, the bounds from the physical constraints on the original system (8a)–(8c) (see Assumption H1) may be rewritten by using the change of variables (12a)–(12c) as

(H2) There exist positive constants q_{ij} and d_i, such that

$$\|\bar{\boldsymbol{g}}_1(z_1, t)\| \le q_{11}\|z_1\| + d_1 \tag{18a}$$

$$\|\bar{\boldsymbol{g}}_2(\bar{z}_2, t) \le q_{22}\|z_2\| + k_1 q_{21}\|z_1\| + d_2 \tag{18b}$$

$$\|\bar{\boldsymbol{g}}_3(\bar{z}_3, t)\| \le q_{33}\|z_3\| + k_2 q_{32}\|z_2\| + k_1^2 q_{21}\|z_1\| + d_3 \tag{18c}$$

$$\|\bar{\boldsymbol{g}}_i(\bar{z}_i, t)\| \le q_{i,i}\|z_i\| + \sum_{j=1}^{i-1} k_j^{(i-j)} q_{i,j}\|z_j\| + d_i, \quad i = 4, \ldots, r-1. \tag{18d}$$

To achieve the robustness property with respect to unknown but bounded uncertainty, the controller gains $k_1, \ldots, k_{r-1}$ have to be chosen hierarchically high. Thus, since $\boldsymbol{g}_1$ does not depend on k_1, the value of this coefficient can be chosen so high that the term $k_1 z_1$ in (17a) will be dominate. By block linearization procedure, the term $\boldsymbol{g}_2$ depends on k_1 but not on $k_2, \ldots, k_{r-1}$ (see Appendix B). Then for fixed k_1, the appropriate choice of value of k_2 provides the dominations of term $k_2 z_2$ in the second block of (17b), and so on.

In order to establish property of the sliding mode motion on the surface $z_r = 0$, the following hierarchy of the control gains $k_1, \ldots, k_{r-1}$ with respect to the given bounds on the unknown terms of (17a)–(17c), is proposed:

$$k_1 > q_{11} \tag{19a}$$

$$k_2 > q_{22} + k_1 q_{21}\alpha_{12}, \quad \alpha_{12} = (k_1 - q_{11})^{-1} \tag{19b}$$

$$k_3 > q_{33} + k_2 q_{32}\alpha_{23} + k_1^2 q_{31}\alpha_{13}, \quad \alpha_{23} = (k_2 - q_{22} - k_1 q_{21}\alpha_{12})^{-1}, \quad \alpha_{13} = \alpha_{12}\alpha_{23} \tag{19c}$$

$$k_i > q_{i,i} + \sum_{j=1}^{i-1} k_j^{(i-j)} q_{i,j}\alpha_{j,i}, \quad \alpha_{i-1,i} = \left(k_{i-1} - q_{i-1,i-1} - \sum_{j=1}^{i-2} k_j^{(i-j)} q_{i-1,j}\alpha_{j,i-1}\right)^{-1},$$
$$\alpha_{j,i} = \alpha_{j,i-1}\alpha_{i-1,i} \quad i = 4, \ldots, r-1. \tag{19d}$$

PROPOSITION 3 *Let the Assumption H2 holds, and the values of positive scalars* $k_1, \ldots, k_{r-1}$ *satisfy the inequalities* (19a)–(19d). *Then there exist positive scalars* $\gamma_{i,j}$ *and* h_i,

$i = 1, \dots, r-1, j = i, \dots, r-1$ such that the solutions of system (17a)–(17c) are estimated by

$$\|z_{r-1}(t)\| \le \gamma_{r-1,r-1} \exp\left[\frac{1}{2}\alpha_{r-1}(t-t_0)\right] + h_{r-1} \tag{20a}$$

$$\|z_{r-2}(t)\| \le \gamma_{r-2,r-2} \exp\left[-\frac{1}{2}\alpha_{r-2}(t-t_0)\right] + \gamma_{r-2,r-1} \exp\left[-\frac{1}{2}\alpha_{r-1}(t-t_0)\right] + h_{r-2} \tag{20b}$$

$$\|z_i(t)\| \le \sum_{j=i}^{r-1} \gamma_{i,j} \exp\left[-\frac{1}{2}\alpha_j(t-t_0)\right] + h_i, \quad i = 1, \dots, r-3 \tag{20c}$$

and these solutions are uniformly ultimately bounded, i.e.

$$\lim_{t\to\infty} \sup \|z_i(t)\| \le h_i, \quad i = 1, \dots, r-1. \tag{21}$$

The detailed proof which is constructive since it establish property of the sliding mode motion on the surface $z_r = 0$, and provides the required values of the controller gains $k_1, \dots, k_{r-1}$ is derived in Appendix D. It is interesting to note that with increasing the values of $k_1, \dots, k_{r-1}$ the values of bounds h_i became arbitrary small. But in this case the domain of sliding mode stability (C1) (see Appendix C) can be decreased since function $\bar{f}_r$ depends as well on gains $k_1, \dots, k_{r-1}$.

3 CONCLUSIONS

In this paper the decomposition block control method has been formulated for the control of a uncertain nonlinear system which can be transformed into a BC-form. This method combined with high gain technique enables to design constructively a sliding mode based controller whose gains are selected at each step of the algorithm in order to satisfy condition based on the given uncertainty bounds. The fact that this controller guarantees overall stability has been shown using Lyapunov techniques. In order to render the method practicable, future research should be directed to design an observer for nonlinear uncertain system.

References

[1] Utkin, V. I. (1992) *Sliding Modes in Control and Optimisation.* Berlin, Heidelberg: Springer Verlag.
[2] Drazenovic, B. (1969) The invariance conditions in variable structure systems, *Automatica*, **5**(3), 287–295.
[3] Isidori, A. (1995) *Nonlinear Control Systems* (3rd ed.). New York: Springer-Verlag.
[4] Krstic, M., Kanellakopulos, I. and Kokotovic, P. (1995) *Nonlinear and Adaptive Control Design*, New York: Wiley Interscience.
[5] Drakunov, S. V., Izosimov, D. B., Luk'yanov, A. G., Utkin, V. A. and Utkin, V. I. (1990) Block control principle I, *Automation and Remote Control*, **51**(5) (P.1), 601–609.
[6] Luk'yanov, A. G. and Utkin, V. I. (1981) Methods of reducing equations for dynamic systems to a regular form, *Automation and Remote Control*, **42**(4) (P.1), 413–420.
[7] Dodds, S. J. and Loukianov, A. G. (1997) Time-varying systems with discontinuous controls, *Automation and Remote Control*, **58**(5), 735–748.
[8] Louk'yanov, A. G., Utkin, V. I., Dodds, S. I. and Vittek, J. (2001) Decomposed control design for non-stationary plants subject to disturbances, *International Journal of Control*, **74**(13), 1335–1352.
[9] Escoto-Hernández, J., Loukianov, A. G., Castillo-Toledo, B. and Utkin, V. I. (2001) Observer based decomposition control of linear delayed systems, *Proc. of the 40th IEEE Conference on Decision and Control*, Orlando, USA, pp. 1867–1872.

[10] Loukianov, A. G. (1998) Nonlinear block control with sliding mode, *Automation and Remote Control*, **59**(7), 916–933.
[11] Loukianov, A. G., Castillo-Toledo, B. and Dodds, S. J. Robust stabilization of a class of uncertain system via block decomposition and VSC, accepted for publication in *International Journal of Robust Control.*
[12] Utkin, V. I., Chen, D. S. and Chang, H. C. (2000) Block control principle for mechanical systems, *Journal of Dynamic Systems, Measurement, and Control*, **122**(1), 1–10.
[13] Utkin, V. I., Luk'yanov, A. G., Castillo-Toledo, B. and Chavez-Gomes, E. (2000) Asymptotic tracking in non-linear systems with discontinuous control, In: Yu, X. and Xu, J. (Eds.), *Advances in Variable Structure Systems: Analysis, Integration and Application.* World Scientific Publishing, Singapore, pp. 383–392.
[14] Loukianov, A. G., Wang, J. and Dodds, S. J. (1995) Non-linear state observer design for block control with sliding mode control of manipulator, *Proc. of die IFAC Workshop "Motion Control"*, Munich, Germany, pp. 632–639.
[15] Loukianov, A. G., Dodds, S. J. and Vittek, J. (1997) A robust automotive controller design, *Proc. of the Third IEEE International Conference on Control Applications*, Hartford, Connecticut, USA, pp. 806–811.
[16] Loukianov, A. G., Cañedo, J. M., Serrano, O., Utkin, V. I. and Celikovsky, S. (2001) Adaptive sliding mode block control of induction motors, *Proc. of the American Control Conference*, Washington, USA, pp. 149–154.
[17] Sanchez, E. N., Loukianov, A. G. and Felix, R. A. (2000) Stepper motor trajectory tracking combining dynamic triangular neural networks and sliding modes, In: Yu, X. and Xu, J. (Eds.), *Advances in Variable Structure Systems: Analysis, Integration and Application.* World Scientific Publishing, Singapore, pp. 293–302.
[18] Loukianov, A. G., Utkin, V. I., Cañedo, J. M., Ramirez, J. M. and Cabrera-Vázquez, J. (2000) Control of the system synchronous generator-exciter via VSC, *Proc. of the IEEE Conference on Design and Control*, Sydney, Australia, pp. 3057–3062.
[19] Khalil, H. K. (1996) *Nonlinear Systems* (2nd. ed.) Prentice-Hall, New-Jersey.

APPENDIX A

Block Decomposition Algorithm

The procedure of reducing the system (1) to the NBC-form described by

$$\begin{aligned}\dot{x}_r &= f_r(x_r, t) + B_r(x_r, t)x_{r-1} + g_r(x_r, t)\\ \dot{x}_i &= f_i(\bar{x}_i, t) + B_i(\bar{x}_i, t)x_{i-1} + g_i(\bar{x}_i, t), i = 2, \dots, r-1\\ \dot{x}_1 &= f_1(x, t) + B_1(x, t)u + g_1(x, t)\end{aligned} \tag{A1}$$

where $x = (x_1, \dots, x_r)^T, \bar{x}_i = (x_i, \dots, x_r)^T, x_i \in X_i \subset R^{\bar{n}_i}i, \dim X_i = \operatorname{rank} B_i = \bar{n}_i, i = 1, \dots, r,$ with the structure

$$m > \bar{n}_1 > \bar{n}_2 \cdots > \bar{n}_r, \quad \sum_{i=1}^{r} \bar{n}_i = n$$

consists of series of steps.

Step 1 Assume that the matrix $B(x, t)$ has an $(\bar{n}_1 \times m)$ block, $B_1(x_1, x_{12}, t)$, such that $\operatorname{rank} B_1 = \operatorname{rank} B = \bar{n}_1 \le m \ \forall x \in X$ and $\forall t \ge 0$ where $B(\cdot) = \begin{bmatrix} B_{12}(\cdot) \\ B_1(\cdot) \end{bmatrix}$, $x = (x_1, x_{12})^T$, $x_1 \in X_1 \subset R^{\bar{n}_1}, x_{12} \in X_{12} \subset R^{n-\bar{n}_1}$. At this point we introduce the following instrumental assumptions which will be carried for each step of the procedure.

(A11) The Pfaffian system

$$\Omega_1(\mathrm{d}) = \mathrm{d}x_{12} + A_1(x_1, x_{12}, t)\,\mathrm{d}x_1 = 0, \quad A_1(x, t) = -B_{12}(x, t)B_1^{+}(x, t) \tag{A2}$$

is completely integrable, that is the following conditions, Luk'yanov [6]:

$$\frac{\partial a_\alpha^j}{\partial x_\beta} - \sum_{k=1}^{n-\bar{n}_1} a_\beta^k \frac{\partial a_\alpha^j}{\partial x_k} = \frac{\partial a_\beta^j}{\partial x_\alpha} - \sum_{k=1}^{n-\bar{n}_1} a_\alpha^k \frac{\partial a_\beta^j}{\partial x_k}, \quad j = 1, \dots, n - \bar{n}_1 \tag{A3}$$

obtained by using the well known Frobenius Theorem, hold, where $\Omega_1(\mathrm{d})$ is a differential one-form, $\boldsymbol{A}_1(x) = \{a_\alpha^j\}, j = 1, \dots, n - \bar{n}_1, \alpha$ and β denote various pair wise combinations from the set of numbers $\{(n - \bar{n}_1 + 1), \dots, n\}, t$ is a parameter, and $\boldsymbol{B}_1^+ = \boldsymbol{B}_1^T[\boldsymbol{B}_1, \boldsymbol{B}_1^T]^{-1}$.
(A12) The unknown mapping $\boldsymbol{g}(\boldsymbol{x}, t)$ can be decomposed in the form

$$\boldsymbol{g}(\boldsymbol{x}, t) = \boldsymbol{g}^m(\boldsymbol{x}_1, \boldsymbol{x}_{12}, t) + \boldsymbol{g}^u(\boldsymbol{x}_{12}, t)$$

where $\boldsymbol{g}^m(\boldsymbol{x}, t)$ satisfies the matching condition, namely

$$\boldsymbol{g}^m(\boldsymbol{x}, t) \in \operatorname{span} \boldsymbol{B}(\boldsymbol{x}, t).$$

Under Assumption A11, it is possible to show that a solution of the Eq. (A2) is given by

$$\boldsymbol{x}_{12} = \bar{\varphi}_1(\boldsymbol{x}_1, t, \boldsymbol{c}), \quad \bar{\varphi}_1 = (\varphi_{11}, \dots, \bar{\varphi}_{1,n-\bar{n}_1})^T, \quad \operatorname{rank}\left(\frac{\partial \bar{\varphi}_1}{\partial \boldsymbol{c}} = n - \bar{n}_1\right)$$

where $\boldsymbol{c} = (c_1, \dots, c_{n-\bar{n}_1})^T$ is a vector of integration constants. Using Implicit Function Theorem, the vector $\boldsymbol{c}$ can be derived as $\boldsymbol{c} = \varphi_1(\boldsymbol{x}_1, \boldsymbol{x}_{12}, t)$, $\varphi_1 = (\varphi_{11}, \dots, \varphi_{1,n-\bar{n}_1})^T$, and be taken a local change of state space coordinates:

$$\boldsymbol{x}_2' = \varphi_1(\boldsymbol{x}_1, \boldsymbol{x}_{12}, t)$$

transforming the system (1) under Assumption A12 into

$$\dot{\boldsymbol{x}}_2' = \boldsymbol{f}_2'(\boldsymbol{x}_1, \boldsymbol{x}_2', t) + \boldsymbol{g}_2'(\boldsymbol{x}_2', t) \tag{A4a}$$
$$\dot{\boldsymbol{x}}_1 = \boldsymbol{f}_1(\boldsymbol{x}_1, \boldsymbol{x}_2', t) + \boldsymbol{B}_1(\boldsymbol{x}_1, \boldsymbol{x}_2', t)\boldsymbol{u} + \boldsymbol{g}_1(\boldsymbol{x}_1, \boldsymbol{x}_2', t) \tag{A4b}$$

where $\boldsymbol{x}_1 \in \boldsymbol{X}_1 \subset \boldsymbol{R}^{\bar{n}_1}, \boldsymbol{x}_2' \in \boldsymbol{X}_2' \subset \boldsymbol{R}^{n-\bar{n}_1}$, and $\operatorname{rank} \boldsymbol{B}_1(\boldsymbol{x}_1, \boldsymbol{x}_2', t) = \bar{n}_1$. The following assumption is fundamental to derive the BC-form.
(A13) The mapping $\boldsymbol{f}_2'$ in the subsystem (A4a) is affine on its first argument, having the form

$$\boldsymbol{f}_2'(\boldsymbol{x}_1, \boldsymbol{x}_2', t) = \boldsymbol{f}_2''(\boldsymbol{x}_2', t) + \boldsymbol{B}_2'(\boldsymbol{x}_2', t)\boldsymbol{x}_1 \tag{A5}$$

Now the following three possible cases are considered:

(i) $\operatorname{rank} \boldsymbol{B}_2'(\boldsymbol{x}_2', t) = 0$.
(ii) $\operatorname{rank} \boldsymbol{B}_2'(\boldsymbol{x}_2', t) = \bar{n}_2 = n - \bar{n}_1$.
(iii) $\operatorname{rank} \boldsymbol{B}_2'(\boldsymbol{x}_2', t) = \bar{n}_2 < n - \bar{n}_1$.

The first case is equivalent to have an uncontrollable system. For the purposes of this work, it is assumed in the sequel that the system is locally controllable. In the second case, after defining $\boldsymbol{x}_2 = \boldsymbol{x}_2', \boldsymbol{f}_2 = \boldsymbol{f}_2'', \boldsymbol{B}_2 = \boldsymbol{B}_2'$ and $\boldsymbol{g}_2 = \boldsymbol{g}_2'$, the NBS-form is

$$\dot{\boldsymbol{x}}_2 = \boldsymbol{f}_2(\boldsymbol{x}_2, t) + \boldsymbol{B}_2(\boldsymbol{x}_2, t)\boldsymbol{x}_1 + \boldsymbol{g}_2(\boldsymbol{x}_2, t)$$
$$\dot{\boldsymbol{x}}_1 = \boldsymbol{f}_1(\boldsymbol{x}_1, \boldsymbol{x}_2, t) + \boldsymbol{B}_1(\boldsymbol{x}_1, \boldsymbol{x}_2, t)\boldsymbol{u} + \boldsymbol{g}_1(\boldsymbol{x}_1, \boldsymbol{x}_2, t)$$

If $\bar{n}_2 < n - \bar{n}_1$, however, a second step is necessary in which the system (A4a) with (A5) is further decomposed and transformed with $\boldsymbol{x}_1$ regarded as a fictitious control vector, as well as system (1) on the first step.

Consider the system obtained at $(q-1)$th step:

$$\dot{\boldsymbol{x}}'_q = \boldsymbol{f}'_q(\boldsymbol{x}'_q, t) + \boldsymbol{B}'_q(\boldsymbol{x}'_q, t)\boldsymbol{x}_{q-1} + \boldsymbol{g}'_q(\boldsymbol{x}'_q, t) \tag{A6a}$$

$$\begin{aligned} \dot{\boldsymbol{x}}_i &= \boldsymbol{f}_i(\boldsymbol{x}_i, \dots, \boldsymbol{x}'_q, t) + \boldsymbol{B}_i(\boldsymbol{x}_i, \dots, \boldsymbol{x}'_q, t)\boldsymbol{x}_{i-1} + \boldsymbol{g}_i(\boldsymbol{x}_i, \dots, \boldsymbol{x}'_q, t), \quad i = 2, \dots, q-1, \\ \dot{\boldsymbol{x}}_1 &= \boldsymbol{f}_1(\boldsymbol{x}_1, \dots, \boldsymbol{x}'_q, t) + \boldsymbol{B}_1(\boldsymbol{x}_1, \dots, \boldsymbol{x}'_q, t)\boldsymbol{u} + \boldsymbol{g}_1(\boldsymbol{x}_1, \dots, \boldsymbol{x}'_q, t) \end{aligned} \tag{A6b}$$

where $\boldsymbol{x} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}'_q)^T, \boldsymbol{x}'_q \in \boldsymbol{X}'_q \subset \boldsymbol{R}^{n-\bar{n}_1-\cdots-\bar{n}_{q-1}}, \boldsymbol{x}_j \in \boldsymbol{X}_j \subset \boldsymbol{R}^{\bar{n}_i}, \operatorname{rank} \boldsymbol{B}_j = \bar{n}_j$.

Assume now that $\operatorname{rank} \boldsymbol{B}'_q \neq 0$, that is, the original system is locally controllable. In the case $\operatorname{rank} \boldsymbol{B}'_q = \bar{n}_q = n - \bar{n}_1 - \cdots - \bar{n}_{q-1}$, we define $\boldsymbol{x}_q = \boldsymbol{x}'_q, \boldsymbol{f}_q = \boldsymbol{f}'_q, \boldsymbol{B}_q = \boldsymbol{B}'_q, \boldsymbol{g}_q = \boldsymbol{g}'_q$, and the algorithm terminates with Eqs. (A6a)–(A6b) having the desired BS-form. If $\operatorname{rank} \boldsymbol{B}'_q = \bar{n}_q < n - \bar{n}_1 - \cdots - \bar{n}_{q-1}$ we proceed with the qth step.

Step q The subsystem (A6a) with input $\boldsymbol{x}_{q-1}$, is partitioned as

$$\begin{aligned} \dot{\boldsymbol{x}}_{q2} &= \boldsymbol{f}_{q2}(\boldsymbol{x}_q, \boldsymbol{x}_{q2}, t) + \boldsymbol{B}_{q2}(\boldsymbol{x}_q, \boldsymbol{x}_{q2}, t)\boldsymbol{x}_{q-1} + \boldsymbol{g}_{q2}(\boldsymbol{x}_q, \boldsymbol{x}_{q2}, t) \\ \dot{\boldsymbol{x}}_q &= \boldsymbol{f}_q(\boldsymbol{x}_q, \boldsymbol{x}_{q2}, t) + \boldsymbol{B}_q(\boldsymbol{x}_q, \boldsymbol{x}_{q2}, t)\boldsymbol{x}_{q-1} + \boldsymbol{g}_q(\boldsymbol{x}_q, \boldsymbol{x}_{q2}, t) \end{aligned}$$

where $\boldsymbol{x}'_q = (\boldsymbol{x}_q, \boldsymbol{x}_{q2})^T, \boldsymbol{x}_q \in \boldsymbol{X}_q \subset \boldsymbol{R}^{\bar{n}_q}, \boldsymbol{x}_{q2} \in \boldsymbol{X}_{q2} \subset \boldsymbol{R}^{n-\bar{n}_1-\cdots-\bar{n}_q}, \operatorname{rank} \boldsymbol{B}_q = \operatorname{rank} \boldsymbol{B}'_q = \bar{n}_q$.

For this step, the Assumptions A11 and A12 are generalized as follows:
(Aq1) The corresponding Pfaffian system

$$\Omega_q(\mathrm{d}) = \mathrm{d}\boldsymbol{x}_{q2} + \boldsymbol{A}_q(\boldsymbol{x}_{q2}, \boldsymbol{x}_q, t)\,\mathrm{d}\boldsymbol{x}_q = 0, \quad \boldsymbol{A}_q = -\boldsymbol{B}_{q2}\boldsymbol{B}_q^+$$

is completely integrable.
(Aq2) The unknown mapping, $\boldsymbol{g}'_q(\boldsymbol{x}_q, t)$ can be decomposed in the form

$$\boldsymbol{g}'_q(\boldsymbol{x}'_q, t) = \boldsymbol{g}^m_q(\boldsymbol{x}_{q2}, \boldsymbol{x}_q, t) + \boldsymbol{g}^u_q(\boldsymbol{x}_{q2}, t)$$

where $\boldsymbol{g}^m_q(\boldsymbol{x}_{q2}, \boldsymbol{x}_q, t)$ satisfies the matching condition, namely

$$\boldsymbol{g}^m_q(\boldsymbol{x}'_q, t) \in \operatorname{span} \boldsymbol{B}'_q(\boldsymbol{x}'_q, t).$$

Proceeding as in the first step, under the previous assumptions, we may find a local change of coordinates given by

$$\boldsymbol{x}'_{q+1} = \varphi_q(\boldsymbol{x}_{q2}, \boldsymbol{x}_q, t)$$

such that the system is described by

$$\begin{aligned} \dot{\boldsymbol{x}}'_{q+1} &= \boldsymbol{f}'_{q+1}(\boldsymbol{x}_q, \boldsymbol{x}'_{q+1}, t) + \boldsymbol{g}'_{q+1}(\boldsymbol{x}_q, \boldsymbol{x}'_{q+1}, t) \\ \dot{\boldsymbol{x}}_i &= \boldsymbol{f}_i(\boldsymbol{x}_i, \dots, \boldsymbol{x}'_{q+1}, t) + \boldsymbol{B}_i(\boldsymbol{x}_i, \dots, \boldsymbol{x}'_{q+1}, t)\boldsymbol{x}_{i-1} + \boldsymbol{g}_i(\boldsymbol{x}_i, \dots, \boldsymbol{x}'_{q+1}, t), \quad i = 2, \dots, q \\ \dot{\boldsymbol{x}}_1 &= \boldsymbol{f}_1(\boldsymbol{x}_1, \dots, \boldsymbol{x}'_{q+1}, t) + \boldsymbol{B}_1(\boldsymbol{x}_1, \dots, \boldsymbol{x}'_{q+1}, t)\boldsymbol{u} + \boldsymbol{g}_1(\boldsymbol{x}_1, \dots, \boldsymbol{x}'_{q+1}, t) \end{aligned}$$

with rank $\boldsymbol{B}_j = \bar{n}_j, j = 1, \ldots, q$. In the same way, the Assumption A13 for step q is stated as: (Aq3) The mapping $\boldsymbol{f}'_{q+1}(\boldsymbol{x}_q, \boldsymbol{x}'_{q+1}, t)$ is affine on its first argument, namely

$$\boldsymbol{f}'_{q+1}(\boldsymbol{x}_q, \boldsymbol{x}'_{q+1}, t) = \boldsymbol{f}''_{q+1}(\boldsymbol{x}'_{q+1}, t) + \boldsymbol{B}'_{q+1}(\boldsymbol{x}'_{q+1}, t)\boldsymbol{x}_q$$

From the previous algorithm, we may state the following result:

LEMMA 1 *Assume that the system is locally controllable, and at each step of the NBC-form algorithm the Assumptions Aq*1, *Aq*2 *and Aq*3 *hold. Then, there exists an integer $r \leq n$ such that the system* (1) *takes the NBC-form.*

Remark A1 For the purposes of the work, it is more convenient to rewrite the obtained NBC form (A1) by renumber the states by taking $i = r - j + 1$, so we obtain

$$\begin{aligned} \dot{\boldsymbol{x}}_i &= \boldsymbol{f}_i(\bar{\boldsymbol{x}}_i, t) + \boldsymbol{B}_i(\bar{\boldsymbol{x}}_i, t)\boldsymbol{x}_{i+1} + \boldsymbol{g}_i(\bar{\boldsymbol{x}}_i, t), \quad i = 1, \ldots, r-1 \\ \dot{\boldsymbol{x}}_r &= \boldsymbol{f}_r(\bar{\boldsymbol{x}}_r, t) + \boldsymbol{B}_r(\bar{\boldsymbol{x}}_r, t)\boldsymbol{u} + \boldsymbol{g}_r(\bar{\boldsymbol{x}}_r, t) \end{aligned}$$

where now rank $\boldsymbol{B}_i = n_i$ and $n_1 \leq n_2 \leq \cdots \leq n_r \leq m$.

APPENDIX B

Proof of Proposition 1 (Block Linearizing Transformation Algorithm)

In order to prove the proposition the method of induction will be used, considering $\boldsymbol{x}_{i+1}, i = 1, \ldots, r-1$, as a fictitious control vector in each ith block of (8a)–(8c).

Step 1 Let the fictitious control $\boldsymbol{x}_2$ in the first block (8a) rewritten as

$$\dot{z}_1 = \boldsymbol{f}_1(\boldsymbol{x}_1, t) + \boldsymbol{B}_1(\boldsymbol{x}_1, t)\boldsymbol{x}_2 + \boldsymbol{g}_1(\boldsymbol{x}_1, t) \tag{B1}$$

be selected of the form

$$\boldsymbol{x}_2 = -\boldsymbol{B}_1^+(\boldsymbol{x}_1, t)\boldsymbol{f}_1(\boldsymbol{x}_1, t) + \boldsymbol{B}_1^+(\boldsymbol{x}_1, t)[-k_1 z_1 + \boldsymbol{E}_{11} z_2] \tag{B2}$$

where $z_1 = \boldsymbol{x}_1, z_2$ is a $n_2 \times 1$ new variables vector, $k_1 > 0, \boldsymbol{E}_{1,1} = [\boldsymbol{I}_{n_1} \quad 0]$, $\boldsymbol{E}_{1,1} \in \boldsymbol{R}^{n_1 \times n_2}, \boldsymbol{I}_{n_1}$ is the identity matrix. The transformed first block (B1) in new variables z_1 and z_2, with input (B2) has the desired form (13a), *i.e.*

$$\dot{z}_1 = -k_1 z_1 + \boldsymbol{E}_{11} z_2 + \bar{\boldsymbol{g}}_1(z_1, t).$$

Now the transformation (B2) is extended by

$$\boldsymbol{M}_1(\boldsymbol{x}_1, t)\boldsymbol{x}_2 = \boldsymbol{E}_{12} z_2, \;\; \boldsymbol{E}_{1,2} = [0 \quad \boldsymbol{I}_{n_2 - n_1}], \;\; \boldsymbol{E}_{1,2} \in \boldsymbol{R}^{(n_2 - n_1) \times n_2}, \;\; \begin{bmatrix} \boldsymbol{E}_{11} \\ \boldsymbol{E}_{12} \end{bmatrix} = \boldsymbol{I}_{n_2} \tag{B3}$$

such that the square matrix $\tilde{\boldsymbol{B}}_2(\boldsymbol{x}_1, t) \equiv \begin{bmatrix} \boldsymbol{B}_1(\boldsymbol{x}_1, t) \\ \boldsymbol{M}_1(\boldsymbol{x}_1, t) \end{bmatrix}$ with $(n_2 - n_1) \times n_2$ matrix $\boldsymbol{M}_1(\boldsymbol{x}_1, t)$, has rank n_2. Using (B2) and (B3), the variable z_2 can be obtained of the form (12b), that is

$$z_2 = \tilde{\boldsymbol{B}}_2(\bar{\boldsymbol{x}}_1, t)\boldsymbol{x}_2 + \begin{bmatrix} \boldsymbol{f}_1(\boldsymbol{x}_1, t) + k_1\Phi_1(\boldsymbol{x}_1, t) \\ 0 \end{bmatrix} \equiv \Phi_2(\bar{\boldsymbol{x}}_2, t).$$

Step i At this stage it possible to show that if we have, after $(i-1)$ steps, the transformed blocks of the system (8a)–(8c) with new variables $z_1, z_2, \ldots, z_{i-1}$ (under structure $\bar{n}_{i-1} < \bar{n}_i$) of the form

$$\begin{aligned} \dot{z}_1 &= -k_1 z_1 + \boldsymbol{E}_{1,1} z_2 + \bar{\boldsymbol{g}}_1(z_1, t) \\ &\vdots \\ \dot{z}_{i-1} &= -k_{i-1} z_{i-1} + \boldsymbol{E}_{i-1,1} z_i + \bar{\boldsymbol{g}}_{i-1}(\bar{z}_{i-1}, t) \end{aligned} \tag{B4}$$

with

$$z_i = \Phi_i(\bar{\boldsymbol{x}}_i, t) \tag{B5}$$

then on the ith step of the transformation procedure, we will have the transformed ith block with new state vector z_i similar to (B4). To carry out this, take the derivative of (B5) along the trajectories of (8a)–(8c), results

$$\dot{z}_i = \bar{\boldsymbol{f}}_i(\bar{\boldsymbol{x}}_i, t) + \bar{\boldsymbol{B}}_i(\bar{\boldsymbol{x}}_i, t)\boldsymbol{x}_{i+1} + \boldsymbol{g}_2(\bar{\boldsymbol{x}}_i, t) \tag{B6}$$

where $\bar{\boldsymbol{f}}_i = \sum_{j=1}^{i-1}[(\partial\Phi_i/\partial \boldsymbol{x}_j)\boldsymbol{f}_j + \boldsymbol{B}_j\boldsymbol{x}_{j+1}] + (\partial\Phi_i/\partial \boldsymbol{x}_i)\boldsymbol{f}_i + (\partial\Phi_i/\partial t)$, and $\bar{\boldsymbol{B}}_i = \tilde{\boldsymbol{B}}_{i-1}\boldsymbol{B}_i$, $\operatorname{rank}\bar{\boldsymbol{B}}_i = \operatorname{rank}\boldsymbol{B}_i = n_i$. For the case $\bar{n}_i < \bar{n}_{i+1}$, the fictitious control vector, $\boldsymbol{x}_{i+1}$ in (B6) can be selected similar to (B2), of the form

$$\boldsymbol{x}_{i+1} = -\bar{\boldsymbol{B}}_i^+(\bar{\boldsymbol{x}}_i, t)\bar{\boldsymbol{f}}_i(\bar{\boldsymbol{x}}_i, t) + \bar{\boldsymbol{B}}_i^+(\bar{\boldsymbol{x}}_i, t)[-k_i z_i + \boldsymbol{E}_{i,1} z_{i+1}] \tag{B7}$$

where $\bar{\boldsymbol{B}}_j^+$ is pseudo inverse matrix of $\bar{\boldsymbol{B}}_i = \tilde{\boldsymbol{B}}_i\boldsymbol{B}_i$, $\boldsymbol{E}_{i,1} = [I_{n_i} \quad 0]$, $\boldsymbol{E}_{i,1} \in R^{n_i \times n_{i+1}}$. Thus, Eq. (B6) with (B7) takes the same form as Eqs. (B4), that is

$$\dot{z}_i = -k_i z_i + \boldsymbol{E}_{i,1} z_{i+1} + \bar{\boldsymbol{g}}_i(\bar{z}_i, t).$$

For this step, the transformation (B7) is extended as follows:

$$\boldsymbol{M}_i(\boldsymbol{x}_i, t)\boldsymbol{x}_{i+1} = \boldsymbol{E}_{i,2} z_{i+1}, \quad \boldsymbol{E}_{i,2} = [0 \quad \boldsymbol{I}_{n_{i+1}-n_i}] \in \boldsymbol{R}^{(n_{i+1}-n_i)\times n_{i+1}} \tag{B8}$$

with $\operatorname{rank}\tilde{\boldsymbol{B}}_{i+1}(\bar{\boldsymbol{x}}_i, t) = n_{i+1}, \tilde{\boldsymbol{B}}_{i+1}(\bar{\boldsymbol{x}}_i, t) = \begin{bmatrix} \bar{\boldsymbol{B}}_i(\bar{\boldsymbol{x}}_i, t) \\ \boldsymbol{M}_i(\bar{\boldsymbol{x}}_i, t) \end{bmatrix}, \boldsymbol{M}_i(\boldsymbol{x}_i, t)$ is a $(n_{i+1} - n_i) \times n_{i+1}$ matrix. Thus, using (B7) and (B8), we can obtain the recursive transformation (12c), *i.e.*

$$z_{i+1} = \tilde{\boldsymbol{B}}_{i+1}(\bar{\boldsymbol{x}}_i, t)\boldsymbol{x}_{i+1} + \begin{bmatrix} \bar{\boldsymbol{f}}_i(\bar{\boldsymbol{x}}_i, t) + k_i\Phi_i(\bar{\boldsymbol{x}}_i, t) \\ 0 \end{bmatrix} := \Phi_{i+1}(\bar{\boldsymbol{x}}_{i+1}, t).$$

Step r On the last step, calculating the time derivative of $z_r = \Phi_r(\bar{x}_r, t)$, gives the last block

$$\dot{z}_r = \bar{f}_r(z, t) + \bar{B}_r(z, t)u + \bar{g}_r(z, t)$$

where $\bar{f}_r = \sum_{j=1}^{r-1}[(\partial\Phi_r/\partial x_j)f_j + B_j x_{j+1}] + (\partial\Phi_r/\partial x_r)f_r + (\partial\Phi_r/\partial t)$, and $\bar{B}_r = \tilde{B}_r B_r$, $\operatorname{rank}\bar{B}_r = \operatorname{rank} B_r = n_r < m$.

Thus transformation (12a)–(12c) reduces the system (8a)–(8c) to (13a)–(13c). ■

APPENDIX C

Proof of the Proposition 2

Taking the derivative of $V_r = (1/2)z_r^T z_r$ along the trajectories of (13c), gives

$$\dot{V}_r = z_r^T(\bar{f}_r + \bar{g}_r) - U_0 z_r^T \bar{B}_r \operatorname{sign}(\bar{B}_r^T z_r).$$

Using the following relations $s^T \operatorname{sign}(s) = \|s\|_1$ and $\|s\|_1 \geq \|s\|_2$, we have

$$\dot{V}_r = z_r^T \bar{B}_r \bar{B}_r^+(\bar{f}_r + \bar{g}_r) - U_0\|z_r^T \bar{B}_r\|_2 \leq -\|z_r^T \bar{B}_r\|_2[U_0 - \|\bar{B}_r^+(\bar{f}_r + \bar{g}_r)\|_2]$$

In the following domain:

$$\|\bar{B}_r^+(\bar{f}_r + \bar{g}_r)\|_2 \leq r_0, \quad r_0 < U_0 \tag{C1}$$

the derivative

$$\dot{V}_r \leq -q_0\|z_r^T \bar{B}_r\|_2, \quad q_0 = U_0 - r_0,$$

is negative, that guaranties convergence of the state vector to the manifold $z_r = 0$. In order to demonstrate that this convergence is finite, first we assume that $\|\bar{B}_r^+\|_2 \leq b_0$, and using the following relation:

$$\|z_r^T\|_2 = \|z_r^T \bar{B}_r \bar{B}_r^+\|_2 \leq \|z_r^T \bar{B}_r\|_2 \|\bar{B}_r^+\|_2 \leq b_0\|z_r^T \bar{B}_r\|_2$$

we have $\|z_r^T \bar{B}_r\|_2 \geq (1/b_0)\|z_r^T\|_2$. Therefore, $\dot{V}_r < -\eta\|z_r\|_2$, $\eta = q_0/b_0$, or

$$\dot{V}_r \leq -\eta\sqrt{2V_r}. \tag{C2}$$

Using the Comparison Lemma, Khalil [19], a solution of (C1) can be estimate as $V_r(t) < (1/2)(\sqrt{2V_r(t_0)} - \eta(t - t_0))^2$. Thus, $\|z_r(t)\|_2 < \|z_r(t_0)\|_2 - \eta(t - t_0)$. Therefore, $z_r(t)$ vanishes in some finite time, $t_s < t_0 + (1/\eta)\|z_r(t_0)\|_2$, and sliding mode starts on the manifold $z_r = 0$ after this time. ■

APPENDIX D

Proof of Proposition 3

First, choose a Lyapunov function candidate V for the system (17a)–(17c) as a sum of Lyapunov function candidates for the each block of (17a)–(17c), namely

$$V = \sum_{i=1}^{r-1} V_i, \quad V_i = \frac{1}{2} z_i^T z_i, \ \ i = 1, \dots, r-1$$

and let us calculate the derivatives $\dot{V}_i$, $i = 1, \dots, r-1$ step by step from the first block to the last block of (17a)–(17c).

At the first step, differentiating the Lyapunov function candidate $V_1 = (1/2)z_1^T z_1$ along the trajectories of (17a) and using Assumption H2, namely (18a), we get

$$\begin{aligned}\dot{V}_1 &= -k_1 z_1^T z_1 + z_1^T[\boldsymbol{E}_{11} z_2 + \bar{g}_1(z_1, t)] \\ &\leq -k_1 \|z_1\|^2 + \|z_1\|(\|z_2\| + q_{11}\|z_1\| + d_1) \\ &= -\|z_1\|[(k_1 - q_{11})\|z_1\| - \|z_2\| - d_1]\end{aligned}$$

which is negative in the region $\|z_1\| > [1/(k_1 - q_{11})]\|z_2\| + d_1/(k_1 - q_{11})$. Therefore, the state ultimately enter the domain in subspace (z_1, z_2) defined by

$$\|z_1\| \leq \alpha_{12}\|z_2\| + \beta_{12} \tag{D1}$$

where the parameters α_{12} and β_{12} defined as

$$\alpha_{12} = (k_1 - q_{11})^{-1} \quad \text{and} \quad \beta_{12} = \alpha_{12} d_1$$

are positive if the condition $k_1 > q_{11}$, (19a) holds.

At the second step, following similar lines to those taken for the first block, the derivative $\dot{V}_2$ of the Lyapunov function candidate $V_2 = (1/2)z_2^T z_2$ calculated along the trajectories of the second block of (17b), under conditions (18a), (18b) and (D1), is given by

$$\begin{aligned}\dot{V}_2 &= -k_2 z_2^T z_2 + z_2^T[\boldsymbol{E}_{21} z_3 + \bar{g}_2(z_1, z_2, t)] \\ &\leq -k_2\|z_2\|^2 + \|z_2\|(\|z_3\| + k_1 q_{21}\|z_1\| + q_{22}\|z_2\| + d_2) \\ &= -\|z_2\|[(k_2 - q_{22})\|z_2\| - \|z_3\| - k_1 q_{21}\|z_1\| - d_2] \\ &\leq -\|z_2\|[(k_2 - q_{22} - k_1 q_{21}\alpha_{12})\|z_2\| - \|z_3\| - k_1 q_{21}\beta_{12} - d_2]\end{aligned}$$

which is negative if $(k_2 - q_{22} - k_1 q_{21}\alpha_{12})\|z_2\| - \|z_3\| - k_1 q_{21}\beta_{12} - d_2 > 0$. Hence, the state ultimately enter the domain in the subspace (z_1, z_2, z_3) defined by

$$\|z_2\| \leq \alpha_{23}\|z_3\| + \beta_{23}$$

and consequently

$$\|z_1\| \leq \alpha_{13}\|z_3\| + \beta_{13}$$

where the scalar parameters $\alpha_{23}, \beta_{23}, \alpha_{13}$ and β_{13} defined as

$$\alpha_{23} = (k_2 - q_{22} - k_1 q_{21}\alpha_{12})^{-1}, \quad \beta_{23} = \alpha_{23}(k_1 q_{21}\beta_{12} + d_2),$$
$$\alpha_{13} = \alpha_{12}\alpha_{23}, \quad \beta_{13} = \alpha_{12}\beta_{23} + \beta_{12}$$

are positive if the values of k_1 and k_2 satisfy the inequalities $k_1 > q_{11}$, (19a) and $k_2 > q_{22} + k_1 q_{21}\alpha_{12}$, (19b).

Proceeding in the same fashion for the ith block of the system (17a)–(17c), then the convergence domain in the subspace $(z_1, z_2, \ldots, z_{i-2}, z_{i-1}, z_i)$, is

$$\begin{aligned} \|z_1\| &\le \alpha_{1,i}\|z_i\| + \beta_{1,i} \\ \|z_2\| &\le \alpha_{2,i}\|z_i\| + \beta_{2,i} \\ &\vdots \\ \|z_{i-1}\| &\le \alpha_{i-1,i}\|z_i\| + \beta_{i-1,i} \end{aligned} \tag{D2}$$

where $\alpha_{j,i} = \alpha_{j,i-1}\alpha_{i-1,i}, \alpha_{i-1,i} = (k_{i-1} - q_{i-1,i-1} - \sum_{j=1}^{i-2} k_j^{(i-j)} q_{i-1,j}\alpha_{j,i-1})^{-1}$, and $\beta_{j,i} = \alpha_{j,i-1}\beta_{i-1,i} + \beta_{j,i-1}$, $j = 1, \ldots, i-1$.

At the next step, taking again the derivative of the Lyapunov function $V_i = (1/2)z_i^T z_i$ along the trajectories of the ith block of (17a)–(17c), and using (18d), we obtain

$$\begin{aligned} \dot{V}_i &= -k_i z_i^T z_i + z_i^T[\boldsymbol{E}_{i,1} z_{i+1} + \bar{\boldsymbol{g}}_i(z_i, \ldots, z_i, t)] \\ &\le -k_i\|z_i\|^2 + \|z_i\|\left(\|z_{i+1}\| + q_{i,i}\|z_i\| + \sum_{j=1}^{i-1} k_j^{i-j} q_{i,j}\|z_j\| + d_i\right). \end{aligned}$$

Using now (D2), we can majorize $\dot{V}_i$ as

$$\dot{V}_i \le -\|z_i\|\left[\left(k_i - q_{i,i} - \sum_{j=1}^{i-1} k_j^{(i-j)} q_{i,j}\alpha_{j,i}\right)\|z_i\| - \|z_{i+1}\| - \sum_{j=1}^{i-1} k_j^{(i-j)} q_{i,j}\beta_{j,i} - d_i\right]. \tag{D3}$$

From this equation it follows that

$$\|z_i\| \le \alpha_{i,i+1}\|z_{i+1}\| + \beta_{i,i+1} \tag{D4}$$

where the parameters

$$\begin{aligned} \alpha_{i,i+1} &= \left(k_i - q_{i,i} - \sum_{j=1}^{i-1} k_j^{(i-j)} q_{i,j}\alpha_{j,i}\right)^{-1} \quad \text{and} \\ \beta_{i,i+1} &= \alpha_{i,i+1}\left(\sum_{j=1}^{i-1} k_j^{(i-j)} q_{i,j}\beta_{j,i} - d_i\right), \quad i = 4, \ldots, r-1 \end{aligned} \tag{D5}$$

are positive if the condition $k_i > q_{i,i} + \sum_{j=1}^{i-1} k_j^{(i-j)} q_{i,j}\alpha_{j,i}$, (19d) holds. Substitution of (D4) in (D2) gives the following set of inequalities for the subspace $(z_1, z_2, \ldots, z_{i-2}, z_{i-1}, z_i, z_{i+1})$:

$$\begin{aligned} \|z_1\| &\le \alpha_{1,i+1}\|z_{i+1}\| + \beta_{1,i+1} \\ \|z_2\| &\le \alpha_{2,i+1}\|z_{i+1}\| + \beta_{2,i+1} \\ &\vdots \\ \|z_{i-1}\| &\le \alpha_{i-1,i+1}\|z_{i+1}\| + \beta_{i-1,i+1} \\ \|z_i\| &\le \alpha_{i,i+1}\|z_{i+1}\| + \beta_{i,i+1} \end{aligned} \tag{D6}$$

where

$$\alpha_{j,i+1} = \alpha_{j,i}\alpha_{i,i+1} \quad \text{and} \quad \beta_{j,i+1} = \alpha_{j,i}\beta_{i,i+1} + \beta_{j,i}, \;\; j = 1, \ldots, i, \;\; i = 4, \ldots, r-1.$$

At the last step we have the domain of convergence in the subspace $(z_1, z_2, \ldots, z_{r-1})$ defined by the following inequalities:

$$\|z_i\| \le \alpha_{i,r-1}\|z_{r-1}\| + \beta_{i,r-1}, \quad i = 1, \ldots, r-2.$$

These expressions are used to evaluate the derivative of the Lyapunov function candidate $V_{r-1} = (1/2)z_{r-1}^T z_{r-1}$ along the trajectories of (17c), that is

$$\begin{aligned} \dot{V}_{r-1} &= -k_{r-1}z_{r-1}^T z_{r-1} + z_{r-1}^T g_{r-1}(z_1, \ldots, z_{r-1}, t) \\ &\le -k_{r-1}\|z_{r-1}\|^2 + \|z_{r-1}\|\left(q_{r-1,r-1} + \sum_{j=1}^{r-2} k_j^{(r-1-j)} q_{r-1,j}\|z_j\| + d_{r-1}\right) \\ &\le -\left(k_{r-1} - q_{r-1,r-1} - \sum_{j=1}^{r-2} k_j^{(r-1-j)} q_{r-1,j}\alpha_{j,r-1}\right)\|z_{r-1}\|^2 \\ &\quad + \left(\sum_{j=1}^{r-2} k_j^{(r-1-j)} q_{r-1,j}\beta_{j,r-1} + d_{r-1}\right)\|z_{r-1}\| \end{aligned}$$

If k_{r-1} is chosen such that the condition (19d) for k_{r-1} presented as

$$k_{r-1} > q_{r-1,r-1} + \sum_{j=1}^{r-2} k_j^{(r-1-j)} q_{r-1,j}\alpha_{j,r-1}$$

holds, then we obtain

$$\dot{V}_{r-1} = -2\alpha_{r-1}V_{r-1} + \beta_{r-1}\sqrt{2V_{r-1}} \tag{D7}$$

with positive

$$\alpha_{r-1} = k_{r-1} - q_{r-1,r-1} - \sum_{j=1}^{r-2} k_j^{(r-1-j)} q_{r-1,j}\alpha_{j,r-1}$$

and

$$\beta_{r-1} = \sum_{j=1}^{r-2} k_j^{(r-1-j)} q_{r-1,j} \beta_{j,r-1} + d_{r-1}.$$

By the Comparison Lemma, we have

$$\|z_{r-1}(t)\| \leq \gamma_{r-1,r-1} \exp\left[\frac{1}{2}\alpha_{r-1}(t - t_0)\right] + h_{r-1} \tag{D8}$$

and thus

$$\limsup_{t\to\infty} \|z_{r-1}(t)\| \leq h_{r-1} \tag{D9}$$

where $\gamma_{r-1,r-1} = \|z_{r-1}(t_0)\| - h_{r-1}$, and $h_{r-1} = \beta_{r-1}/\alpha_{r-1}$.

Therefore, using the obtained upper (D8) and the ultimate (D9) bounds on the solution $z_{r-1}(t)$, and the inequalities (D3), and going back, from the $(r-1)$th block to the first block of (17a)–(17c), we can find step-by-step upper estimations and ultimate bounds on the solutions $z_{r-2}(t), z_{r-3}(t), \ldots, z_1(t)$.

At the next step, we use the inequality (D3) for $\dot{V}_{r-2}$ which takes the form

$$\begin{aligned} \dot{V}_{r-2} &\leq -\alpha_{r-2}\|z_{r-2}\|^2 + (\|z_{r-1}\| + \beta_{r-2})\|z_{r-2}\| \\ &\leq -2\alpha_{r-2}V_{r-2} + (\|z_{r-1}\| + \beta_{r-2})\sqrt{2V_{r-2}} \end{aligned} \tag{D10}$$

where the positive parameters α_{r-2} and β_{r-2} are calculated by using (D5), of the form

$$\alpha_{r-2} = \frac{1}{\alpha_{r-2,r-1}} = k_{r-2} - q_{r-2,r-2} - \sum_{j=1}^{r-3} k_j^{(r-2-j)} q_{r-2,j}\alpha_{j,r-2}$$

$$\beta_{r-2} = \frac{\beta_{r-2,r-1}}{\alpha_{r-2,r-1}} = \sum_{j=1}^{r-3} k_j^{(r-2-j)} q_{r-2,j}\beta_{j,r-2} + d_{r-2}.$$

Substituting (D8) in (D10) and applying again the Comparison Lemma, yields (20b), that is

$$\|z_{r-2}(t)\| \leq \gamma_{r-2,r-2} \exp\left[-\frac{1}{2}\alpha_{r-2}(t - t_0)\right] + \gamma_{r-2,r-1} \exp\left[-\frac{1}{2}\alpha_{r-1}(t - t_0)\right] + h_{r-2} \tag{D11}$$

and thus

$$\limsup_{t\to\infty} \|z_{r-2}(t)\| \leq h_{r-2} \tag{D12}$$

where

$$\gamma_{r-2,r-2} = \|z_{r-2}(t_0)\| - \gamma_{r-2,r-1} = h_{r-2}, \quad \gamma_{r-2,r-1} = \frac{\|z_{r-1}(t_0)\| - h_{r-1}}{\alpha_{r-1} - \alpha_{r-2}},$$

and

$$h_{r-2} = \frac{\beta_{r-2} + h_{r-1}}{\alpha_{r-2}} = \alpha_{r-2,r-1}h_{r-1} + \beta_{r-2,r-1}.$$

The obtained estimation (D11) and bound (D12), and the inequality (D3) are used at the next step to obtain the upper and ultimate bounds on the solution $z_{r-3}(t)$.

Proceeding in the same fashion, the inequality (D3) for the (i)th can be represented similar to (D10) of the form

$$\dot{V}_i \leq -\alpha_i \|z_i\|^2 + (\|z_{i+1}\| + \beta_i)\|z_i\| \leq -2\alpha_i V_i + (\|z_{i+1}\| + \beta_i)\sqrt{2V_i} \qquad \text{(D13)}$$

where

$$\alpha_i = \frac{1}{\alpha_{i,i+1}} = k_i - q_{i,i} - \sum_{j=1}^{i-1} k_j^{(i-1)} q_{i,j}$$

and

$$\beta_{i,i+1} = \frac{\beta_{i,i+1}}{\alpha_{i,i+1}} = \left(\sum_{j=1}^{i-1} k_j^{(i-j)} q_{i,j} \beta_{j,i} - d_i \right).$$

Then substituting the obtained on the previously step the estimation for the solution $z_{i+1}(t)$ of the form

$$\|z_{i+1}(t)\| \leq \sum_{j=i+1}^{r-1} \gamma_{i+1,j} \exp\left[-\frac{1}{2}\alpha_j (t - t_0) \right] + h_i$$

in (D13) and applying again the Comparison Method yields, the estimation for the solution $z_i(t)$ is derived of the form (20c), that is

$$\|z_i(t)\| \leq \sum_{j=i}^{r-1} \gamma_{i,j} \exp\left[-\frac{1}{2}\alpha_j (t - t_0) \right] + h_i, \quad i = r-3, r-4, \ldots, 2, 1$$

and thus

$$\limsup_{t\to\infty} \|z_i(t)\| \leq h_i, \quad i = r-3, r-4, \ldots, 2, 1. \qquad \blacksquare$$