ESCAPE TIME FROM POTENTIAL WELLS OF STRONGLY NONLINEAR OSCILLATORS WITH SLOWLY VARYING PARAMETERS

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The effect of negative damping to an oscillatory system is to force the amplitude to increase gradually and the motion will be out of the potential well of the oscillatory system eventually. In order to deduce the escape time from the potential well of quadratic or cubic nonlinear oscillator, the multiple scales method is firstly used to obtain the asymptotic solutions of strongly nonlinear oscillators with slowly varying parameters, and secondly the character of modulus of Jacobian elliptic function is applied to derive the equations governing the escape time. The approximate potential method, instead of Taylor series expansion, is used to approximate the potential of an oscillation system such that the asymptotic solution can be expressed in terms of Jacobian elliptic function. Numerical examples verify the efficiency of the present method.

1. Introduction

The effect of negative damping or external excitation to an oscillatory system may force the amplitude to increase gradually and the motion will eventually be out of the potential well of the oscillatory system. Escape from a potential well is a ubiquitous phenomenon in science. Examples are known in mechanics [16], chemistry [7], physics [2, 10], electronics [9], and so forth. Many efforts have been done to control a nonlinear dynamical system against escape from a potential well, which is often identified with system failure [5, 8, 17]. Bosley and Kevorkian derived an easily evaluated condition to predict the distance at which the electron escapes from the ponderomotive potential well [1]. An important parameter to characterize the dynamics of system is escape time. Coffey et al. obtained the escape time for rigid Brownian rotators [6]. Kevorkian and Li calculated the escape time as an application example of the Kuzmak-Luke method [12] and then Li generalized it to cubic nonlinear oscillator with slowly varying parameters [15]. Only the case of symmetric oscillation with a stationary oscillation center was studied there. This paper will develop the procedure in [12, 15] to drive the escape time from potential wells of quadratic and cubic nonlinear oscillators. Symmetric and asymmetric oscillations with stationary or varying oscillation centers are studied in detail.

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We begin with the following strongly nonlinear oscillator with slowly varying parameters:

$$\frac{d^2y}{dt^2}+\varepsilon k(y,\tilde{t})\frac{dy}{dt}+g(y,\tilde{t})=0, \tag{1.1}$$

where $\tilde{t}=\varepsilon t$ is the slow scale. Assume that functions k and g are arbitrary nonlinear functions of their arguments and (1.1) has periodic solutions when $\varepsilon=0$. Firstly, the multiple scales method is applied to obtain the asymptotic solutions of (1.1). Secondly, the character of modulus of Jacobian elliptic function is used to deduce the escape time from the potential well of quadratic or cubic nonlinear oscillator. Different signs of coefficients of quadratic and cubic polynomials will deduce different equations to determine the escape time. For a strongly nonlinear spring in Example 4.3, the approximate potential method, proposed by the authors in [4, 14], is applied to approximate the potential of oscillatory system such that the asymptotic solution can be expressed in terms of Jacobian elliptic function. Three examples are given to illustrate the main idea of this paper. Comparisons of asymptotic and numerical results are also made to show the efficiency of present method.

2. Asymptotic solution of strongly nonlinear oscillator

We assume that the solution of (1.1) can be developed in the form

$$y(t,\varepsilon)=y_0(t^+,\tilde{t})+\varepsilon y_1(t^+,\tilde{t})+\varepsilon^2y_2(t^+,\tilde{t})+\cdots, \tag{2.1}$$

where $\tilde{t}=\varepsilon t$ is the slow scale. The fast scale t^+ must be chosen such that the period is independent of $\tilde{t}$ when measured on the t^+ scale (see [11, Section 3.6] for more details). Following Kuzmak [13], the fast scale t^+ is defined as $dt^+/dt=\omega(\tilde{t})$ with an unknown $\omega(\tilde{t})$ to be determined by the periodicity of solution of (1.1). Substituting (2.1) into (1.1) and equating coefficients of like power of ε yield the following equations:

$$\omega^2(\tilde{t})\frac{\partial^2y_0}{\partial t^{+2}}+g(y_0,\tilde{t})=0, \tag{2.2}$$

$$\omega^2(\tilde{t})\frac{\partial^2y_n}{\partial t^{+2}}+g_y(y_0,\tilde{t})y_n=F_n(y_0,y_1,\ldots,y_{n-1},\tilde{t}), \tag{2.3}$$

where $n=1,2,\ldots$. F_1 can be worked out in the form

$$F_1=-2\omega\frac{\partial^2y_0}{\partial t^+\partial\tilde{t}}-\frac{d\omega}{dt}\frac{\partial y_0}{\partial t^+}-\omega k(y_0,\tilde{t})\frac{\partial y_0}{\partial t^+}. \tag{2.4}$$

Note that there is a periodic solution to the homogeneous equation (2.3) in the form

$$y_{\mathrm{I}}=\frac{\partial y_0}{\partial\varphi}, \tag{2.5}$$

where $\varphi = t^+ + \varphi_0$, φ_0 is constant and is determined by initial conditions [14]. The other solution linearly independent of y_{I} can be found by the reduction of order

$$y_{\mathrm{II}} = y_{\mathrm{I}} \int_0^{\varphi} \frac{1}{y_{\mathrm{I}}^2} d\psi. \tag{2.6}$$

Unfortunately, the solution y_{II} is no longer periodic to general nonlinear system. Using variation of parameters, we obtain the general solution of the inhomogeneous equation (2.3) in the form

$$\begin{aligned} y_n &= C_n(\tilde{t})\, y_{\mathrm{I}} + D_n(\tilde{t})\, y_{\mathrm{II}} - \frac{y_{\mathrm{I}}}{\omega^2} \int_0^{\varphi} F_n y_{\mathrm{II}} d\psi + \frac{y_{\mathrm{II}}}{\omega^2} \int_0^{\varphi} F_n y_{\mathrm{I}}\, d\psi \\ &= y_{\mathrm{I}} \left[C_n(\tilde{t}) + \int_0^{\varphi} \frac{d\psi}{y_{\mathrm{I}}^2} \left(D_n(\tilde{t}) + \frac{1}{\omega^2} \int_0^{\psi} F_n y_{\mathrm{I}}\, d\gamma \right) \right], \end{aligned} \tag{2.7}$$

where coefficients C_n and D_n can be determined by the periodicity of higher-order solutions. To have y_n periodic in φ, the inner integral and the outer integral in (2.7) must be periodic in ψ and φ, respectively. We thus have, with the periodic normalized to be T,

$$\int_0^T F_n y_{\mathrm{I}}\, d\varphi = 0, \tag{2.8}$$

$$\int_0^T \frac{d\varphi}{y_{\mathrm{I}}^2} \left(D_n(\tilde{t}) + \frac{1}{\omega^2} \int_0^{\varphi} F_n y_{\mathrm{I}}\, d\psi \right) = 0. \tag{2.9}$$

This paper just concerns applications of leading-order approximations. For more details of higher-order solution, readers can refer to [12] or [14].

Substituting (2.4) and (2.5) into (2.8) with $n = 1$ yields

$$\int_0^T \left(2\omega f_{\varphi\tilde{t}} f_{\varphi} + \left(\frac{d\omega}{d\tilde{t}} + \omega k(f,\tilde{t}) \right) f_{\varphi}^2 \right) d\varphi = 0. \tag{2.10}$$

This can be written as

$$\frac{d}{d\tilde{t}} \left(\omega \int_0^T f_{\varphi}^2\, d\varphi \right) + \omega \int_0^T k(f,\tilde{t})\, f_{\varphi}^2\, d\varphi = 0. \tag{2.11}$$

Integrating (2.11) gives the following equation to determine $\omega(\tilde{t})$:

$$\omega(\tilde{t}) = \frac{c}{\int_0^T f_{\varphi}^2 d\varphi} \exp\left(\int_0^{\tilde{t}} \frac{\int_0^T k(f,\tau) f_{\varphi}^2 d\varphi}{\int_0^T f_{\varphi}^2 d\varphi}\, d\tau \right), \tag{2.12}$$

where c is an integral constant. If the damping k depends on y, the calculation of $\omega(\tilde{t})$ will be rather involved. An approach of average damping is proposed in [14]. Instead of k, we use the leading term of its Taylor series expansion around $f = y_r$, the oscillatory center, that is, we assume

$$k(y,\tilde{t}) = k(y_r,\tilde{t}) + k_y(y_r,\tilde{t})(y - y_r) + \frac{1}{2} k_{yy}(y_r,\tilde{t})(y - y_r)^2 + \cdots, \tag{2.13}$$

where y_r is the oscillatory center of system (1.1) and is determined by $g(y_r,\tilde{t}) = 0$. Because the system oscillates around the center y_r, the second term of (2.13) vanishes on average. Therefore, substitution of $k(y,\tilde{t}) \approx k(y_r,\tilde{t})$ into (2.12) should give a good approximation for $\omega(\tilde{t})$. The result is

$$\omega(\tilde{t}) = \frac{c}{\int_0^T f_\varphi^2 d\varphi} \exp\left(\int_0^{\tilde{t}} k(y_r,\tau)\,d\tau\right). \tag{2.14}$$

Numerical examples in Section 4 show that the results are quite satisfactory.

3. Calculations of escape time from potential well

3.1. Quadratic nonlinear oscillator. We now apply the results summarized in the previous section to the following quadratic nonlinear system:

$$\frac{d^2y}{dt^2} + \varepsilon k(y,\tilde{t})\frac{dy}{dt} + a(\tilde{t})y + b(\tilde{t})y^2 = 0. \tag{3.1}$$

Suppose that the solution of (3.1) can be developed in the form of asymptotic expression (2.1). The leading-order equation corresponding to (2.2) has the form

$$\omega^2(\tilde{t})\frac{\partial^2 y_0}{\partial t^{+2}} + a(\tilde{t})y_0 + b(\tilde{t})y_0^2 = 0. \tag{3.2}$$

Its energy integral is

$$\frac{\omega^2(\tilde{t})}{2}\left(\frac{\partial y_0}{\partial t^+}\right)^2 + V(y_0,a,b) = E_0(\tilde{t}), \tag{3.3}$$

where

$$V(y_0,a,b) = \frac{1}{2}a(\tilde{t})y_0^2 + \frac{1}{3}b(\tilde{t})y_0^3 \tag{3.4}$$

is the potential, and $E_0(\tilde{t})$ is the slowly varying energy of the system. It can be seen from (3.4) that the potential V has a minimum at $y_0 = 0$ for the case of $a(\tilde{t}) > 0$. So (3.1) has periodic solutions around $y_0 = 0$ and the oscillatory center is at $y_r = 0$. For the case of $a(\tilde{t}) < 0$, the potential V has a minimum at $y_0 = -a(\tilde{t})/b(\tilde{t})$. Equation (3.1) has periodic solutions around $y_0 = -a(\tilde{t})/b(\tilde{t})$ and the oscillatory center $y_r = -a(\tilde{t})/b(\tilde{t})$ is moving slowly with time.

3.1.1. Case of $a(\tilde{t}) > 0$. For this case, the solution of (3.2) can be expressed in terms of Jacobian elliptic cosine function [3]

$$y_0 = A_0(\tilde{t})cn^2[K(\nu)\varphi,\nu(\tilde{t})] + B_0(\tilde{t}), \tag{3.5}$$

where $\varphi = t^+ + \varphi_0$, and $K(\nu)$ is the complete elliptic integral of the first kind associated with the modulus $\sqrt{\nu}$. Substituting (3.5) into (3.2) yields

$$\begin{aligned} &2\omega^2K^2A_0(1-\nu)+aB_0+B_0^2+A_0\left[4\omega^2K^2(2\nu-1)+a+2bB_0\right]cn^2(u,\nu) \\ &\quad +A_0\left(bA_0-6\omega^2K^2\nu\right)cn^4(u,\nu)=0, \end{aligned} \tag{3.6}$$

where $u = K(\nu)\varphi$. From (3.6), we obtain algebraic equations

$$\begin{aligned} 2\omega^2K^2A_0(1-\nu)+aB_0+B_0^2&=0, \\ A_0\left[4\omega^2K^2(2\nu-1)+a+2bB_0\right]&=0, \\ A_0\left(bA_0-6\omega^2K^2\nu\right)&=0. \end{aligned} \tag{3.7}$$

Then, we have

$$A_0=\frac{3a\nu}{2b\sqrt{\nu^2-\nu+1}}, \tag{3.8}$$

$$B_0=-\frac{a}{2b}\left(\frac{2\nu-1}{\sqrt{\nu^2-\nu+1}}+1\right), \tag{3.9}$$

$$\omega^4=\frac{a^2}{16K^4\left(\nu^2-\nu+1\right)}. \tag{3.10}$$

Substituting (3.5) into (2.14), we get another form of $\omega(\tilde{t})$

$$\omega^5=\frac{cb^2}{144K^5\nu^2J(\nu)}\exp\left(-\int_0^{\tilde{t}}k(0,\tau)d\tau\right), \tag{3.11}$$

where

$$\begin{aligned} J(\nu)&=\int_0^K sn^2(u,\nu)cn^2(u,\nu)dn^2(u,\nu)du \\ &=\frac{1}{15\nu^2}\left[(1-\nu)(\nu-2)K(\nu)+2\left(\nu^2-\nu+1\right)E(\nu)\right]. \end{aligned} \tag{3.12}$$

From (3.10) and (3.11), we have an equation for ν

$$\frac{\nu^2J(\nu)}{\left(\nu^2-\nu+1\right)^{5/4}}=\frac{2cb^2}{9a^{5/2}}\exp\left(-\int_0^{\tilde{t}}k(0,\tau)d\tau\right), \tag{3.13}$$

where constant c can be determined by initial values of the system.

It can be seen from (3.4) that the potential well is "$\sim$-shaped" or "$\backsim$-shaped" when $a(\tilde{t}) > 0$. In this case, if there exists the effect of a negative damping in the system, the amplitude will be gradually increasing, and the motion will be forced out of the potential well and cease to be periodic. Obviously from (3.8), if a/b is constant, the increasing amplitude implies an increasing modulus ν. Once ν reaches 1, y_0 is no longer a periodic function of φ. This can be used to determine the escape time from potential well. Denote

the escape time as T_0, from (3.13) we have

$$\frac{cb^2(T_0)}{a^{5/2}(T_0)}\exp\left(-\int_0^{\varepsilon T_0} k(0,\tau)d\tau\right)=\frac{3}{5}. \tag{3.14}$$

Here, the fact that $J(\nu)\to 2/15$ as $\nu\to 1$ has been used.

3.1.2. Case of $a(\tilde{t})<0$ and $b(\tilde{t})>0$. For this case, the solution of (3.2) also can be expressed as (3.5) with

$$\begin{aligned} A_0 &= \frac{-3a\nu}{2b\sqrt{\nu^2-\nu+1}},\\ B_0 &= -\frac{a}{2b}\left(\frac{1-2\nu}{\sqrt{\nu^2-\nu+1}}+1\right),\\ \omega^4 &= \frac{a^2}{16K^4(\nu^2-\nu+1)}. \end{aligned} \tag{3.15}$$

Substituting (3.5) into (2.14), we get another form of $\omega(\tilde{t})$

$$\omega^5=\frac{cb^2}{144K^5\nu^2J(\nu)}\exp\left(-\int_0^{\tilde{t}} k\left(-\frac{a(\tau)}{b(\tau)},\tau\right)d\tau\right). \tag{3.16}$$

Then the equation governing ν becomes

$$\frac{\nu^2J(\nu)}{(\nu^2-\nu+1)^{5/4}}=\frac{2cb^2}{9(-a)^{5/2}}\exp\left(-\int_0^{\tilde{t}} k\left(-\frac{a(\tau)}{b(\tau)},\tau\right)d\tau\right). \tag{3.17}$$

Now the escape time T_0 can be determined by

$$\frac{cb^2(T_0)}{(-a)^{5/2}(T_0)}\exp\left(-\int_0^{\varepsilon T_0} k\left(-\frac{a(\tau)}{b(\tau)},\tau\right)d\tau\right)=\frac{3}{5}. \tag{3.18}$$

3.2. Cubic nonlinear oscillator. Consider the following cubic nonlinear oscillator:

$$\frac{d^2y}{dt^2}+\varepsilon k_1(y,\tilde{t})\frac{dy}{dt}+a_1(\tilde{t})y+b_1(\tilde{t})y^3=0. \tag{3.19}$$

The potential corresponding to (3.4) is

$$V(y_0,a_1,b_1)=\frac{1}{2}a_1(\tilde{t})y_0^2+\frac{1}{4}b_1(\tilde{t})y_0^4. \tag{3.20}$$

It can be seen from (3.20) that the potential V has a minimum at $y_0=0$ for the case of $a_1(\tilde{t})>0$ and the system oscillates around the center $y_r=0$. For the case of $a_1(\tilde{t})<0$ and $b_1(\tilde{t})>0$, the potential V has two minimums at $y_0=\pm\sqrt{-a_1(\tilde{t})/b_1(\tilde{t})}$. The system has two families of oscillations centered about $y_r=\pm\sqrt{-a_1(\tilde{t})/b_1(\tilde{t})}$, which are moving slowly with time.

3.2.1. Case of $a_1(\tilde{t}) > 0$ and $b_1(\tilde{t}) < 0$. Similar to the quadratic nonlinear oscillator, we can get approximate solution of leading order

$$y_0 = \sqrt{\frac{-2a_1\nu}{b_1(1+\nu)}}\, sn(K\varphi,\nu), \tag{3.21}$$

$$\omega(\tilde{t}) = \frac{-c_1 b_1(1+\nu)}{2a_1\nu K(\nu)L(\nu)} \exp\left(-\int_0^{\tilde{t}} k_1(0,\tau)d\tau\right), \tag{3.22}$$

where

$$L(\nu) = \int_0^K cn^2(u,\nu)dn^2(u,\nu)du = \frac{1}{3\nu}\left[(1+\nu)E(\nu) - (1-\nu)K(\nu)\right]. \tag{3.23}$$

The equation governing ν becomes

$$\frac{\nu^2 L^2(\nu)}{(1+\nu)^3} = \frac{c_1^2 b_1^2}{4a_1^3} \exp\left(-2\int_0^{\tilde{t}} k_1(0,\tau)d\tau\right), \tag{3.24}$$

where constant c_1 can be determined by the initial values of the system.

The escape time T_0 can be solved from (3.24):

$$\frac{c_1^2 b_1^2(T_0)}{a_1^3(T_0)} \exp\left(-2\int_0^{\varepsilon T_0} k_1(0,\tau)d\tau\right) = \frac{2}{9}. \tag{3.25}$$

Here, the fact that $L(\nu) \to 2/3$ as $\nu \to 1$ has been used. For more details of this subsection, readers can refer to [15].

3.2.2. Case of $a_1(\tilde{t}) < 0$ and $b_1(\tilde{t}) > 0$. Here we are concerned only with the oscillation around the right-hand side center. Similar to the above sections, we can get the asymptotic solution of leading order

$$\begin{aligned} y_0 &= \sqrt{\frac{2a_1}{b_1(\nu-2)}}\, dn(K\varphi,\nu), \\ \omega(\tilde{t}) &= \frac{c_1 b_1(\nu-2)}{2a_1\nu^2 K(\nu)M(\nu)} \exp\left(-\int_0^{\tilde{t}} k_1\left(\sqrt{-\frac{a(\tau)}{b(\tau)}},\tau\right)d\tau\right), \end{aligned} \tag{3.26}$$

where

$$M(\nu) = \int_0^K sn^2(u,\nu)cn^2(u,\nu)du = \frac{1}{3\nu^2}\left[(2-\nu)E(\nu) - 2(1-\nu)K(\nu)\right]. \tag{3.27}$$

The equation governing ν becomes

$$\frac{\nu^4 M^2(\nu)}{(\nu-2)^3} = \frac{c_1^2 b_1^2}{4a_1^3} \exp\left(-2\int_0^{\tilde{t}} k_1\left(\sqrt{-\frac{a(\tau)}{b(\tau)}},\tau\right)d\tau\right). \tag{3.28}$$

The escape time T_0 can be determined by (3.28):

$$\frac{c_1^2 b_1^2(T_0)}{a_1^3(T_0)} \exp\left(-2\int_0^{\varepsilon T_0} k_1\left(\sqrt{-\frac{a_1(\tau)}{b_1(\tau)}}, \tau\right) d\tau\right) = -\frac{4}{9}. \tag{3.29}$$

Here, the fact that $M(v) \to 1/3$ as $v \to 1$ has been used.

4. Examples

Example 4.1. Consider the following quadratic nonlinear oscillator:

$$\frac{d^2 y}{dt^2} - \varepsilon\left(\frac{1}{1+\varepsilon t} - y^3\right)\frac{dy}{dt} + (1+\varepsilon t)^2 y - (1+\varepsilon t)^{5/2} y^2 = 0, \tag{4.1}$$

$$y(0) = 0.5, \qquad \dot{y}(0) = 0. \tag{4.2}$$

From initial conditions we can obtain $\varphi_0 = 1$, $v(0) = 0.5$, and $c = 0.27312$. The escape time T_0 can be solved from (3.14):

$$T_0 = \frac{1}{\varepsilon}\left(\frac{3}{5c} - 1\right) = \frac{1.197}{\varepsilon}. \tag{4.3}$$

Comparisons of asymptotic results from (4.3) and numerical results are shown in Table 4.1. In this paper, numerical results are obtained by software *Mathematica.*

Example 4.2. Consider the following quadratic nonlinear oscillator:

$$\frac{d^2 y}{dt^2} - \varepsilon(3 + \varepsilon t + y - y^2)\frac{dy}{dt} - (1+\varepsilon t)^2 y + (1+\varepsilon t) y^2 = 0, \tag{4.4}$$

$$y(0) = 0.1, \qquad \dot{y}(0) = 0. \tag{4.5}$$

From initial conditions we can obtain $\varphi_0 = 1$, $v(0) = 0.878$, and $c = 0.57514$. The escape time T_0 can be solved from (3.18):

$$T_0 = \frac{0.1819}{\varepsilon}. \tag{4.6}$$

Comparisons of asymptotic results from (4.6) and numerical results are shown in Table 4.2.

Example 4.3. Consider the strongly nonlinear oscillator

$$\frac{d^2 y}{dt^2} - \varepsilon\frac{1}{1+\varepsilon t}\frac{dy}{dt} + (1+\varepsilon t)\sin y - \frac{2}{\pi}(1+\varepsilon t) y = 0, \tag{4.7}$$

$$y(0) = \frac{\pi}{3}, \qquad \dot{y}(0) = 0. \tag{4.8}$$

The leading-order equation corresponding to (2.2) has the form

$$\omega^2(\tilde{t})\frac{\partial^2 y_0}{\partial t^{+2}} + (1+\varepsilon t)\sin y_0 - \frac{2}{\pi}(1+\varepsilon t) y_0 = 0. \tag{4.9}$$

Table 4.1. Escape time from potential well of system (4.1).

ε	Asymptotic result	Numerical result	Error (%)
0.1	11.97	14.83	19
0.01	119.7	124.4	3.8
0.001	1197	1236	3.2
0.0001	11 970	12 301	2.7

Its energy integral is

$$\frac{\omega^2(\tilde{t})}{2}\left(\frac{\partial y_0}{\partial t^+}\right)^2 + V(y_0) = E_0(\tilde{t}), \tag{4.10}$$

where

$$V(y_0) = -(1+\varepsilon t)\cos y_0 - \frac{1}{\pi}(1+\varepsilon t)y_0^2 + 1 + \varepsilon t \tag{4.11}$$

is the potential. With this potential, the integral of (4.10) cannot be expressed in terms of any elemental or known functions. Approximate approaches must be used. The approximate potential method was first proposed in [14] to deal with a strongly nonlinear oscillator resulting from the free-electron laser (FEL), where the potential was approximated by a polynomial of degree three. Note that the potential $V(y_0)$ has a minimum point at $y_0 = 0$ and two maximum points at $y = \pm\pi/2$. We may seek a polynomial of degree four to approximate it (see [4] for more details). Denoting it by

$$\bar{V}(y) = \frac{1}{2}a_1(\tilde{t})y^2 + \frac{1}{4}b_1(\tilde{t})y^4, \tag{4.12}$$

where the coefficients are chosen such that

$$\bar{V} = V, \qquad \bar{V}' = 0 \quad \text{at } y = 0,\ y = \frac{\pi}{2}, \tag{4.13}$$

we get

$$a_1(\tilde{t}) = \frac{4(4-\pi)}{\pi^2}(1+\varepsilon t), \qquad b_1(\tilde{t}) = -\frac{16(4-\pi)}{\pi^4}(1+\varepsilon t). \tag{4.14}$$

Substituting $\bar{V}$ for V in (4.10) and integrating it, we can obtain the approximate solution of (3.21) and the escape time of (3.25). From initial conditions, we can obtain $\varphi_0 = 1$, $v(0) = 0.285714$, and $c = 0.431393$. From (3.25), the approximate escape time T_0 is

$$T_0 = \frac{1.529}{\varepsilon}. \tag{4.15}$$

Comparisons of asymptotic results from (4.15) and numerical results are shown in Table 4.3.

If the potential (4.11) is expanded as Taylor series of fourth order, the escape time corresponding to (4.15) is $\tilde{T}_0 = 1.082/\varepsilon$, which has an unacceptable error of about 26%.

Table 4.2. Escape time from potential well of system (4.4).

ε	Asymptotic result	Numerical result	Error (%)
0.01	18.19	18.49	1.6
0.001	181.9	180.4	0.8
0.0001	1819	1779	2.3

Table 4.3. Escape time from potential well of system (4.7).

ε	Asymptotic result	Numerical result	Error (%)
0.1	15.29	17.25	11
0.01	152.9	147.5	3.7
0.001	1529	1467	4.2
0.0001	15 290	14 635	4.5

The reason is that Taylor series expansion is valid only for small oscillation, while the approximate potential works for relatively large oscillation.

5. Conclusions

(1) The multiple scales method and character of modulus of Jacobian elliptic function are applied to drive the escape time from the potential wells of quadratic and cubic nonlinear oscillators. The method of approximate potential makes the result more accurate than that of Taylor series expansion.

(2) Examples show that the asymptotic results are in good agreement with the numerical results. It should be noted that the amount of computations is about the same though ε decreases. However, if one uses a numerical method to solve (4.1), (4.4), and (4.7), as ε decreases these systems become stiff and the computing time increases rapidly and may produce large system errors.

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