RESEARCH OF NONLINEAR VIBRATIONS OF ORTHOTROPIC PLATES WITH A COMPLEX FORM

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This paper deals with effects of large amplitude on the free and forced flexural vibrations of elastic orthotropic plates of arbitrary shape. R-function method (RFM) is applied to obtain the basis functions need for expansion of sought solution into Fourier series. The initial nonlinear system of differential equations with partial derivatives is reduced to system of ordinary nonlinear differential equations by Galerkin procedure. The solving-obtained system is carried out by Runge-Kutta or Galerkin methods. The numerical results for the plate of complex form and also rectangular form and different boundary conditions have been presented and compared with other known results.

1. Introduction

The large amplitude vibration of orthotropic plates was investigated by many scientists [3, 4, 8, 9, 11]. However most papers are devoted to research of nonlinear vibrations of plates and shallow shells with simple enough form. There are only few works in which the plates and shells, with a shape different from rectangle, circle, ring, or ellipse, are considered. From the authors' point of view, deficiency of such works is connected with difficulties of construction of analytical expressions for basic functions. These functions are needed to reduce a nonlinear system of differential equations with partial derivatives to a system of the ordinary differential equations for time. One of the universal approaches, which can be used for solving this problem, is founded on the usage of the R-functions theory [12, 15]. This theory allows constructing a complete set of the coordinate functions for different types of boundary conditions and practically of an arbitrary domain. Many papers [7, 12–14, 16] showed the use of the R-functions theory for investigation of linear vibrations of plates and shallow shells with different plan forms, a curvature and different types of boundary conditions. In this paper the R-functions theory together with variational methods is applied to research of nonlinear vibrations of orthotropic plates of an arbitrary plan form and different types of boundary conditions.

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2. Mathematical statement of the problem

Thin orthotropic plates with geometric-type nonlinearity subjected to periodic lateral loading are considered. It is assumed that the material of the plate is of uniform thickness and specially orthotropic, with the principal axes of orthotropy being parallel to the x and y directions. Under the assumption that the effect of both the longitudinal and rotatory inertia forces can be neglected, the basic equations governing the nonlinear vibrations of plates can be reduced to the following set of equations [19, 20]:

$$L_{11}u + L_{12}v = Nl_1(w), \tag{2.1}$$

$$L_{21}u + L_{22}v = Nl_2(w), \tag{2.2}$$

$$L_{31}w = Nl_3(u,v,w) + 12(1-\mu_1\mu_2)\bar{q} - \lambda^2\frac{\partial^2 w}{\partial t^2}, \tag{2.3}$$

where $\xi = x/a$, $\eta = y/a$, $\bar{w} = w/h$, $\bar{u} = ua/h^2$, $\bar{v} = va/h^2$,

$$\begin{gathered} L_{11} = C_1\frac{\partial^2}{\partial\xi^2} + C_2\frac{\partial^2}{\partial\eta^2}, \qquad L_{12} = L_{21} = C_3\frac{\partial^2}{\partial\xi\,\partial\eta}, \qquad L_{22} = C_2\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}, \\ L_{31} = C_1\frac{\partial^4}{\partial\xi^4} + 2C_3\frac{\partial^4}{\partial\xi^2\,\partial\eta^2} + \frac{\partial^4}{\partial\eta^4}, \end{gathered} \tag{2.4}$$

$$Nl_1(w) = -\left(\frac{1}{2}\frac{\partial}{\partial\xi}\left(C_1\left(\frac{\partial w}{\partial\xi}\right)^2 + \mu_1\left(\frac{\partial w}{\partial\eta}\right)^2\right) + C_2\frac{\partial}{\partial\eta}\left(\frac{\partial w}{\partial\xi}\frac{\partial w}{\partial\eta}\right)\right), \tag{2.5}$$

$$Nl_2(w) = -\left(\frac{1}{2}\frac{\partial}{\partial\eta}\left(\left(\frac{\partial w}{\partial\eta}\right)^2 + \mu_1\left(\frac{\partial w}{\partial\xi}\right)^2\right) + C_2\frac{\partial}{\partial\xi}\left(\frac{\partial w}{\partial\xi}\frac{\partial w}{\partial\eta}\right)\right), \tag{2.6}$$

$$Nl_3(u,v,w) = 12(1-\mu_1\mu_2)\left(N_\xi\frac{\partial^2 w}{\partial\xi^2} + 2T\frac{\partial^2 w}{\partial\xi\,\partial\eta} + N_\eta\frac{\partial^2 w}{\partial\eta^2}\right), \tag{2.7}$$

$$C_1 = \frac{E_1}{E_2}, \qquad C_2 = \frac{G(1-\mu_1\mu_2)}{E_2}, \qquad C_3 = \frac{G(1-\mu_1\mu_2)}{E_2} + \mu_1 = C_2 + \mu_1. \tag{2.8}$$

In (2.3) value λ^2 is defined as

$$\lambda^2 = \frac{\gamma}{g}\frac{a^4\cdot 12(1-\mu_1\mu_2)}{E_2h^2}. \tag{2.9}$$

And the nondimensional expressions for N_ξ, N_η, T have the following form:

$$\begin{gathered} N_\xi = \frac{C_1}{1-\mu_1\mu_2}\cdot\left(\frac{\partial u}{\partial\xi} + \mu_2\frac{\partial v}{\partial\eta}\right) + \frac{C_1}{2(1-\mu_1\mu_2)}\cdot\left(\left(\frac{\partial\bar{w}}{\partial\xi}\right)^2 + \mu_2\left(\frac{\partial\bar{w}}{\partial\eta}\right)^2\right), \\ N_\eta = \frac{1}{1-\mu_1\mu_2}\cdot\left(\frac{\partial v}{\partial\eta} + \mu_1\frac{\partial u}{\partial\xi}\right) + \frac{1}{2(1-\mu_1\mu_2)}\cdot\left(\left(\frac{\partial w}{\partial\eta}\right)^2 + \mu_1\left(\frac{\partial w}{\partial\xi}\right)^2\right), \\ T = \frac{G}{E_2}\cdot\left(\frac{\partial u}{\partial\eta} + \frac{\partial v}{\partial\xi}\right) + \frac{G}{E_2}\cdot\left(\frac{\partial w}{\partial\eta}\frac{\partial w}{\partial\xi}\right). \end{gathered} \tag{2.10}$$

The motion equations are supplemented by boundary conditions, a type of which is defined by a way of plate-edge fixing.

3. Research method for nonlinear forced vibration of orthotropic plate

As shown in references [7, 13, 14], determination of eigenfunctions appropriate to linear vibrations of an orthotropic plate can be fulfilled practically for arbitrary geometry and enough general boundary conditions by R-functions method (RFM). Due to application of RFM, the eigenfunctions are founded in an analytical form. This fact allows using obtained eigenfunctions as a basis for solving the nonlinear tasks, in particular, problems of large amplitude of vibration plates.

Let us suppose that for an orthotropic plate the eigenfunctions appropriate to the frequencies of free linear vibrations plate are known (see Section 4). Let eigenfunctions $W_1(\xi,\eta)$, $W_2(\xi,\eta)$ be appropriated to the first two frequencies. It is supposed that lateral load may be presented as

$$q(x,y,t) = P(t)\cdot W_1(x,y). \tag{3.1}$$

The deflection function is represented in the following form:

$$W(\xi,\eta,t) = y_1(t)W_1(\xi,\eta) + y_2(t)W_2(\xi,\eta). \tag{3.2}$$

After substitution of (3.2) into the first two equations (2.1), (2.2), one receives

$$\begin{aligned} L_{11}u + L_{12}v &= y_1^2(t)Nl_1(W_1) + y_2^2(t)Nl_1(W_2) + y_1(t)y_2(t)Nl_1l_1(W_1,W_2), \\ L_{21}u + L_{22}v &= y_1^2(t)Nl_2(W_1) + y_2^2(t)Nl_2(W_2) + y_1(t)y_2(t)Nl_1l_2(W_1,W_2), \end{aligned} \tag{3.3}$$

where the operators $Nl_1(W_i)$, $Nl_2(W_i)$ $(i = 1,2)$ are defined by expressions (2.5), (2.6), and the operators $Nl_1l_1(W_1W_2)$, $Nl_2l_2(W_1W_2)$ are represented below:

$$\begin{aligned} &Nl_1l_1(W_1W_2) \\ &\quad = -\left(\frac{\partial}{\partial\xi}\left(C_1\frac{\partial W_1}{\partial\xi}\frac{\partial W_2}{\partial\eta} + \nu_1\frac{\partial W_1}{\partial\eta}\frac{\partial W_2}{\partial\eta}\right) + C_2\frac{\partial}{\partial\eta}\left(\frac{\partial W_1}{\partial\xi}\frac{\partial W_2}{\partial\eta} + \frac{\partial W_2}{\partial\xi}\frac{\partial W_1}{\partial\eta}\right)\right), \\ &Nl_2l_2(W_1,W_2) \\ &\quad = -\left(\frac{\partial}{\partial\eta}\left(\frac{\partial W_1}{\partial\eta}\frac{\partial W_2}{\partial\eta} + \nu_1\frac{\partial W_1}{\partial\xi}\frac{\partial W_2}{\partial\xi}\right) + C_2\frac{\partial}{\partial\xi}\left(\frac{\partial W_2}{\partial\xi}\frac{\partial W_1}{\partial\eta} + \frac{\partial W_1}{\partial\xi}\frac{\partial W_2}{\partial\eta}\right)\right). \end{aligned} \tag{3.4}$$

Equations (3.3) will be satisfied identically, if the functions $u(\xi,\eta,t)$ and $v(\xi,\eta,t)$ are selected as

$$\begin{aligned} u(\xi,\eta,t) &= u_1(\xi,\eta)y_1^2(t) + u_2(\xi,\eta)y_2^2(t) + u_3(\xi,\eta)y_1(t)y_2(t), \\ v(\xi,\eta,t) &= v_1(\xi,\eta)y_1^2(t) + v_2(\xi,\eta)y_2^2(t) + v_3(\xi,\eta)y_1(t)y_2(t), \end{aligned} \tag{3.5}$$

where (u_1,v_1), (u_2,v_2), and (u_3,v_3) are solutions of the following set of equations:

$$\begin{aligned} L_{11}u_1 + L_{12}v_1 &= Nl_1(W_1), \\ L_{21}u_1 + L_{22}v_1 &= Nl_2(W_1), \\ L_{11}u_2 + L_{12}v_2 &= Nl_1(W_2), \\ L_{21}u_2 + L_{22}v_2 &= Nl_2(W_2), \\ L_{11}u_3 + L_{12}v_3 &= Nl_1l_1(W_1,W_2), \\ L_{21}u_3 + L_{22}v_3 &= Nl_2l_2(W_1,W_2). \end{aligned} \tag{3.6}$$

Equations (3.6) are supplemented by appropriate boundary conditions. The obtained equations formally coincide with the equations of a plane problem in the theory of elasticity for an orthotropic body. Thus the expressions $Nl_1(W_1)$, $Nl_2(W_1)$, $Nl_1(W_2)$, $Nl_2(W_2)$, $Nl_1l_1(W_1,W_2)$, and $Nl_2l_2(W_1,W_2)$ may be considered as mass forces. A solution of the plane problem in theory of elasticity for orthotropic plate is carried out by RFM as well (see Section 5). After finding solutions (u_1,v_1), (u_2,v_2), (u_3,v_3), the functions (3.5) may be determined and substituted into (2.3) together with expressions for deflection (3.2) and load (3.1). As a result the ordinary differential equation is received. Let us apply method of Bubnov-Galerkin to the obtained equation. Projecting this equation on the eigenfunctions $W_1(\xi,\eta)$ and $W_2(\xi,\eta)$ and taking into account their orthogonality, one can receive the following system of the nonlinear differential equations:

$$\begin{aligned} y_1'' + \alpha_0 y_1 + \alpha_1 y_1^3 + \alpha_2 y_1^2 y_2 + \alpha_3 y_1 y_2^2 + \alpha_4 y_2^3 &= \alpha_5 P(t), \\ y_2'' + \beta_0 y_2 + \beta_1 y_1^3 + \beta_2 y_1^2 y_2 + \beta_3 y_1 y_2^2 + \beta_4 y_2^3 &= 0, \end{aligned} \tag{3.7}$$

where

$$\begin{gathered} \alpha_0 = 1, \qquad \alpha_1 = \frac{a_{13}}{\omega_0^2 \|W_1\|^2}, \qquad \alpha_2 = \frac{a_{14}}{\omega_0^2 \|W_1\|^2}, \\ \alpha_3 = \frac{a_{15}}{\omega_0^2 \|W_1\|^2}, \qquad \alpha_4 = \frac{a_{16}}{\omega_0^2 \|W_1\|^2}, \qquad \alpha_5 = \frac{a_{17}}{\omega_0^2 \|W_1\|^2}, \\ \beta_0 = \frac{\omega_{2L}^2}{\omega_{1L}^2}, \qquad \beta_1 = \frac{a_{23}}{\omega_0^2 \|W_2\|^2}, \qquad \beta_2 = \frac{a_{24}}{\omega_0^2 \|W_2\|^2}, \\ \beta_3 = \frac{a_{25}}{\omega_0^2 \|W_2\|^2}, \qquad \beta_4 = \frac{a_{26}}{\omega_0^2 \|W_2\|^2}. \end{gathered} \tag{3.8}$$

The expressions for coefficients a_{ij} $(i = 1,2;\ j = 0,1,2,3,4,5,6)$ in (3.8) are

$$
\begin{aligned}
&a_{i1} = \|w_i\|^2, \qquad a_{i2} = \lambda^2\omega_{iL}^2\|w_i\|^2,\\
&a_{i3} = -12(1-\mu_1\mu_2)\iint_\Omega\left(N_x(\vec{U}_1)\frac{\partial^2 w_1}{\partial\xi^2} + 2T(\vec{U}_1)\frac{\partial^2 w_1}{\partial\xi\,\partial\eta} + N_y(\vec{U}_1)\frac{\partial^2 w_1}{\partial\eta}\right)w_i d\Omega,\\
&a_{i4} = -12(1-\mu_1\mu_2)\iint_\Omega\left(N_x(\vec{U}_1)\frac{\partial^2 w_2}{\partial\xi^2} + 2T(\vec{U}_1)\frac{\partial^2 w_2}{\partial\xi\,\partial\eta} + N_y(\vec{U}_1)\frac{\partial^2 w_2}{\partial\eta}\right)w_i d\Omega\\
&\qquad - 12(1-\mu_1\mu_2)\iint_\Omega\left(N_x(\vec{U}_3)\frac{\partial^2 w_1}{\partial\xi^2} + 2T(\vec{U}_3)\frac{\partial^2 w_1}{\partial\xi\,\partial\eta} + N_y(\vec{U}_3)\frac{\partial^2 w_1}{\partial\eta}\right)w_i d\Omega,\\
&a_{i5} = -12(1-\mu_1\mu_2)\iint_\Omega\left(N_x(\vec{U}_2)\frac{\partial^2 w_1}{\partial\xi^2} + 2T(\vec{U}_2)\frac{\partial^2 w_1}{\partial\xi\,\partial\eta} + N_y(\vec{U}_2)\frac{\partial^2 w_1}{\partial\eta}\right)w_i d\Omega\\
&\qquad - 12(1-\mu_1\mu_2)\iint_\Omega\left(N_x(\vec{U}_3)\frac{\partial^2 w_2}{\partial\xi^2} + 2T(\vec{U}_3)\frac{\partial^2 w_2}{\partial\xi\,\partial\eta} + N_y(\vec{U}_3)\frac{\partial^2 w_2}{\partial\eta}\right)w_i d\Omega,\\
&a_{i6} = -12(1-\mu_1\mu_2)\iint_\Omega\left(N_x(\vec{U}_2)\frac{\partial^2 w_2}{\partial\xi^2} + 2T(\vec{U}_2)\frac{\partial^2 w_2}{\partial\xi\,\partial\eta} + N_y(\vec{U}_2)\frac{\partial^2 w_2}{\partial\eta}\right)w_i d\Omega,\\
&a_{17} = 12(1-\mu_1\mu_2)\|w_1\|^2 P(t), \qquad a_{27} = 0.
\end{aligned}
\tag{3.9}
$$

Solving the system (3.7), supplemented by the initial conditions

$$y_1(0) = A_{\max}, \qquad y_1'(0) = 0, \qquad y_2(0) = 0, \qquad y_2'(0) = 0, \tag{3.10}$$

can be carried out, for example, by Runge-Kutta method.

4. Solving the linear vibrations problem of orthotropic plates

Let us consider the problem of determination of the basis functions W_1, W_2 in more detail. For their finding, it is necessary to solve the problem of free linear vibrations of orthotropic plate; that is, it is necessary to solve the equation

$$L_{31}W = \lambda^2\omega_L^2 W. \tag{4.1}$$

The combination of variational method of Ritz and RFM [12, 13, 15] is applied for solving the problem. According to RFM, it is necessary to make variational statement of the problem (4.1):

$$\partial J = 0, \tag{4.2}$$

where

$$J(W) = \iint_{\Omega} \left(C_1 \left(\frac{\partial^2 W}{\partial \xi^2} \right)^2 + 2\mu_1 \frac{\partial^2 W}{\partial \xi^2} \frac{\partial^2 W}{\partial \eta^2} + \left(\frac{\partial^2 W}{\partial \eta^2} \right)^2 + 4C_2 \left(\frac{\partial^2 W}{\partial \xi \partial \eta} \right)^2 \right) \times \partial\Omega - \omega_0^2 \iint_{\Omega} W^2 \partial\Omega. \tag{4.3}$$

Here $\omega_0^2 = \lambda^2 \omega_L^2$.

A minimum of this functional is sought on a set of coordinate functions, which are constructed by the RFM. To form the sequences of the coordinate functions, it is necessary to construct the first structure of solution [12, 15] for boundary value problems (4.1). As shown in [7, 12, 13, 15], the solution structure depends on the type of boundary conditions. For example, the structure of solution, which satisfies to boundary conditions for clamped plate, is determined by the expression

$$w = \omega^2 P_1. \tag{4.4}$$

Here P_1 is an indefinite component of structure [7, 12, 13, 15]. Note that all boundary conditions are satisfied independently of P_1 choice. Equation of domain boundary $\omega(x, y) = 0$ is constructed by R-functions as a uniform analytical expression. It should be noted that the constructed equation $\omega(x, y) = 0$ contains only elementary functions. The function $\omega(x, y)$ also satisfies the following conditions:

$$\begin{aligned} \omega(x, y) &= 0, \quad \forall (x, y) \in \partial\Omega, \\ \omega(x, y) &> 0, \quad \forall (x, y) \in \Omega, \\ \frac{\partial \omega(x, y)}{\partial n} &= 1, \quad \forall (x, y) \in \partial\Omega. \end{aligned} \tag{4.5}$$

In the case of simply supported edge of orthotropic plate, the structure of a solution is the following [7, 13]:

$$w = \omega P_1 - \frac{\omega^2}{2(A_1 - \omega)} \left(A_1 \left\{ 2D_1^{(\omega)} P_1 + P_1 D_2^{(\omega)} \omega \right\} + 2A_2 T_1^{(\omega)} P_1 - \frac{1}{\rho} A_3 P_1 \right) + \omega^3 P_2, \tag{4.6}$$

where ρ is a continuation of a boundary curvature inside of the domain, and the operators $D_m^{(\omega)}$, $T_m^{(\omega)}$ $(m = 1, 2)$ are defined as [12]

$$\begin{aligned} D_m^{(\omega)} f &= \left(\frac{\partial \omega}{\partial x} \frac{\partial}{\partial x} + \frac{\partial \omega}{\partial y} \frac{\partial}{\partial y} \right)^m \cdot f, \\ T_m^{(\omega)} f &= \sum (-1)^{m-i} C_m^i \frac{\partial^m f}{\partial x^{m-i} \partial y^i} \left(\frac{\partial \omega}{\partial x} \right)^i \left(\frac{\partial \omega}{\partial y} \right)^{m-i}. \end{aligned} \tag{4.7}$$

The coefficients A_1, A_2, A_3 are defined by the following expressions:

$$A_1 = -\left[D_1\left(\frac{\partial\omega}{\partial x}\right)^4 + D_2\left(\frac{\partial\omega}{\partial y}\right)^4 + 2D_3\left(\frac{\partial\omega}{\partial x}\right)^2\left(\frac{\partial\omega}{\partial y}\right)^2\right],$$
$$A_2 = \left[(D_1 - D_3)\left(\frac{\partial\omega}{\partial x}\right)^2 + (D_3 - D_2)\left(\frac{\partial\omega}{\partial y}\right)^2\right]\left(\frac{\partial\omega}{\partial x}\right)\left(\frac{\partial\omega}{\partial y}\right), \tag{4.8}$$
$$A_3 = -\mu_2 D_1 + [2D_3 - (D_1 + D_2)]\left(\frac{\partial\omega}{\partial x}\right)^2\left(\frac{\partial\omega}{\partial y}\right)^2,$$

where D_1, D_2, D_3 are rigidity coefficients, which are defined by the following formulas:

$$D_1 = \frac{E_1 h^3}{12(1-\mu_1\mu_2)}, \qquad D_2 = \frac{E_2 h^3}{12(1-\mu_1\mu_2)}, \qquad D_3 = D_1\mu_2 + 2D_k, \qquad D_k = \frac{Gh^3}{12}. \tag{4.9}$$

In the case when the plate is simply supported, it is possible to use the structure of solution which takes into account only principal (kinematic) conditions. For orthotropic and isotropic plates this solution structure can be represented in the following form:

$$w = \omega P_1, \tag{4.10}$$

where the functions P_1 and $\omega(x, y)$ have the same sense, as earlier in the formulas (4.4), (4.5).

If the plate is clamped on one part of boundary $\partial\Omega_1$ and is simply supported on the remaining part of the boundary $\partial\Omega_2 = \partial\Omega\backslash\partial\Omega_1$, the solution structure satisfying only main conditions is [13] the following:

$$w(x, y) = \omega\omega_1 P_1, \tag{4.11}$$

where $\omega = 0$ is an equation all over the boundary; $\omega_1 = 0$ is an equation of the clamped edge of the domain. Further components P_1 (P_2) in the structural formulas are expanded in a series of some complete set of functions $\{\psi_k\}$: degree polynomials, trigonometric, splines, polynomials of Chebyshev, and so forth; that is,

$$P_i = \sum_{k=1}^{n} a_k^{(i)}\psi_k. \tag{4.12}$$

After substitution of (4.12) into the structural formulas, the unknown function Wwill be received in an analytical form. For example, for the clamped plate,

$$W = \sum_{k=1}^{n} a_k\omega_1^2\psi_k = \sum_{k=1}^{n} a_k\varphi_k, \tag{4.13}$$

where $\varphi_k = \omega^2\psi_k$ is a complete set of coordinate functions satisfying the given boundary conditions, and a_k, $k = 1, 2, \ldots, n$, are unknown constant coefficients, which may be

founded from a condition of a functional minimum (4.2), that is, as a result of solving the linear algebraic equations system

$$\frac{\partial J}{\partial a_i} = 0, \quad i = 1,2,\ldots,n. \tag{4.14}$$

5. Solving the plane problem in the theory of elasticity by RFM

Let us consider a method of solving a plane problem in the theory of elasticity for definition of functions (u_1, v_1), (u_2, v_2), (u_3, v_3), that is, solving (3.6). The variational statement of the problem is fulfilled by Lagrange variational principle [10]. The functional by Lagrange reaches its minimum on the solution of the posed problem, that is,

$$\partial I(\vec{U}) = 0. \tag{5.1}$$

In the extended form the functional $I(\vec{U})$ for displacements can be written as

$$\begin{aligned} I(\vec{U}_i) = {} & \iint_\Omega \left(C_1 \left(\frac{\partial u_i}{\partial \xi} \right)^2 + \left(\frac{\partial v_i}{\partial \eta} \right)^2 + C_2 \left(\frac{\partial v_i}{\partial \xi} + \frac{\partial u_i}{\partial \eta} \right)^2 + 2\mu_1 \frac{\partial u_i}{\partial \xi} \frac{\partial v_i}{\partial \eta} \right) d\Omega \\ & + I_r(\vec{U}_i) + I_k(\vec{U}_i), \end{aligned} \tag{5.2}$$

where $\vec{U}_i = (u_i, v_i)$ $(i = 1,2,3)$. Expressions for $I_r(\vec{U}_i)$, $I_k(\vec{U}_i)$, in case $i = 1,2$, are

$$\begin{aligned} I_r(\vec{U}_i) &= 2 \iint_\Omega (Nl_1(W_i) u_i + Nl_2(W_i) v_i) \partial\Omega, \\ I_k(\vec{U}_i) &= \oint_{\partial\Omega} \left(C_1 (l^2 + \mu_2 m^2) \left(\frac{\partial W_i}{\partial \xi} \right)^2 + (m^2 + \mu_1 l^2) \left(\frac{\partial W_i}{\partial \eta} \right)^2 + 2C_2 lm \frac{\partial W_i}{\partial \xi} \frac{\partial W_i}{\partial \eta} \right) u_{in} \partial s \\ &\quad + \oint_{\partial\Omega} \left(C_1 (\mu_2 - 1) lm \left(\frac{\partial W_i}{\partial \xi} \right)^2 + (1 - \mu_1) lm \left(\frac{\partial W_i}{\partial \eta} \right)^2 \right. \\ &\qquad \left. + 2C_3 (l^2 - m^2) \frac{\partial W_i}{\partial \xi} \frac{\partial W_i}{\partial \eta} \right) v_{in} \partial s. \end{aligned} \tag{5.3}$$

For $i = 3$,

$$\begin{aligned} I_r(\vec{U}_3) &= 2 \iint_\Omega (Nl_1 l_1(W_1, W_2) u_3 + Nl_2 l_2(W_1, W_2) v_3) \partial\Omega, \\ I_k(\vec{U}_3) &= \oint_{\partial\Omega} \left(2C_1 (l^2 + \mu_2 m^2) \frac{\partial W_1}{\partial \xi} \frac{\partial W_2}{\partial \xi} + 2(m^2 + \mu_1 l^2) \frac{\partial W_1}{\partial \eta} \frac{\partial W_2}{\partial \eta} \right) u_{in} \partial s \\ &\quad + \oint_{\partial\Omega} \left(2C_1 (\mu_2 - 1) lm \frac{\partial W_1}{\partial \xi} \frac{\partial W_2}{\partial \xi} + 2(1 - \mu_1) lm \frac{\partial W_1}{\partial \eta} \frac{\partial W_2}{\partial \eta} \right. \\ &\qquad \left. + 2C_3 (l^2 - m^2) \left(\frac{\partial W_1}{\partial \xi} \frac{\partial W_2}{\partial \eta} + \frac{\partial W_2}{\partial \xi} \frac{\partial W_1}{\partial \eta} \right) \right) v_{in} \partial s, \end{aligned} \tag{5.4}$$

where $u_{in} = u_i l + v_i m$, $v_{in} = -u_i m + v_i l$.

Here l and m are directional cosines of a normal vector to the boundary $\partial\Omega$ of the domain Ω. While calculating the directional cosines are substituted by the expressions

$$l = -\omega_{,1}; \qquad m = -\omega_{,2}, \tag{5.5}$$

where $\omega(x, y) = 0$ is the normalized equation of the boundary, that is, the function $\omega(x, y)$ satisfies conditions (4.5).

Let us remark that in case of clamped immovable edge the tangent displacement on the boundary is equal to zero; therefore contour integral $I_k(\vec{U}_i)$ in a functional (5.2) will vanish.

The descretization of the functional is carried out on a set of functions satisfying, at least, kinematics boundary conditions. In case of a clamped edge the structural formulas are

$$u_i = \omega P_u^i, \quad v_i = \omega P_v^i, \quad i = 1, 2, 3. \tag{5.6}$$

The indefinite components P_u^i, P_v^i, as in solving the task on eigenvalues, are expanded in a series of some complete set of functions. The coefficients of these expansions are founded from condition of a functional minimum (5.2), that is, as a result of solving the linear algebraic equations system.

The proposed investigation method of nonlinear free vibrations of orthotropic plates with an arbitrary shape was realized with "POLE-RL" program system [15, 16], designed at AN Podgorny Institute for Mechanical Engineering Problems of NAS of Ukraine under guidance of the academician Rvachev.

6. Numerical results

Example 6.1. The proposed algorithm was tested for simply supported and clamped square plate with immovable edge. The boundary conditions for functions u, v over the boundary were accepted as follows:

$$u = 0, \quad v = 0, \quad \forall (x, y) \in \partial\Omega. \tag{6.1}$$

In case of a clamped edge, the boundary conditions for deflection functions are

$$w = 0, \quad \frac{\partial w}{\partial n} = 0; \tag{6.2}$$

in case of a simply supported edge,

$$w = 0, \qquad M_n = 0. \tag{6.3}$$

Let the plates be manufactured of glass-epoxy and aragonite crystal materials (Table 6.1).

The obtained results for different materials and for isotropic plates are compared with similar results that were obtained earlier and presented in [3, 6, 9, 10, 17]. This comparison is presented in Tables 6.2 and 6.3.

Table 6.1

Material	E_2/E_1	G_{12}/E_2	$\nu_{12} = \nu_{21}\, E_2/E_1$
Glass epoxy (A)	3	0.5	0.25
Aragonite crystal (B)	0.543103	0.262931	0.23319

Table 6.2. Nonlinear free-vibration frequency ratio ω_N/ω_L for a simply supported isotropic square plate.

w/h	[3]	[6]	[10]	[17]	[9]	RFM
0.2	1.03	1.02	1.02	1.02	1.01	1.03
0.4	1.10	1.10	1.09	1.07	1.05	1.08
0.6	1.21	1.21	1.20	1.15	1.12	1.19
0.8	1.35	1.35	1.34	1.27	1.20	1.32
1.0	1.52	1.51	1.51	1.38	1.30	1.52

Table 6.3. Nonlinear free-vibration frequency ratio ω_N/ω_L for a simply supported orthotropic square plate.

w/h	Material A		Material B	
	[6]	RFM	[6]	RFM
0.2	1.03	1.03	1.03	1.03
0.4	1.11	1.10	1.10	1.09
0.6	1.23	1.23	1.21	1.20
0.8	1.39	1.41	1.36	1.35
1.0	1.57	1.65	1.52	1.55

Example 6.2. To illustrate the resources of the proposed method, the task of nonlinear vibrations of a complex planform plate (Figure 6.1) is considered.

Let us consider two cases (clamped and simply supported) of boundary conditions and also two types of materials. The elastic constants of these materials are presented in Table 6.1. To concretize the structural formulas (4.4), (4.10), and (5.6), the equation of domain boundary is constructed by R-functions. It is easy to check that the function $\omega(x,y)$ can be represented as

$$\omega(x,y) = (F_1 \wedge_0 F_2) \wedge_0 (F_3 \vee_0 F_4) \wedge_0 (F_5 \vee_0 F_6), \tag{6.4}$$

where the functions F_i $(i = 1,2,3,4,5,6)$ are defined by the following analytical expressions:

$$\begin{gathered} F_1 = \frac{a^2 - x^2}{2a} \geq 0, \qquad F_2 = \frac{b^2 - y^2}{2b} \geq 0, \\ F_3 = d_2 + y \geq 0, \qquad F_4 = c_2 + x \geq 0, \qquad F_5 = d_1 - y \geq 0, \qquad F_6 = c_1 - x \geq 0. \end{gathered} \tag{6.5}$$

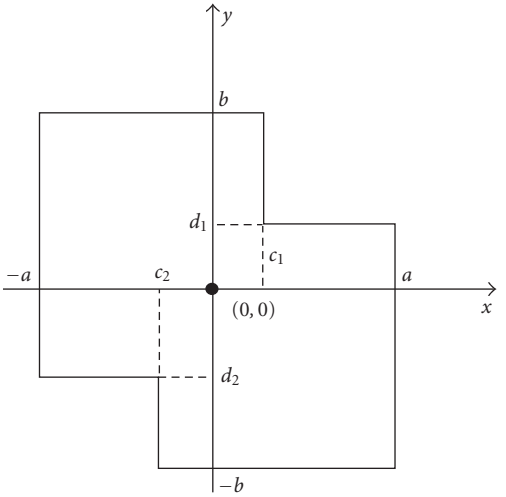

Figure 6.1

In the formula (6.4) the symbols $\wedge_0$, $\vee_0$ are signs of R-operations [12], defined according to the formulas

$$X \wedge_0 Y = X + Y - \sqrt{X^2 + Y^2}, \qquad X \vee_0 Y = X + Y + \sqrt{X^2 + Y^2}. \tag{6.6}$$

It is easy to check that the constructed function $\omega(x, y)$ satisfies conditions (4.5). For approximation of the indefinite components in the constructed structural formulas, it is possible to use a system of degree polynomials

$$1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^4, x^3y, \ldots. \tag{6.7}$$

In Table 6.4 the obtained results for linear fundamental frequencies are presented for the given plate at variation of the ratio $\alpha = c_1/2a = c_2/2a = d_1/2a = d_2/2a$ and a number of coordinate functions (n = NCF) approximating the deflection W in expression (4.13). If parameter $\alpha \to 1$, then the form of the plate shown in Figure 6.1 tends to square form. To research practical convergence of the obtained results, the different degree of approximating polynomials was chosen. It was established that for solving linear vibrations problem, the tenth degree of polynomials that corresponds to 66 of the coordinate functions for W, may be used. A further increase of coordinate functions number does not change the obtained results in the third sign after comma.

The amplitude-frequency dependence for the clamped orthotropic plates, which have the following relations of geometrical parameters: $b/a = 1$; $c_1/a = c_2/a = 1/2$; $d_1/a = d_2/a = 1/2$, is represented in Table 6.5.

The amplitude-frequency dependence of a simply supported plate with immovable plane edge for the same geometrical sizes of plate (Figure 6.1) is represented in Table 6.6.

To check the reliability of the result obtained for amplitude-frequency dependence of the plate with complex form, let us carry out series of calculations for this plate (Figure 6.1), for instance, manufactured from material B with clamped boundary condition,

Table 6.4. Convergence of a nondimensional linear frequency parameter $\Lambda = \omega_{11}^{0} \cdot a^2(\rho/E_2h^2)^{1/2}$ of fundamental mode for orthotropic plate with complex form (Figure 6.1).

Material	NCF	α			
		0.25	0.4	0.45	1(square)
Clamped					
A	55	11.294	9.212	9.174	9.171
	66	11.291	9.204	9.173	9.171
	91	11.291	9.204	9.173	9.171
B	55	15.437	12.370	12.322	12.320
	66	15.423	12.367	12.321	12.320
	91	15.423	12.367	12.321	12.319
Isotropic	55	13.697	10.935	10.891	10.889
	66	13.691	10.930	10.890	10.847
	91	13.691	10.930	10.890	10.847
Simply supported					
A	55	7.560	5.166	5.004	4.871
	66	7.488	5.158	4.997	4.871
	91	7.488	5.158	4.997	4.871
B	55	10.870	7.112	6.748	6.587
	66	10.792	7.101	6.740	6.587
	91	10.792	7.101	6.740	6.587
Isotropic	55	9.663	6.415	6.122	5.973
	66	9.594	6.405	6.116	5.973
	91	9.594	6.405	6.116	5.973

Table 6.5. Free-vibration frequency ratio ω_N/ω_L for a clamped orthotropic plate with a complex form (Figure 6.1).

w/h	Material A	Material B	Isotropic
0.2	1.004	1.003	1.003
0.4	1.018	1.011	1.011
0.6	1.037	1.024	1.026
0.8	1.065	1.043	1.045
1.0	1.100	1.066	1.070
1.2	1.142	1.094	1.099
1.4	1.189	1.126	1.133
1.6	1.241	1.161	1.171
1.8	1.297	1.201	1.212
2.0	1.357	1.243	1.257

when parameter $\alpha = c_1/2a = c_2/2a = d_1/2a = d_2/2a$ approaches to 1 and the given plate takes planform of square plate. Results of these investigations are presented in Table 6.7.

Example 6.3. The plate with the plan represented in Figure 6.2 is considered. The plate is under uniform load, which is changed in time by harmonic law.

Table 6.6. Nonlinear free-vibration frequency ratio ω_N/ω_L for a simply supported orthotropic plate (Figure 6.1).

w/h	Material A	Material B	Isotropic
0.2	1.011	1.008	1.008
0.4	1.043	1.031	1.032
0.6	1.095	1.069	1.071
0.8	1.164	1.121	1.123
1.0	1.246	1.184	1.187
1.2	1.340	1.256	1.261
1.4	1.444	1.336	1.343
1.6	1.554	1.424	1.432
1.8	1.671	1.516	1.526
2.0	1.793	1.614	1.625

Table 6.7. Nonlinear free-vibration frequency ratio ω_N/ω_L for a clamped orthotropic plates (Figure 6.1, Material B).

w/h	α				
	0.25	0.4	0.45	0.49	1
0.2	1.003	1.003	1.003	1.003	1.003
0.4	1.011	1.014	1.014	1.014	1.016
0.6	1.024	1.031	1.031	1.031	1.032
0.8	1.043	1.054	1.054	1.054	1.057
1.0	1.066	1.083	1.084	1.083	1.088
1.2	1.094	1.118	1.118	1.118	1.124
1.4	1.126	1.157	1.158	1.158	1.166
1.6	1.162	1.201	1.202	1.202	1.212
1.8	1.201	1.249	1.251	1.250	1.263
2.0	1.243	1.301	1.303	1.302	1.317

Let us consider three kinds of materials: glass epoxy, boron epoxy, and graphite epoxy. Physical elastic constants for these materials are presented in Table 6.8.

Calculation was carried out for two types of boundary conditions:

(a) clamped plate with movable edge:

$$w = 0, \qquad \frac{\partial w}{\partial n} = 0, \qquad \frac{\partial^2 \Phi}{\partial \tau^2} = 0, \qquad \frac{\partial^2 \Phi}{\partial n \partial \tau} = 0, \tag{6.8}$$

where Φ is force function [19].

(b) simply supported plate with movable edge:

$$W = 0, \qquad M_n = 0, \qquad \frac{\partial^2 \Phi}{\partial \tau^2} = 0, \qquad \frac{\partial^2 \Phi}{\partial n \partial \tau} = 0. \tag{6.9}$$

The function $\omega(x, y)$ is constructed by R-functions [12]:

$$\omega(x, y) = (f_3 \vee_0 f_4) \wedge_0 (f_1 \wedge_0 f_2) \wedge_0 (f_5 \wedge_0 f_6). \tag{6.10}$$

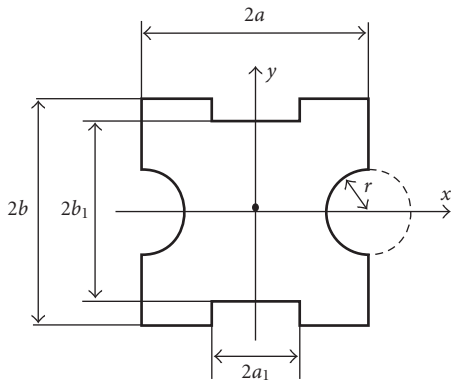

Figure 6.2. Planform of the plate.

Table 6.8

Material	E_1/E_2	G_{12}/E_2	$\nu_{12} = \nu_{21}\, E_2/E_1$
Glass epoxy	3	0.6	0.25
Boron epoxy	10	1/3	0.22
Graphite epoxy	40	0.5	0.25

Table 6.9. Linear frequency parameter, $\omega_{11}^0 \cdot a^2(\rho/E_2h^2)^{1/2}$, of fundamental mode.

Material	Clamped plate	Simply supported plate
Glass epoxy	21.2	14.6
Boron epoxy	32.3	18.7
Graphite epoxy	58.6	31.8
Isotropic	15.2	11.6

Functions f_i $(i = 1,2,\ldots,6)$ are determined by the following expressions:

$$f_1 = \frac{1}{2a}(a^2 - x^2), \qquad f_2 = \frac{1}{2b}(b^2 - y^2), \qquad f_3 = -\frac{1}{2a_1}(a_1^2 - x^2),$$
$$f_4 = \frac{1}{2b_1}(b_1^2 - y^2), \qquad f_5 = \frac{1}{2r}((x-a)^2 + y^2 - r^2), \tag{6.11}$$
$$f_6 = \frac{1}{2r}((x+a)^2 + y^2 - r^2).$$

There were chosen the following values of geometrical parameters of a plate: $b/a = 1$; $b_1/a = 0.375$; $a_1/a = 0.125$; $r/a = 0.125$. Values of the basic linear frequency for all kinds of material are presented in Table 6.9. The amplitude-frequency dependences for nonlinear free vibrations are presented in Figures 6.3 and 6.4. Numerical results were obtained using power polynomial approximation of the indefinite components. The approximate

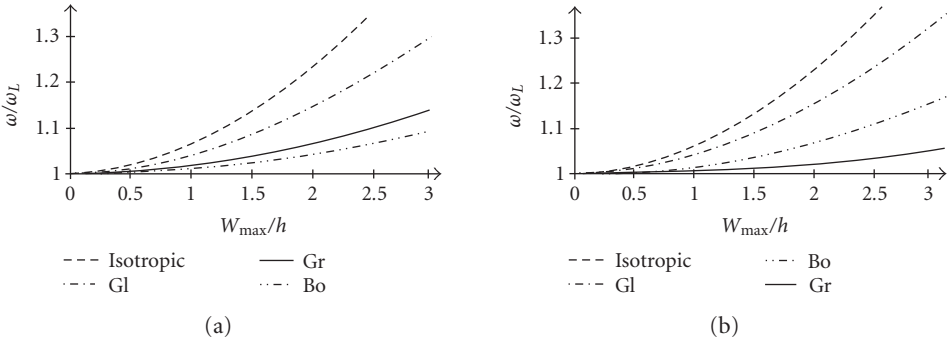

Figure 6.3. (a) Effect of elastic properties on nonlinear frequency of a clamped orthotropic plate. (b) Effect of elastic properties on nonlinear frequency of a simply supported orthotropic plate.

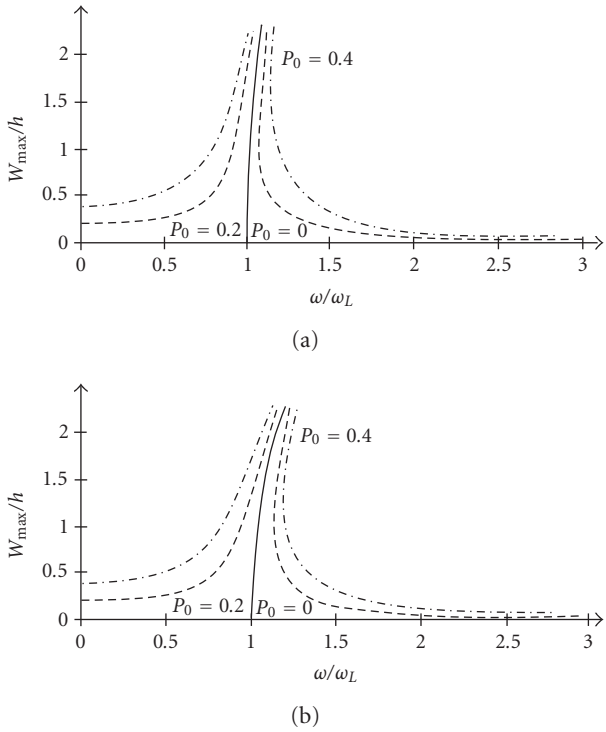

Figure 6.4. (a) Frequency response function for a clamped plate (graphite epoxy). (b)Frequency response function for a simply supported plate (glass epoxy).

polynomials were chosen up to 14 degrees which correspond to 45 terms of series. The symmetry of a problem was taken into account. Calculation of Ritz matrix elements was carried out by 10-dot Gauss's formulas.

7. Conclusion

In this work, the research method for forced nonlinear vibrations of an arbitrary form orthotropic plates was proposed. This approach is based on R-functions theory, variational methods, and Runge-Kutt method. The software "POLE-RL" is applied to obtain the numerical results. The investigations are carried out for plates of different shapes, boundary conditions, and materials. The amplitude-frequency dependences are obtained and presented by graphics. The obtained results for a square plate are compared with known results. This comparison confirms effectiveness and reliability of the proposed method and created a software.

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