SOLUTION OF NONLINEAR FREDHOLM-HAMMERSTEIN INTEGRAL EQUATIONS BY USING SEMIORTHOGONAL SPLINE WAVELETS

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Received 16 June 2004

Compactly supported linear semiorthogonal B-spline wavelets together with their dual wavelets are developed to approximate the solutions of nonlinear Fredholm-Hammerstein integral equations. Properties of these wavelets are first presented; these properties are then utilized to reduce the computation of integral equations to some algebraic equations. The method is computationally attractive, and applications are demonstrated through an illustrative example.

1. Introduction

Wavelets theory is a relatively new and emerging area in mathematical research. It has been applied in a wide range of engineering disciplines; particularly, wavelets are very successfully used in signal analysis for waveform representations and segmentations, time-frequency analysis, and fast algorithms for easy implementation [6]. Wavelets permit the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms [2, 3]. Wavelets can be separated into two distinct types, orthogonal and semiorthogonal [5]. Publications on integral equation methods have shown a marked preference for orthogonal wavelets [11]. This is probably because the original wavelets, which were widely used for signal processing, were primarily orthogonal. In signal processing applications, unlike integral equation methods, the wavelet itself is never constructed since only its scaling function and coefficients are needed. However, orthogonal wavelets either have infinite support or a nonsymmetric, and in some cases fractal, nature. These properties can make them a poor choice for characterization of a function. In contrast, the semiorthogonal wavelets have finite support, both even and odd symmetry, and simple analytical expressions, ideal attributes of a basis function [11].

Several numerical methods for approximating the solution of Hammerstein integral equations are known. For Fredholm-Hammerstein integral equations, the classical method of successive approximations was introduced in [12]. A variation of the Nyström method was presented in [10]. A collocation-type method was developed in [9]. In [4], Brunner applied a collocation-type method to nonlinear Volterra-Hammerstein integral

Mathematical Problems in Engineering 2005:1 (2005) 113–121
DOI: 10.1155/MPE.2005.113

equations and integrodifferential equations, and discussed its connection with the iterated collocation method. Guoqiang [8] introduced and discussed the asymptotic error expansion of a collocation-type method for Volterra-Hammerstein integral equations. The methods in [8, 9] transform a given integral equation into a system of nonlinear equations, which has to be solved with some kind of an iterative method. In [9] the definite integrals involved in the solution may be evaluated analytically only in favorable cases, while in [8] the integrals involved in the solution have to be evaluated at each time step of the iteration.

In the present paper, we apply compactly supported linear semiorthogonal B-spline wavelets, specially constructed for the bounded interval to solve the nonlinear Fredholm-Hammerstein integral equations of the form

$$y(x) = f(x) + \int_0^1 K(x,t)g[t,y(t)]dt, \quad 0 \le x \le 1, \tag{1.1}$$

where f, g, and K are given continuous functions, with $g(t,y)$ nonlinear in y. The use of semiorthogonal compactly supported spline wavelets is justified by their interesting properties. Among them, the following can be explicitly cited [1]: they satisfy all the properties on a bounded interval that are verified by the usual wavelets on the real line, but they do not present the difficulties related to the boundary conditions, when applying such wavelets to problems in finite bounded domains, unlike most of the continuous orthogonal wavelets. Also, the semiorthogonal compactly supported spline wavelets have closed-form expressions. In [11], the two categories of wavelets, orthogonal and semiorthogonal, are compared, and it is shown that semiorthogonal wavelets are best suited for integral equation applications.

Our method consists of reducing (1.1) to a set of algebraic equations by expanding the unknown function as linear B-spline wavelets with unknown coefficients. The properties of these wavelets are then utilized to evaluate the unknown coefficients. The paper is organized as follows. In Section 2, we describe the formulation of the B-spline scaling functions and wavelets on $[0,1]$ required for our subsequent development. In Section 3, the proposed method is used to approximate the solution of nonlinear Fredholm-Hammerstein integral equation. In Section 4, we report our numerical finding and demonstrate the accuracy of the proposed numerical scheme by considering a numerical example.

2. B-spline scaling functions and wavelets on $[0,1]$

When semiorthogonal wavelets are constructed from B-splines of order m, the lowest octave level $j = j_0$ is determined in [7] by

$$2^{j_0} \ge 2m - 1 \tag{2.1}$$

so as to give a minimum of one complete wavelet on the interval $[0,1]$. In this paper, we will use a wavelet generated by a linear spline—the second-order cardinal B-spline basis function. From (2.1), the second-order B-spline of lowest level, which must be an integer, is determined to be $j_0 = 2$. This constrains all octave levels to $j \ge 2$.

As is the case with all semiorthogonal wavelets, the second-order B-splines also serve as scaling functions. The second-order B-splines/scaling functions are given by

$$\phi_{j,k}(x) = \begin{cases} x_j - k, & k \le x_j \le k+1, \\ 2-(x_j-k), & k+1 \le x_j \le k+2,\ k = 0,\dots,2^j-2, \\ 0, & \text{otherwise}, \end{cases} \tag{2.2}$$

with the respective left- and right-hand side boundary scaling functions

$$\phi_{j,k}(x) = \begin{cases} 2-(x_j-k), & 0 \le x_j \le 1,\ k = -1, \\ 0, & \text{otherwise}, \end{cases} \tag{2.3}$$

$$\phi_{j,k}(x) = \begin{cases} x_j - k, & k \le x_j \le k+1,\ k = 2^j-1, \\ 0, & \text{otherwise}. \end{cases} \tag{2.4}$$

The actual coordinate position x is related to x_j according to $x_j = 2^j x$. The second-order B-spline wavelets are given by

$$\psi_{j,k}(x) = \frac{1}{6}\begin{cases} x_j - k, & k \le x_j \le k+\frac{1}{2}, \\ 4-7(x_j-k), & k+\frac{1}{2} \le x_j \le k+1, \\ -19+16(x_j-k), & k+1 \le x_j \le k+\frac{3}{2}, \\ 29-16(x_j-k), & k+\frac{3}{2} \le x_j \le k+2,\ k = 0,\dots,2^j-3, \\ -17+7(x_j-k), & k+2 \le x_j \le k+\frac{5}{2}, \\ 3-(x_j-k), & k+\frac{5}{2} \le x_j \le k+3, \\ 0, & \text{otherwise}, \end{cases} \tag{2.5}$$

with the respective left- and right-hand side boundary wavelets

$$\psi_{j,k}(x) = \frac{1}{6}\begin{cases} -6+23x_j, & 0 \le x_j \le \frac{1}{2}, \\ 14-17x_j, & \frac{1}{2} \le x_j \le 1, \\ -10+7x_j, & 1 \le x_j \le \frac{3}{2},\ k = -1, \\ 2-x_j, & \frac{3}{2} \le x_j \le 2, \\ 0, & \text{otherwise}, \end{cases} \tag{2.6}$$

$$\psi_{j,k}(x) = \frac{1}{6}\begin{cases} 2-(k+2-x_j), & k \le x_j \le k+\frac{1}{2}, \\ -10+7(k+2-x_j), & k+\frac{1}{2} \le x_j \le k+1, \\ 14-17(k+2-x_j), & k+1 \le x_j \le k+\frac{3}{2}, k = 2^j-2, \\ -6+23(k+2-x_j), & k+\frac{3}{2} \le x_j \le k+2, \\ 0, & \text{otherwise.} \end{cases} \tag{2.7}$$

For example, for $j = 2$, the inner scaling functions are obtained by putting $k = 0,1,2$ in (2.2) as

$$\phi_{2,0}(x) = \begin{cases} 4x, & 0 \le x < \frac{1}{4}, \\ 2-4x, & \frac{1}{4} \le x < \frac{1}{2}, \\ 0, & \text{otherwise,} \end{cases} \tag{2.8}$$

$$\phi_{2,1}(x) = \begin{cases} 4x-1, & \frac{1}{4} \le x \le \frac{1}{2}, \\ 1-4x, & \frac{1}{2} \le x \le \frac{3}{4}, \\ 0, & \text{otherwise,} \end{cases} \tag{2.9}$$

$$\phi_{2,2}(x) = \begin{cases} 4x-2, & \frac{1}{2} \le x \le \frac{3}{4}, \\ -4x, & \frac{3}{4} \le x \le 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.10}$$

Also, for $j = 2$, the left- and right-hand side boundary scaling functions are obtained by putting $j = 2$, $k = -1$, and $k = 3$ in (2.3) and (2.4), respectively, as

$$\phi_{2,-1}(x) = \begin{cases} 1-4x, & 0 \le x \le \frac{1}{4}, \\ 0, & \text{otherwise,} \end{cases} \tag{2.11}$$

$$\phi_{2,3}(x) = \begin{cases} 4x-3, & \frac{3}{4} \le x \le 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.12}$$

Similarly, for $j = 2$, the inner wavelet functions are obtained by putting $j = 2$, $k = 0$, and $k = 1$ in (2.5) and the left- and right-hand side boundary wavelets are obtained by putting $j = 2$, $k = -1$, and $k = 2$ in (2.6) and (2.7), respectively.

2.1. Function approximation. A function $f(x)$ defined over $[0,1]$ may be represented by B-spline wavelets as

$$f(x) = \sum_{k=-1}^{3} c_k \phi_{2,k} + \sum_{i=2}^{\infty} \sum_{j=-1}^{2^{(i-1)}} d_{i,j} \psi_{i,j}, \tag{2.13}$$

where $\phi_{2,k}$ and $\psi_{i,j}$ are scaling and wavelets functions, respectively. If the infinite series in (2.13) is truncated, then (2.13) can be written as

$$f(x) = \sum_{k=-1}^{3} c_k \phi_{2,k} + \sum_{i=2}^{M} \sum_{j=-1}^{2^{(i-1)}} d_{i,j} \psi_{i,j} = C^T \Psi, \tag{2.14}$$

where C and Ψ are $(2^{(M+1)}+1)\times 1$ vectors given by

$$C = \left[c_{-1}, c_0, \ldots, c_3, d_{2,-1}, \ldots, d_{2,2}, d_{3,-1}, \ldots, d_{3,6}, \ldots, d_{M,-1}, \ldots, d_{M,2^{(M-1)}}\right]^T, \tag{2.15}$$

$$\Psi = \left[\phi_{2,-1}, \phi_{2,0}, \ldots, \phi_{2,3}, \psi_{2,-1}, \ldots, \psi_{2,2}, \psi_{3,-1}, \ldots, \psi_{3,6}, \ldots, \psi_{M,-1}, \ldots, \psi_{M,2^{(M-1)}}\right]^T, \tag{2.16}$$

with

$$c_k = \int_0^1 f(x) \tilde{\phi}_{2,k}(x) dx, \quad k = -1, 0, \ldots, 3, \tag{2.17}$$

$$d_{i,j} = \int_0^1 f(x) \tilde{\psi}_{i,j}(x) dx, \quad i = 2, 3, 4, \ldots, M,\ j = -1, 0, 1, \ldots, 2^{(i-1)}, \tag{2.18}$$

where $\tilde{\phi}_{2,k}(x)$ and $\tilde{\psi}_{i,j}(x)$ are dual functions of $\phi_{2,k}$, and $\psi_{i,j}$, respectively. These can be obtained by linear combinations of $\phi_{2,k}$, $k = -1, \ldots, 3$, and $\psi_{i,j}$, $i = 2, \ldots, M$, $j = -1, \ldots, 2^{(M-1)}$, as follows. Let

$$\Phi = \left[\phi_{2,-1}(x), \phi_{2,0}(x), \phi_{2,1}(x), \phi_{2,2}(x), \phi_{2,3}(x)\right]^T, \tag{2.19}$$

$$\bar{\Psi} = \left[\psi_{2,-1}(x), \psi_{2,0}(x), \ldots, \psi_{M,2^{(M-1)}}(x)\right]^T. \tag{2.20}$$

Using (2.8)–(2.12) and (2.19) we get

$$\int_0^1 \Phi \Phi^T dx = P_1 = \begin{bmatrix} \frac{1}{12} & \frac{1}{24} & 0 & 0 & 0 \\ \frac{1}{24} & \frac{1}{6} & \frac{1}{24} & 0 & 0 \\ 0 & \frac{1}{24} & \frac{1}{6} & \frac{1}{24} & 0 \\ 0 & 0 & \frac{1}{24} & \frac{1}{6} & \frac{1}{24} \\ 0 & 0 & 0 & \frac{1}{24} & \frac{1}{12} \end{bmatrix}, \tag{2.21}$$

and from (2.5)–(2.7) and (2.20) we have

$$\int_0^1 \bar{\Psi}\bar{\Psi}^T dx = P_2 = \begin{bmatrix} N_{4\times 4} & & & \\ & \frac{1}{2}N_{8\times 8} & & \\ & & \ddots & \\ & & & \frac{1}{2^{M-2}}N_{2^M\times 2^M} \end{bmatrix}, \tag{2.22}$$

where P_1 and P_2 are 5×5 and $(2^{M+1}-4)\times(2^{M+1}-4)$ matrices, respectively, and N is a five-diagonal matrix given by

$$N = \begin{bmatrix} \frac{2}{27} & \frac{1}{96} & -\frac{-1}{864} & 0 & 0 & \cdots & 0 \\ \frac{1}{96} & \frac{1}{16} & \frac{5}{432} & -\frac{1}{864} & 0 & \cdots & 0 \\ -\frac{1}{864} & \frac{5}{432} & \frac{1}{16} & \frac{5}{432} & -\frac{1}{864} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -\frac{1}{864} & \frac{5}{432} & \frac{1}{16} & \frac{5}{432} & -\frac{1}{864} \\ 0 & \cdots & 0 & -\frac{1}{864} & \frac{5}{432} & \frac{1}{16} & \frac{1}{96} \\ 0 & \cdots & 0 & 0 & -\frac{1}{864} & \frac{1}{96} & \frac{2}{27} \end{bmatrix}. \tag{2.23}$$

Suppose $\tilde{\Phi}$ and $\tilde{\bar{\Psi}}$ are the dual functions of Φ and $\bar{\Psi}$, respectively, given by

$$\tilde{\Phi} = \left[\tilde{\phi}_{2,-1}(x), \tilde{\phi}_{2,0}(x), \tilde{\phi}_{2,1}(x), \tilde{\phi}_{2,2}(x), \tilde{\phi}_{2,3}(x)\right]^T, \tag{2.24}$$

$$\tilde{\bar{\Psi}} = \left[\tilde{\psi}_{2,-1}(x), \tilde{\psi}_{2,0}(x), \ldots, \tilde{\psi}_{M,2^M-2}(x)\right]^T. \tag{2.25}$$

Using (2.17)–(2.20), (2.24), and (2.25) we have

$$\int_0^1 \tilde{\Phi}\Phi^T dx = I_1, \qquad \int_0^1 \tilde{\bar{\Psi}}\bar{\Psi}^T dx = I_2, \tag{2.26}$$

where I_1 and I_2 are 5×5 and $(2^{(M+1)}-4)\times(2^{(M+1)}-4)$ identity matrices, respectively. Then (2.21), (2.22), and (2.26) give

$$\tilde{\Phi} = P_1{}^{-1}\Phi, \qquad \tilde{\bar{\Psi}} = P_2{}^{-1}\bar{\Psi}. \tag{2.27}$$

3. Nonlinear Fredholm-Hammerstein integral equations

In this section, we solve nonlinear Fredholm-Hammerstein integral equations of the form in (1.1) by using B-spline wavelets. For this purpose, we first assume

$$z(x) = g(x, y(x)), \quad 0 \le x \le 1. \tag{3.1}$$

We now use (2.14) to approximate $y(x)$, $z(x)$ as

$$y(x) = D^T\Psi(x), \qquad z(x) = E^T\Psi(x), \tag{3.2}$$

where $\Psi(x)$ is defined in (2.15) and D and E are $(2^{(M+1)}+1)\times 1$ unknown vectors defined similarly to C in (2.16). We also expand $f(x)$, $K(x,t)$ by B-spline dual wavelets $\tilde{\Psi}$ defined as in (2.24) and (2.25) as

$$f(x) = \Lambda^T\tilde{\Psi}(x), \qquad K(x,t) = \tilde{\Psi}^T(t)\Theta\tilde{\Psi}(x), \tag{3.3}$$

where

$$\Theta_{(i,j)} = \int_0^1 \left[\int_0^1 K(x,t)\Psi_i(t)dt\right]\Psi_j(x)dx. \tag{3.4}$$

From (3.1), (3.2), and (3.3) we get

$$\begin{aligned}\int_0^1 K(x,t)g(t,y(t))dt &= \int_0^1 E^T\Psi(t)\tilde{\Psi}^T(t)\Theta\tilde{\Psi}(x)dt \\ &= E^T\left[\int_0^1 \Psi(t)\tilde{\Psi}^T(t)dt\right]\Theta\tilde{\Psi}(x) \\ &= E^T\Theta\tilde{\Psi}(x).\end{aligned} \tag{3.5}$$

Applying (3.1)–(3.5) in (1.1), we get

$$D^T\Psi(x) - \Lambda^T\tilde{\Psi}(x) - E^T\Theta\tilde{\Psi}(x) = 0; \tag{3.6}$$

multiplying (3.6) by $\Psi^T(x)$ and integrating from 0 to 1, we have

$$D^TP - \Lambda^T - E^T\Theta = 0, \tag{3.7}$$

in which P is a $(2^{(M+1)}+1)\times(2^{(M+1)}+1)$ square matrix given by

$$P = \int_0^1 \Psi(x)\Psi^T(x)dx = \begin{bmatrix} P_1 & \\ & P_2 \end{bmatrix}. \tag{3.8}$$

To find the solution $y(x)$ in (3.2), we first collocate the following equation in $x_i = i/2^{M+1}$, $i = 0, 1, \ldots, 2^{M+1}$:

$$g(x, D^T\Psi(x)) = E^T\Psi(x). \tag{3.9}$$

Equation (3.7) generates a set of $2^{(M+1)}+1$ algebraic equations. The total number of unknowns for vectors D and E in (3.2) is $2[2^{(M+1)}+1]$. These can be obtained by using (3.7) and (3.9).

Table 4.1. Exact and approximate solutions.

x	Approximate $M = 2$	Approximate $M = 4$	Approximate $M = 6$	Exact
0	1	1	1	1
0.1	0.995688	0.995012	0.995005	0.995004
0.2	0.983165	0.983077	0.983094	0.983095
0.3	0.955652	0.955324	0.955336	0.955336
0.4	0.921345	0.921066	0.921062	0.921061
0.5	0.877261	0.877575	0.877582	0.877583
0.6	0.825418	0.825343	0.825337	0.825336
0.7	0.764678	0.764859	0.764844	0.764842
0.8	0.696157	0.696694	0.696705	0.696707
0.9	0.621445	0.621603	0.621617	0.621619
1	0.540280	0.540312	0.540303	0.540302

4. Illustrative example

Consider the equation

$$y(x) = 1 + 3\sin^2(x) + \int_0^1 K(x,t)y^2(t)dt, \quad 0 \le x \le 1, \tag{4.1}$$

where

$$K(x,t) = \begin{cases} -3\sin(x-t), & 0 \le t \le x, \\ 0, & x < t \le 1. \end{cases} \tag{4.2}$$

The solution for $y(x)$ is obtained by the method in Section 3. The computational results for $M = 2$, $M = 4$, and $M = 6$ together with the exact solution $y(x) = \cos(x)$ are given in Table 4.1.

5. Conclusion

In the present work, a technique has been developed for solving nonlinear Fredholm-Hammerstein integral equations. The method is based upon compactly supported linear semiorthogonal B-spline wavelets. The dual wavelets for these B-spline wavelets were also given. The problem has been reduced to solving a system of nonlinear algebraic equations. An illustrative example was included to demonstrate the validity and applicability of the technique.

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