Boundary Value Problems on the Half Line in the Theory of Colloids

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We present existence results for some boundary value problems defined on infinite intervals. In particular our discussion includes a problem which arises in the theory of colloids.

Key words: Boundary value problem, half line, colloids, existence

1 INTRODUCTION

In the theory of colloids [4, 7] it is possible to relate particle stability with the charge on the colloidal particle. We model the particle and its attendant electrical double layer using Poisson’s equation for a flat plate. If $\Psi$ is the potential, $\rho$ the charge density, $D$ the dielectric constant and $y$ the displacement, then we have

$$\frac{d^2 \Psi}{dy^2} = -\frac{4\pi \rho}{D}.$$ 

We assume the ions are point charged and their concentrations in the double layer satisfies the Boltzmann distribution

$$c_i = c_i^* \exp\left(\frac{-z_i e \Psi}{\kappa T}\right),$$

where $c_i$ is the concentration of ions of type $i$, $c_i^* = \lim_{\psi \to 0} c_i$, $\kappa$ the Boltzmann constant, $T$ the absolute temperature, $e$ the electrical charge, and $z$ the valency of the ion. In the neutral case, we have

$$\rho = c_+ z_+ e + c_- z_+ e \quad \text{or} \quad \rho = ze(c_+ - c_-)$$

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where \( z = z_+ - z_- \). Then we have using

\[
c_+ = c \exp\left( \frac{-ze\Psi}{\kappa T} \right) \quad \text{and} \quad c_- = c \exp\left( \frac{ze\Psi}{\kappa T} \right),
\]

that

\[
\frac{d^2\Psi}{dy^2} = \frac{8\pi cze}{D} \sinh\left( \frac{ze\Psi}{\kappa T} \right)
\]

where the potential initially takes some positive value \( \Psi(0) = \Psi_0 \) and tends to zero as the distance from the plate increases i.e. \( \Psi(\infty) = 0 \). Using the transformation

\[
\phi(y) = \frac{ze\Psi(y)}{\kappa T} \quad \text{and} \quad x = \sqrt{\frac{4\pi cze^2}{\kappa TD}} y,
\]

the problem becomes

\[
\begin{align*}
\frac{d^2\phi}{dx^2} &= 2 \sinh \phi, \quad 0 < x < \infty \\
\phi(0) &= c_1 \\
\lim_{x \to \infty} \phi(x) &= 0,
\end{align*}
\]

where \( c_1 = ze\Psi_0 / \kappa T > 0 \). From a physical point of view we wish the solution \( \phi \) in (1.1) to also satisfy \( \lim_{x \to \infty} \phi'(x) = 0 \).

In this paper using the notion of upper and lower solutions (see [1, 2, 6]) we establish general existence results which guarantee the existence of \( BC(0, \infty) \) solutions to

\[
\begin{align*}
\frac{1}{p(t)} (p(t)y'(t))' &= q(t)f(t, y(t)), \quad 0 < t < \infty \\
-a_0y(0) + b_0 \lim_{t \to 0^+} p(t)y'(t) &= c_0, \quad a_0 > 0, \quad b_0 \geq 0 \\
\lim_{t \to \infty} y(t) &= 0;
\end{align*}
\]

where \( BC(0, \infty) \) denotes the space of continuous, bounded functions from \( [0, \infty) \) to \( \mathbb{R} \). Our theory not only complements some of the known results, e.g., [5, 8], but also automatically produces the existence of a solution to (1.1). To establish these results we recall, for the convenience of the reader, the existence principle [3] we will use in Section 2. Consider the boundary value problem

\[
\begin{align*}
\frac{1}{p} (py')' &= qf(t, y), \quad 0 < t < \infty \\
-a_0y(0) + b_0 \lim_{t \to 0^+} p(t)y'(t) &= c_0, \quad a_0 > 0, \quad b_0 \geq 0 \\
y(t) \quad \text{bounded on} \ [0, \infty).
\end{align*}
\]
By an upper solution \( \beta \) to (1.3) we mean a function \( \beta \in BC[0, \infty) \cap C^2(0, \infty), p\beta' \in C[0, \infty) \) with
\[
\begin{cases}
\frac{1}{p}(p\beta')' \leq qf(t, \beta), & 0 < t < \infty \\
-a_0\beta(0) + b_0 \lim_{t \to 0^+} p(t)\beta'(t) \leq c_0, \\
\beta(t) \text{ bounded on } [0, \infty)
\end{cases}
\] (1.4)
and by a lower solution \( \alpha \) to (1.3) we mean a function \( \alpha \in BC[0, \infty) \cap C^2(0, \infty), p\alpha' \in C[0, \infty) \) with
\[
\begin{cases}
\frac{1}{p}(p\alpha')' \geq qf(t, \alpha), & 0 < t < \infty \\
-a_0\alpha(0) + b_0 \lim_{t \to 0^+} p(t)\alpha'(t) \geq c_0, \\
\alpha(t) \text{ bounded on } [0, \infty)
\end{cases}
\] (1.5)

**Theorem 1.1** [3] Let \( f: [0, \infty) \times \mathbb{R} \to \mathbb{R} \) be continuous. Suppose the following conditions are satisfied:

\[ q \in C(0, \infty) \text{ with } q > 0 \text{ on } (0, \infty) \] (1.6)

\[ p \in C[0, \infty) \cap C^1(0, \infty) \text{ with } p > 0 \text{ on } (0, \infty) \] (1.7)

\[ \int_0^\mu \frac{ds}{p(s)} < \infty \text{ and } \int_0^\mu p(s)q(s) \, ds < \infty \text{ for any } \mu > 0 \] (1.8)

\[ \{ \text{there exists } \alpha, \beta \text{ respectively lower and upper solutions of (1.3) with } \alpha(t) \leq \beta(t) \text{ for } t \in [0, \infty) \} \] (1.9)

and

\[ \{ \text{there exists a constant } M > 0 \text{ with } |f(t, u)| \leq M \text{ for } t \in [0, \infty) \text{ and } u \in [\alpha(t), \beta(t)]. \} \] (1.10)

Then (1.3) has a solution \( y \in BC[0, \infty) \cap C^2(0, \infty), py' \in C[0, \infty) \) with \( \alpha(t) \leq y(t) \leq \beta(t) \) for \( t \in [0, \infty) \). Also there exist constants \( A_0 \) and \( A_1 \) with \( |p(t)y'(t)| \leq A_0 + A_1 \int_0^t p(s)q(s) \, ds \) for \( t \in (0, \infty) \).

## 2 THE BOUNDARY CONDITION AT INFINITY

Motivated by the colloid example [4, 7] we discuss the boundary value problem
\[
\begin{cases}
\frac{1}{p}(py')' = qf(t, y), & 0 < t < \infty \\
-a_0 y(0) + b_0 \lim_{t \to 0^+} p(t)y'(t) = c_0, & a_0 > 0, \quad b_0 \geq 0, \quad c_0 \leq 0 \\
\lim_{t \to \infty} y(t) = 0.
\end{cases}
\] (2.1)
\textbf{Theorem 2.1} \textit{Let } f : [0, \infty) \times \mathbb{R} \to \mathbb{R} \textit{ be continuous and suppose the following conditions hold:}

\begin{equation}
q \in C(0, \infty) \text{ with } q > 0 \text{ on } (0, \infty) \tag{2.2}
\end{equation}

\begin{equation}
p \in C[0, \infty) \cap C^1(0, \infty) \text{ with } p > 0 \text{ on } (0, \infty) \text{ and } \int_0^\infty \frac{ds}{p(s)} = \infty \tag{2.3}
\end{equation}

\begin{equation}
\int_0^\mu \frac{ds}{p(s)} < \infty \text{ and } \int_0^\mu p(s)q(s)\,ds < \infty \text{ for any } \mu > 0 \tag{2.4}
\end{equation}

\begin{equation}
f(t, 0) \leq 0 \text{ for } t \in (0, \infty) \tag{2.5}
\end{equation}

\begin{equation}
\exists \ r_0 \geq -\frac{c_0}{a_0} \text{ with } f(t, r_0) \geq 0 \text{ for } t \in (0, \infty) \tag{2.6}
\end{equation}

\begin{equation}
\exists \ M > 0 \text{ with } |f(t, u)| \leq M \text{ for } t \in [0, \infty) \text{ and } u \in [0, r_0] \tag{2.7}
\end{equation}

\begin{equation}
\left\{ \begin{array}{l}
\exists \text{ a constant } m > 0 \text{ with } q(t)p^2(t)[f(t, u) - f(t, 0)] \geq m^2u \\
\text{for } t \in (0, \infty) \text{ and } u \in [0, r_0]
\end{array} \right. \tag{2.8}
\end{equation}

\begin{equation}
\int_0^\infty p(x)\exp\left(-m\int_0^x \frac{dx}{p(s)}\right)q(x)|f(x, 0)|\,dx < \infty \tag{2.9}
\end{equation}

\begin{equation}
\lim_{t \to \infty} p^2(t)q(t)f(t, 0) = 0 \tag{2.10}
\end{equation}

\textit{and}

\begin{equation}
\left\{ \begin{array}{l}
\lim_{t \to \infty} \left( B_0 \int_\mu^t \frac{1}{p(s)} \int_\mu^s \frac{1}{p(x)} \,dx \,ds + C_0 \int_\mu^t \frac{ds}{p(s)} \right) = \infty \tag{2.11}
\end{array} \right.
\end{equation}

\textit{for any constants } B_0 > 0, C_0 \in \mathbb{R} \textit{ and } \mu > 0.

Then (2.1) has a solution } y \in C[0, \infty) \cap C^2(0, \infty) \textit{ with } py' \in C[0, \infty) \textit{ and } 0 \leq y(t) \leq r_0 \textit{ for } t \in [0, \infty).

\textbf{Proof} \hspace{1em} \textit{Now Theorem 1.1 (with } \alpha = 0 \textit{ and } \beta = r_0 \textit{) guarantees that}

\begin{equation}
\left\{ \begin{array}{l}
\frac{1}{p}(py')' = q(t)f(t, y), \quad 0 < t < \infty \\
-a_0y(0) + b_0 \lim_{t \to 0^+} p(t)y'(t) = c_0 \\
y(t) \text{ bounded on } [0, \infty)
\end{array} \right. \tag{2.12}
\end{equation}
has a solution \( y \in C[0, \infty) \cap C^2(0, \infty), py' \in C[0, \infty) \) and \( 0 \leq y(t) \leq r_0 \) for \( t \in [0, \infty) \). Let \( g(x) = q(x)f(x, 0) \) and notice that

\[
w(t) = \exp \left( -m \int_0^t \frac{ds}{p(s)} \left[ \frac{(-c_0)}{a_0 + b_0m} \right. \right.
\left. + \frac{(a_0 - b_0m)}{2m(a_0 + b_0m)} \int_0^\infty \frac{dx}{p(x)} \exp \left( -m \int_0^\infty \frac{ds}{p(s)} g(x, dx) \right) \right] \right.
\left. - \frac{1}{2m} \exp \left( m \int_0^t \frac{ds}{p(s)} \right) \int_0^\infty \frac{p(x)}{p(s)} \exp \left( -m \int_0^\infty \frac{ds}{p(s)} g(x, dx) \right) \right.
\left. - \frac{1}{2m} \exp \left( \int_0^\infty \frac{p(x)}{p(s)} \right) \int_0^\infty \frac{p(x)}{p(s)} \exp \left( m \int_0^\infty \frac{ds}{p(s)} g(x, dx) \right) \right.
\left. = \exp \left( -m \int_0^t \frac{ds}{p(s)} \left[ \frac{(-c_0)}{a_0 + b_0m} \right. \right.
\left. - \frac{b_0}{a_0 + b_0m} \int_0^\infty \frac{p(x)}{p(s)} \exp \left( -m \int_0^\infty \frac{ds}{p(s)} g(x, dx) \right) \right] \right.
\left. - \int_0^t \frac{1}{p(s)} \exp \left( -m \int_0^{\infty} \frac{p(x)}{p(s)} \right) \left( \int_0^\infty \frac{p(x)}{p(s)} \exp \left( -m \int_0^\infty \frac{ds}{p(s)} g(x, dx) \right) \right) d\zeta \right)
\]

is a nonnegative solution of

\[
\begin{cases}
\frac{1}{p} \left( \frac{p}{p'(t)} \right)' = w(t), & 0 < t < \infty \\
-aw(0) + b_0 \lim_{t \to 0^+} p(t)w'(t) = c_0 \\
\lim_{t \to \infty} w(t) = 0.
\end{cases}
\tag{2.13}
\]

Notice (2.10) and l'Hôpital's rule guarantees that \( w(\infty) = 0 \).

Now let

\( r(t) = y(t) - w(t) \).

We first show \( r \) cannot have a local positive maximum on \([0, \infty)\). Suppose \( r \) has a local positive maximum at \( t_0 \in [0, \infty) \).

**Case (i)  \( t_0 \in [0, \infty) \).**

For \( t > 0 \) notice from assumption (2.8) that

\[
\frac{1}{p} \left( \frac{p}{p'(t)} \right)'(t) = q(t)[f(t, y(t)) - f(t, 0)] - \frac{m^2}{p^2(t)} w(t) \geq \frac{m^2}{p^2(t)} [y(t) - w(t)].
\tag{2.14}
\]

We also have \( r'(t_0) = 0 \) and \( r''(t_0) < 0 \). However (2.14) yields

\[
r''(t_0) = \frac{1}{p(t_0)} \left( \frac{p}{p'(t)} \right)'(t_0) \geq \frac{m^2}{p^2(t_0)} [y(t_0) - w(t_0)] > 0,
\]

a contradiction.

**Case (ii)  \( t_0 = 0 \).**

Of course if \( b_0 = 0 \) we have a contradiction immediately. So suppose \( b_0 \neq 0 \). Then

\[
\lim_{t \to 0^+} p(t)r'(t) = \frac{a_0}{b_0} [y(0) - w(0)].
\tag{2.15}
\]
Now since \( y(0) - w(0) > 0 \) there exists \( \delta > 0 \) with \( y(t) - w(t) > 0 \) for \( t \in (0, \delta) \). Then (2.14) implies \((pr')' > 0\) on \((0, \delta)\) and this together with (2.15) (i.e. \( \lim_{t \to 0^+} p(t)'(t) > 0 \)) implies \( pr' > 0 \) on \((0, \delta)\), a contradiction.

Thus \( r(t) \) cannot have a local positive maximum on \([0, \delta)\). We now claim that \( r(t) \leq 0 \) on \([0, \infty)\). If \( r(t) \not\leq 0 \) on \([0, \infty)\) then there exists a \( c_1 > 0 \) with \( r(c_1) > 0 \). Now since \( r(t) \) cannot have a positive local maximum on \([0, \infty)\) it follows that \( r(t_2) > r(t_1) \) for all \( t_2 > t_1 \geq c_1 \); otherwise \( r(t) \) would have a local positive maximum on \([0, t_2)\). Thus \( r(t) \) is strictly increasing for \( t \geq c_1 \). Since both \( y(t) \) and \( w(t) \) are bounded on \([0, \infty)\) and \( \lim_{t \to \infty} w(t) = 0 \) then

\[
\lim_{t \to \infty} y(t) = \lim_{t \to \infty} [y(t) - w(t)] = \kappa \in (0, r_0].
\]

(2.16)

Now there exists \( c_2 \geq c_1 \) with \( y(t) \geq \kappa/2 \) for \( t \geq c_2 \). The differential equation and (2.8) imply that for \( t > 0 \) that we have

\[
(p(t)y'(t))' = p(t)q(t)f(t, y(t)) = p(t)q(t)[f(t, y(t)) - f(t, 0)] + p(t)q(t)f(t, 0)
\geq \frac{m^2}{p(t)} y(t) + p(t)q(t)f(t, 0).
\]

Consequently for \( t \geq c_2 \) we have

\[
(p(t)y'(t))' \geq \frac{m^2 \kappa}{2p(t)} + p(t)q(t)f(t, 0) = \frac{1}{p(t)} \left[ \frac{m^2 \kappa}{2} + p^2(t)q(t)f(t, 0) \right].
\]

Assumption (2.10) implies that there is a constant \( c_3 \geq c_2 \) with

\[
(p(t)y'(t))' \geq \frac{m^2 \kappa}{4p(t)} \quad \text{for } t \geq c_3.
\]

Two integrations together with the fact that \( y \geq 0 \) on \([0, \infty)\) yields

\[
y(t) \geq p(c_3)y'(c_3) \int_{c_3}^{t} \frac{dx}{p(x)} + \frac{m^2 \kappa}{4} \int_{c_3}^{t} \int_{c_3}^{s} \frac{1}{p(s)} ds \frac{dx}{p(x)} ds
\]

(not also from Theorem 1.1 that there exist constants \( A_0 \) and \( A_1 \) with \(|p(t)y'(t)| \leq A_0 + A_1 \int_{0}^{t} p(s)q(s) ds \) \( t \in (0, \infty) \)). Now assumption (2.11) implies that \( y \) is unbounded on \([0, \infty)\), a contradiction. Thus \( r(t) \leq 0 \) on \([0, \infty)\) and the result follows. \( \blacksquare \)

Notice in Theorem 3.1 that the solution \( y \) of (2.1) satisfies \( r(t) \leq 0 \) for \( t \in [0, \infty) \), and so \( y(t) \leq w(t) \) for \( t \in [0, \infty) \).

**Corollary 2.2** Let \( f: [0, \infty) \times R \to R \) be continuous and suppose (2.2)–(2.11) hold. Then (2.1) has a solution \( y \in C[0, \infty) \cap C^2(0, \infty) \) with \( py' \in C[0, \infty) \) and \( 0 \leq y(t) \leq w(t) \) for \( t \in [0, \infty) \), with \( w \) given in Theorem 2.1.

The colloid [4, 7] example motivates our next result.

**Theorem 2.3** Let \( f: [0, \infty) \times R \to R \) be continuous and suppose (2.2)–(2.11) hold. In addition assume the following conditions hold:

\[
f(t, u) \geq 0 \quad \text{for } t \in [0, \infty) \quad \text{and} \quad u \in [0, w(t)]; \quad \text{here } w \text{ is as in Theorem 2.1}
\]

(2.17)
and
\[ \lim_{t \to \infty} p(t) \in (0, \infty). \]  
(2.18)

Then (2.1) has a solution \( y \in C[0, \infty) \cap C^2(0, \infty) \) with \( py' \in C[0, \infty), 0 \leq y(t) \leq w(t) \) for \( t \in [0, \infty) \) and \( \lim_{t \to \infty} y'(t) = 0 \).

**Proof** From Corollary 2.2 we know that there exists a solution \( y \in C[0, \infty) \cap C^2(0, \infty), py' \in C[0, \infty) \) and \( 0 \leq y(t) \leq w(t) \) for \( t \in [0, \infty) \), to (2.1). Also (2.17) and the differential equation yields
\[ (py')(t) = p(t)q(t)f(t, y(t)) \geq 0 \quad \text{for } t > 0, \]  
(2.19)
so \( py' \) is nondecreasing on \((0, \infty)\), and \( \lim_{t \to \infty} p(t)y'(t) \in [-\infty, \infty] \).

Suppose there exists \( t_1 \in [0, \infty) \) with \( p(t_1)y'(t_1) > 0 \). Then
\[ p(t)y'(t) \geq a_0 \equiv p(t_1)y'(t_1) \quad \text{for } t \geq t_1, \]
and so
\[ y(t) \geq y(t_1) + a_0 \int_{t_1}^{t} \frac{ds}{p(s)} \quad \text{for } t \geq t_1. \]  
(2.20)
That is
\[ y(t) \geq a_0 \int_{t_1}^{t} \frac{ds}{p(s)} \quad \text{for } t \geq t_1 \]  
(2.21)
(notice (2.3) implies that the right hand side of (2.21) goes to \( \infty \) as \( t \to \infty \)). This contradicts \( 0 \leq y(t) \leq r_0 \) for \( t \in [0, \infty) \). Thus \( p(t)y'(t) \leq 0 \) for \( t \in (0, \infty) \), and so
\[ \lim_{t \to \infty} p(t)y'(t) = \kappa \in [-\infty, 0] \quad \text{and} \quad \lim_{t \to \infty} y'(t) \in [-\infty, 0]. \]  
(2.22)
In fact \( \kappa \in (-\infty, 0] \) from (2.19). Finally if \( \kappa < 0 \) then there exists \( t_2 > 0 \) with \( p(t)y'(t) \leq \kappa/2 \) for \( t \geq t_2 \). Integrate from \( t_2 \) to \( t \) \((t \geq t_2)\) to get
\[ y(t) \leq y(t_2) + \frac{\kappa}{2} \int_{t_2}^{t} \frac{ds}{p(s)} \leq r_0 + \frac{\kappa}{2} \int_{t_2}^{t} \frac{ds}{p(s)}. \]  
(2.23)
Now (2.23) together with (2.3) contradicts \( y \geq 0 \) on \([0, \infty)\). Consequently \( \lim_{t \to \infty} p(t)y'(t) = 0 \), and this together with (2.18) gives \( \lim_{t \to \infty} y'(t) = \lim_{t \to \infty} p(t)y'(t)/p(t) = 0 \).

**Example 2.1** (Colloid problem [4, 7]).

The boundary value problem
\[
\begin{aligned}
&y'' = 2 \sinh y, \quad 0 < t < \infty \\
&y(0) = c > 0 \\
&\lim_{t \to \infty} y(t) = 0 
\end{aligned}
\]  
(2.24)
has a solution $y \in C[0, \infty) \cap C^2(0, \infty)$ with

$$0 \leq y(t) \leq ce^{-t} \quad \text{for } t \in [0, \infty).$$

To see this we will apply Corollary 2.2 with

$$p = 1, \quad q = 1, \quad a_0 = 1, \quad c_0 = -c, \quad b_0 = 0 \quad \text{and} \quad r_0 = c.$$

Clearly (2.1)–(2.7), (2.8) since $f(t, u) - f(t, 0) = \sinh u \geq u$ for $u \geq 0$, (2.9)–(2.11) hold. Corollary 2.2 guarantees that (2.24) has a solution $y \in C[0, \infty) \cap C^2(0, \infty)$ with $0 \leq y(t) \leq w(t)$ for $t \in [0, \infty)$. It is immediate from (2.13) (since $g = 0$) that

$$w(t) = ce^{-t} \quad \text{for } t \in [0, \infty).$$

Finally we remark that the solution $y$ satisfies $\lim_{t \to \infty} y'(t) = 0$. To see this we need only check that (2.17)–(2.18) hold, but these are immediate.

References


