BIFURCATIONS OF PLANAR SLIDING HOMOCLINICS

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We study bifurcations from sliding homoclinic solutions to bounded solutions on $\mathbb{R}$ for certain discontinuous planar systems under periodic perturbations. Sufficient conditions are derived for such perturbation problems.

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1. Introduction

We start from the planar discontinuous system

$$\begin{aligned}\dot{z} &= f_+(z) + \varepsilon g(z, t, \varepsilon) \quad \text{for } y > 1, \\ \dot{z} &= f_-(z) + \varepsilon g(z, t, \varepsilon) \quad \text{for } y < 1,\end{aligned}\tag{1.1}$$

where $z = (x, y) \in \mathbb{R}^2$, $f_{\pm}, g$ are C^3 -smooth, and g is 1-periodic in t . Here we set

$$q_{\pm}(z, t, \varepsilon) = f_{\pm}(z) + \varepsilon g(z, t, \varepsilon).\tag{1.2}$$

We suppose the following conditions:

- (i) $f_-(0) = 0$, and $Df_-(0)$ has no eigenvalues on the imaginary axis,
- (ii) there are two solutions $\gamma_-(s), \gamma_+(s)$ of $\dot{z} = f_-(z)$, $y \leq 1$ defined on $\mathbb{R}_- = (-\infty, 0]$, $\mathbb{R}_+ = [0, +\infty)$, respectively, such that $\lim_{s \rightarrow \pm\infty} \gamma_{\pm}(s) = 0$ and $\gamma_{\pm}(s) = (x_{\pm}(s), y_{\pm}(s))$ with $y_{\pm}(0) = 1, x_-(0) < x_+(0)$. Moreover, $f_{\pm}(z) = (f_{\pm 1}(z), f_{\pm 2}(z))$ with $f_{\pm 1}(x, 1) > 0, f_{+2}(x, 1) < 0$ for $x_-(0) \leq x \leq x_+(0)$. Furthermore, $f_{-2}(x, 1) > 0$ for $x_-(0) \leq x < x_+(0)$, $f_{-2}(x_+(0), 1) = 0$, and $\partial_x f_{-2}(x_+(0), 1) < 0$.

Assumptions (i) and (ii) mean that (1.1) for $\varepsilon = 0$ has a sliding homoclinic solution γ , created by $\gamma_{\pm}$, to a hyperbolic equilibrium 0. We are interested in the bifurcation of γ to bounded solutions on $\mathbb{R}$ of (1.1) under the perturbation $\varepsilon g(z, t, \varepsilon)$.

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The plan of the paper is as follows. In Section 2, we study (1.1) by using functional methods based on [4] along with the implicit function theorem [5]. In Section 3, we generalize results of Section 2 to systems with multiple discontinuous levels. Final Section 4 is devoted to a concrete system of piece-wise linear systems with periodic perturbations.

Sliding periodic solutions of discontinuous differential equations are investigated in [1–3] with both analytical and numerical methods. Qualitative properties of discontinuous systems are studied in [6]. Bifurcations for planar discontinuous ordinary differential systems with small periodic perturbations from homoclinic solutions transversally intersecting levels of discontinuity are studied in [7] to generalize the well-known Melnikov method for a smooth case [4] to a discontinuous one. We note that bifurcations from sliding homoclinic solutions, studied in this paper, are different to [4, 7].

2. Bifurcation result

In this section, we find conditions under which γ persists in (1.1) for $\varepsilon \neq 0$ small. For this purpose, we consider (1.1) as a system in $\mathbb{R}^3$ defined by

$$\begin{aligned} \dot{z} &= f_+(z) + \varepsilon g(z, t, \varepsilon) \quad \text{for } y > 1, \\ \dot{z} &= f_-(z) + \varepsilon g(z, t, \varepsilon) \quad \text{for } y < 1, \\ \dot{t} &= 1, \end{aligned} \tag{2.1}$$

while on $y = 1$ (cf. [1, 6]), we consider the system

$$\begin{aligned} \dot{x} &= \frac{q_{+2}(x, 1, t, \varepsilon)}{q_{+2}(x, 1, t, \varepsilon) - q_{-2}(x, 1, t, \varepsilon)} q_{+1}(x, 1, t, \varepsilon) \\ &+ \frac{q_{-2}(x, 1, t, \varepsilon)}{q_{-2}(x, 1, t, \varepsilon) - q_{+2}(x, 1, t, \varepsilon)} q_{-1}(x, 1, t, \varepsilon), \end{aligned} \tag{2.2}$$

where $q_{\pm} = (q_{\pm 1}, q_{\pm 2})$. We first study the system

$$\begin{aligned} \dot{z} &= f_-(z) + \varepsilon g(z, t, \varepsilon) \quad \text{for } y \leq 1, \\ \dot{t} &= 1, \quad y(0) = 1, \quad t(0) = \alpha, \quad s \leq 0. \end{aligned} \tag{2.3}$$

LEMMA 2.1. *For any ε small, there is a unique bounded solution $z_-(s, \varepsilon, \alpha)$ of (2.3) on $\mathbb{R}_-$, which is near to $\gamma_-(s)$.*

Proof. We consider the Banach space

$$X = \{v = (x(s), y(s)) \in C_b(\mathbb{R}_-, \mathbb{R}^2) \mid y(0) = 0\} \tag{2.4}$$

with the usual sup-norm $\|\cdot\|$. We put $z = \gamma_- + v$ into (2.3) to get

$$\begin{aligned} \dot{v} &= Df_-(\gamma_-(s))v + \{f_-(\gamma_-(s) + v) - f_-(\gamma_-(s)) - Df_-(\gamma_-(s))v\} + \varepsilon g(\gamma_-(s) + v, s + \alpha, \varepsilon), \\ v_2(0) &= 0, \end{aligned} \tag{2.5}$$

where $v = (v_1, v_2)$. Next, the system

$$\dot{v} = Df_-(\gamma_-(s))v \quad (2.6)$$

has an exponential dichotomy on $\mathbb{R}_-$ (cf. [4]), that is, there are positive constants K, a and a projection $P: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$\begin{aligned} \|V_-(s)PV_-(\theta)^{-1}\| &\leq Ke^{-a(s-\theta)} \quad \text{for } \theta \leq s \leq 0, \\ \|V_-(s)(\mathbb{I} - P)V_-(\theta)^{-1}\| &\leq Ke^{a(s-\theta)} \quad \text{for } s \leq \theta \leq 0, \end{aligned} \quad (2.7)$$

where $V_-(s)$, $V_-(0) = \mathbb{I}$ is the fundamental matrix solution of (2.6). Moreover, since $\dot{\gamma}_-(s)$ solves (2.6), and it is bounded on $\mathbb{R}_-$, and $\dot{\gamma}_-(0)$ is transversal to the x -axis, we can suppose (cf. [4]) that $\text{Im}(\mathbb{I} - P) = \mathbb{R}\dot{\gamma}_-(0)$ and $\text{Im}P$ is the x -axis. Then, (2.5) can be rewritten as a fixed point problem

$$v(s) = \int_{-\infty}^s V_-(s)PV_-(\theta)^{-1}h(\theta)d\theta - \int_s^0 V_-(s)(\mathbb{I} - P)V_-(\theta)^{-1}h(\theta)d\theta \quad (2.8)$$

on the Banach space X , where

$$h(\theta) = f_-(\gamma_-(\theta) + v(\theta)) - f_-(\gamma_-(\theta)) - Df_-(\gamma_-(\theta))v(\theta) + \varepsilon g(\gamma_-(\theta) + v(\theta), \theta + \alpha, \varepsilon). \quad (2.9)$$

Since

$$f_-(\gamma_-(\theta) + v) - f_-(\gamma_-(\theta)) - Df_-(\gamma_-(\theta))v = O(|v|^2), \quad (2.10)$$

for ε small, we can solve (2.8) by using the implicit function theorem to obtain a unique small solution $v(s, \alpha, \varepsilon)$ of (2.8), and so

$$z(s, \alpha, \varepsilon) = \gamma_-(s) + v(s, \alpha, \varepsilon) \quad (2.11)$$

solves (2.3). The proof is finished. $\square$

We put

$$\varphi_-(\alpha, \varepsilon) = x(0, \alpha, \varepsilon), \quad (2.12)$$

where

$$z(s, \alpha, \varepsilon) = (x(s, \alpha, \varepsilon), y(s, \alpha, \varepsilon)). \quad (2.13)$$

Clearly, $\varphi_-(\alpha, 0) = x_-(0)$. Next, we consider (2.2) with the initial condition

$$x(0) = \varphi_-(\alpha, \varepsilon). \quad (2.14)$$

If $h(x, s, \varepsilon)$ is the right-hand side of (2.2), then conditions (i) and (ii) imply that $h(x, s, \varepsilon) > 0$ for any $x_-(0) \leq x \leq x_+(0)$ and ε small. Then assumption (ii) gives the solvability of the equation

$$q_{-2}(x(s_+(\alpha, \varepsilon)), 1, s_+(\alpha, \varepsilon) + \alpha, \varepsilon) = 0 \quad (2.15)$$

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for the function $s_+(\alpha, \varepsilon) > 0$, where $x(s)$ solves (2.2) and (2.14). So, $s_+(\alpha, \varepsilon)$ is the time when the sliding motion of (2.2) is ending. We put

$$\varphi_+(\alpha, \varepsilon) = x(s_+(\alpha, \varepsilon)). \quad (2.16)$$

Finally, we consider the initial value problem

$$\begin{aligned} \dot{z} &= f_-(z) + \varepsilon g(z, t, \varepsilon) \quad \text{for } y \leq 1, \\ \dot{t} &= 1, \quad s \geq s_+(\alpha, \varepsilon), \\ z(s_+(\alpha, \varepsilon)) &= (\varphi_+(\alpha, \varepsilon), 1), \quad t(s_+(\alpha, \varepsilon)) = s_+(\alpha, \varepsilon) + \alpha. \end{aligned} \quad (2.17)$$

That is the initial value problem

$$\begin{aligned} \dot{z} &= f_-(z) + \varepsilon g(z, s + \alpha, \varepsilon) \quad \text{for } y \leq 1, \\ z(s_+(\alpha, \varepsilon)) &= (\varphi_+(\alpha, \varepsilon), 1), \quad s \geq s_+(\alpha, \varepsilon). \end{aligned} \quad (2.18)$$

We note that $\gamma_+(0) = (\varphi_+(\alpha, 0), 1)$ and we look for a solution z of (2.18) near to $\gamma_+(s - s_+(\alpha, \varepsilon)) = \omega_+(s)$. By taking

$$z(s) = \omega_+(s) + \varepsilon w(s) \quad (2.19)$$

in (2.18), we obtain

$$\begin{aligned} \dot{w} &= Df_-(\omega_+(s))w + \frac{1}{\varepsilon} \{f_-(\omega_+(s) + \varepsilon w) - f_-(\omega_+(s)) - Df_-(\omega_+(s))\varepsilon w\} \\ &\quad + g(\omega_+(s) + \varepsilon w, s + \alpha, \varepsilon), \quad s \geq s_+(\alpha, \varepsilon), \\ w(s_+(\alpha, \varepsilon)) &= (\psi_+(\alpha, \varepsilon), 0), \end{aligned} \quad (2.20)$$

where

$$\psi_+(\alpha, \varepsilon) = (\varphi_+(\alpha, \varepsilon) - \varphi_+(\alpha, 0))/\varepsilon. \quad (2.21)$$

By shifting the time $s \mapsto s + s_+(\alpha, \varepsilon)$, $s \geq 0$ in (2.20), we obtain

$$\begin{aligned} \dot{w} &= Df_-(\gamma_+(s))w + \frac{1}{\varepsilon} \{f_-(\gamma_+(s) + \varepsilon w) - f_-(\gamma_+(s)) - Df_-(\gamma_+(s))\varepsilon w\} \\ &\quad + g(\gamma_+(s) + \varepsilon w, s_+(\alpha, \varepsilon) + s + \alpha, \varepsilon), \quad s \geq 0, \\ w(0) &= (\psi_+(\alpha, \varepsilon), 0). \end{aligned} \quad (2.22)$$

We set

$$\eta(\alpha, \varepsilon) = (\psi_+(\alpha, \varepsilon), 0). \quad (2.23)$$

Now we study the problem

$$\begin{aligned} \dot{w} &= Df_-(\gamma_+(s))w + h(s), \\ w(0) &= u, \end{aligned} \quad (2.24)$$

for $h \in C_b(\mathbb{R}_+, \mathbb{R}^2)$ and $u = (u_1, u_2) \in \mathbb{R}^2$. The system

$$\dot{w} = Df_-(\gamma_+(s))w \quad (2.25)$$

has an exponential dichotomy on $\mathbb{R}_+$ (cf. [4]), that is, there are positive constants M, b and a projection $Q : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$\begin{aligned} \|V_+(s)QV_+(\theta)^{-1}\| &\leq Me^{-b(s-\theta)} \quad \text{for } 0 \leq \theta \leq s, \\ \|V_+(s)(\mathbb{I} - Q)V_+(\theta)^{-1}\| &\leq Me^{b(s-\theta)} \quad \text{for } 0 \leq s \leq \theta, \end{aligned} \quad (2.26)$$

where $V_+(s)$, $V_+(0) = \mathbb{I}$ is the fundamental matrix solution of (2.25). Moreover, since $\dot{\gamma}_+(s)$ solves (2.25) and it is bounded on $\mathbb{R}_+$, we can suppose (cf. [4]) that $\text{Im } Q = \mathbb{R}\dot{\gamma}_+(0)$ and $\text{Im}(\mathbb{I} - Q)$ is orthogonal to the line $\mathbb{R}\dot{\gamma}_+(0)$. On the other hand, condition (ii) implies that

$$\dot{\gamma}_+(0) = f_{-2}(x_+(0), y_+(0)) = f_{-2}(x_+(0), 1) = 0. \quad (2.27)$$

So,

$$\dot{\gamma}_+(0) = (\dot{x}_+(0), \dot{y}_+(0)) = (\dot{x}_+(0), 0). \quad (2.28)$$

Consequently, Q is the orthogonal projection onto the x -axis. Let $\Gamma = \dot{\gamma}_+(0)^\perp$ be a nonzero orthogonal vector onto $\dot{\gamma}_+(0)$. Now, for simplicity, we can take $\Gamma = (0, 1)$. So, $\text{Im}(\mathbb{I} - Q) = \mathbb{R}\Gamma$. We note that

$$\mu(t) = V_+^*(s)^{-1}\Gamma \quad (2.29)$$

is a basis of a space of bounded solutions on $\mathbb{R}_+$ of the adjoint system (cf. [4])

$$\dot{w} = -Df_-^*(\gamma_+(s))w. \quad (2.30)$$

We need the following result.

LEMMA 2.2. *Problem (2.24) has a bounded solution w on $\mathbb{R}_+$ if and only if*

$$\int_0^{+\infty} (h(s), \mu(s)) ds = -(\Gamma, u) = -u_2, \quad (2.31)$$

where $(\cdot, \cdot)$ is the usual scalar product on $\mathbb{R}^2$. Moreover, if condition (2.31) holds, then problem (2.24) has a unique bounded solution $w = w(u, h)$ on $\mathbb{R}_+$. Furthermore, there is a constant $c > 0$ such that

$$\|w(u, h)\| \leq c(\|h\| + |u|), \quad (2.32)$$

where $\|\cdot\|$ is the sup-norm on $Y = C_b(\mathbb{R}_+, \mathbb{R}^2)$ and $|\cdot|$ corresponds to $(\cdot, \cdot)$.

Proof. A general form of a bounded solution of equation

$$\dot{w} = Df_- (\gamma_+(s))w + h(s) \quad (2.33)$$

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on $\mathbb{R}_+$ is given by

$$w(s) = c\dot{\gamma}_+(s) + \int_0^s V_+(s)QV_+(\theta)^{-1}h(\theta)d\theta - \int_s^{+\infty} V_+(s)(\mathbb{I} - Q)V_+(\theta)^{-1}h(\theta)d\theta. \quad (2.34)$$

Then using the initial condition $w(0) = u$, we get the equation

$$u = c\dot{\gamma}_+(0) - \int_0^{+\infty} (\mathbb{I} - Q)V_+(\theta)^{-1}h(\theta)d\theta, \quad (2.35)$$

which implies

$$u_2 = (u, \Gamma) = - \int_0^{+\infty} (V_+(s)^{-1}h(s), \Gamma)ds = - \int_0^{+\infty} (h(s), V_+^*(s)^{-1}\Gamma)ds = - \int_0^{+\infty} (h(s), \mu(s))ds. \quad (2.36)$$

So, (2.31) is proved. On the other hand, if (2.31) holds, then (2.35) gives

$$u_1 = c\dot{\gamma}_+(0). \quad (2.37)$$

Consequently, the unique bounded solution of (2.24) on $\mathbb{R}_+$ is given by

$$w(s) = \frac{u_1}{\dot{\gamma}_+(0)}\dot{\gamma}_+(s) + \int_0^s V_+(s)QV_+(\theta)^{-1}h(\theta)d\theta - \int_s^{+\infty} V_+(s)(\mathbb{I} - Q)V_+(\theta)^{-1}h(\theta)d\theta. \quad (2.38)$$

Then, (2.32) follows directly from (2.38). The proof is finished. $\square$

Let $S : Y \rightarrow Y$ be a projection defined by

$$Sh = h(s) - \int_0^{+\infty} \left[\left(h(\theta), \frac{\mu(\theta)}{\|\mu\|_2^2} \right) d\theta \right] \mu(s), \quad (2.39)$$

where $\|\mu\|_2^2 = \int_0^{+\infty} \mu(\theta)^2 d\theta$. Then, (2.22) is splitted as follows

$$\begin{aligned} \dot{w} = Df_-(\gamma_+(s))w + S \left[\frac{1}{\varepsilon} \{ f_-(\gamma_+ + \varepsilon w) - f_-(\gamma_+) - Df_-(\gamma_+)\varepsilon w \} \right. \\ \left. + g(\gamma_+ + \varepsilon w, s_+(\alpha, \varepsilon) + s + \alpha, \varepsilon) \right], \end{aligned} \quad (2.40)$$

$$w(0) = (\psi_+(\alpha, \varepsilon), 0),$$

$$\begin{aligned} \int_0^{+\infty} \left(\frac{1}{\varepsilon} \{ f_-(\gamma_+(s) + \varepsilon w(s)) - f_-(\gamma_+(s)) - Df_-(\gamma_+(s))\varepsilon w(s) \} \right. \\ \left. + g(\gamma_+(s) + \varepsilon w(s), s_+(\alpha, \varepsilon) + s + \alpha, \varepsilon), \mu(s) \right) ds = 0. \end{aligned} \quad (2.41)$$

By using Lemma 2.2 together with the implicit function theorem, we can solve (2.40) to obtain its solution

$$w = w(\alpha, \varepsilon, s). \quad (2.42)$$

Then, by plugging it into (2.41), we arrive at a bifurcation equation

$$B(\alpha, \varepsilon) = \int_0^{+\infty} \left(\frac{1}{\varepsilon} \{ f_-(\gamma_+(s) + \varepsilon w(\alpha, \varepsilon, s)) - f_-(\gamma_+(s)) - Df_-(\gamma_+(s)) \varepsilon w(\alpha, \varepsilon, s) \} \right. \\ \left. + g(\gamma_+(s) + \varepsilon w(\alpha, \varepsilon, s), s_+(\alpha, \varepsilon) + s + \alpha, \varepsilon), \mu(s) \right) ds = 0. \quad (2.43)$$

We have

$$\bar{M}(\alpha) = B(\alpha, 0) = \int_0^{+\infty} (g(\gamma_+(s), s_+(\alpha, 0) + s + \alpha, 0), \mu(s)) ds = 0. \quad (2.44)$$

Any simple root α_0 of $\bar{M}(\alpha)$; that is, $\bar{M}(\alpha_0) = 0$ and $\bar{M}'(\alpha_0) \neq 0$, gives the solvability of $B(\alpha, \varepsilon) = 0$ with respect to $\alpha = \alpha(\varepsilon)$ for any ε small with $\alpha(0) = \alpha_0$.

On the other hand, from the definition of function $s_+(\alpha, \varepsilon)$ in (2.15), we see that $\partial_{\alpha} s_+(\alpha, 0) = 0$. So, simple roots of $\bar{M}(\alpha)$ are in one-to-one correspondence with simple roots of the function

$$M(\beta) = \int_0^{+\infty} (g(\gamma_+(s), \beta + s, 0), \mu(s)) ds. \quad (2.45)$$

Summarizing we arrive at the following result.

THEOREM 2.3. *If there is a simple root β_0 of $M(\beta)$, that is, it holds that $M(\beta_0) = 0$ and $M'(\beta_0) \neq 0$, then homoclinic solution γ bifurcates to a bounded solution on $\mathbb{R}$ of (1.1) with $\varepsilon \neq 0$ small.*

3. Generalization to multiple discontinuous systems

The above approach to (1.1) can be generalized to cases when homoclinic orbit $\gamma(s)$ transversally crosses another curve of discontinuity. For simplicity, we suppose that such a discontinuity in (1.1) occurs at the level $y = 1/2$, that is, in this section, we deal with the system

$$\begin{aligned} \dot{z} &= f_+(z) + \varepsilon g(z, t, \varepsilon) & \text{for } y > 1, \\ \dot{z} &= f_-(z) + \varepsilon g(z, t, \varepsilon) & \text{for } \frac{1}{2} < y < 1, \\ \dot{z} &= F(z) + \varepsilon g(z, t, \varepsilon) & \text{for } y < \frac{1}{2}, \end{aligned} \quad (3.1)$$

where $z = (x, y) \in \mathbb{R}^2$, $f_{\pm}$, F , g are C^3 -smooth and g is 1-periodic in t . We suppose the following conditions:

- (a) $F(0) = 0$ and $DF(0)$ has no eigenvalues on the imaginary axis,
- (b) there are two solutions η_-, η_+ of $\dot{z} = f_{\pm}(z)$, $1/2 \leq y \leq 1$ defined on $[a_-, 0]$, $[0, a_+]$, $a_- < 0 < a_+$, respectively, such that $\eta_{\pm}(s) = (\tilde{x}_{\pm}(s), \tilde{y}_{\pm}(s))$ with $\tilde{y}_{\pm}(0) = 1$, $\tilde{y}_{\pm}(a_{\pm}) = 1/2$, $\tilde{x}_-(0) < \tilde{x}_+(0)$, $\tilde{x}_-(a_-) < \tilde{x}_+(a_+)$. Moreover, $f_{\pm}(z) = (f_{\pm 1}(z), f_{\pm 2}(z))$ with $f_{\pm 1}(x, 1) > 0$, $f_{\pm 2}(x, 1) < 0$ for $\tilde{x}_-(0) \leq x \leq \tilde{x}_+(0)$. Furthermore, $f_{-2}(x, 1) > 0$ for $\tilde{x}_-(0) \leq x < \tilde{x}_+(0)$, $f_{-2}(\tilde{x}_+(0), 1) = 0$, and $\partial_x f_{-2}(\tilde{x}_+(0), 1) < 0$. Finally, we suppose that $f_{-2}(\eta_-(a_-)) > 0$ and $f_{-2}(\eta_+(a_+)) < 0$,

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- (c) there are two solutions $\tilde{y}_-(s), \tilde{y}_+(s)$ of $\dot{z} = F(z)$, $y \leq 1/2$ defined on $\mathbb{R}_- = (-\infty, 0]$, $\mathbb{R}_+ = [0, +\infty)$, respectively, such that $\lim_{s \rightarrow \pm\infty} \tilde{y}_{\pm}(s) = 0$ and $\tilde{y}_{\pm}(0) = \eta_{\pm}(a_{\pm})$. Moreover, $F(z) = (F_1(z), F_2(z))$ with $F_2(\tilde{y}_-(0)) > 0$ and $F_2(\tilde{y}_+(0)) < 0$.

Again, assumptions (a), (b), and (c) imply that (3.1) for $\varepsilon = 0$ has a sliding homoclinic solution $\tilde{y}$, created by $\eta_{\pm}$ and $\tilde{y}_{\pm}$, to a hyperbolic equilibrium 0. We study in this section bifurcation of $\tilde{y}$ in system (3.1) for $\varepsilon \neq 0$ small. We can directly follow a method of Section 2. We first solve the equation

$$q_{-2}(\tilde{\varphi}_+(\alpha, \varepsilon), 1, \alpha, \varepsilon) = 0. \quad (3.2)$$

Since

$$\begin{aligned} q_{-2}(\tilde{x}_+(0), 1, \alpha, 0) &= f_{-2}(\tilde{x}_+(0), 1) = 0, \\ \partial_x q_{-2}(\tilde{x}_+(0), 1, \alpha, 0) &= \partial_x f_{-2}(\tilde{x}_+(0), 1) \neq 0, \end{aligned} \quad (3.3)$$

we can solve (3.2) with $\tilde{\varphi}_+(\alpha, 0) = \tilde{x}_+(0)$. Next, we consider the initial value problem

$$\begin{aligned} \dot{z} &= f_-(z) + \varepsilon g(z, t, \varepsilon) \quad \text{for } \frac{1}{2} \leq y \leq 1, \\ \dot{t} &= 1, \quad s \geq 0, \\ z(0) &= (\tilde{\varphi}_+(\alpha, \varepsilon), 1), \quad t(0) = \alpha, \end{aligned} \quad (3.4)$$

which has a unique solution

$$\tilde{z}(s, \alpha, \varepsilon) = (\tilde{x}(s, \alpha, \varepsilon), \tilde{y}(s, \alpha, \varepsilon)). \quad (3.5)$$

Then condition (b) implies that there is the smallest time $\tilde{s}_+(\alpha, \varepsilon)$ such that

$$\tilde{y}(\tilde{s}_+(\alpha, \varepsilon), \alpha, \varepsilon) = \frac{1}{2}. \quad (3.6)$$

So, $\tilde{s}_+(\alpha, \varepsilon)$ is the first hitting time for the level $y = 1/2$ of the solution of (3.4). We set

$$\xi(\alpha, \varepsilon) = \tilde{x}(\tilde{s}_+(\alpha, \varepsilon), \alpha, \varepsilon). \quad (3.7)$$

Consequently, in order to study the bifurcation of $\tilde{y}$, we need to show that the point $(\xi(\alpha, \varepsilon), 1/2)$ lies on the stable manifold of a unique small 1-periodic solution of (3.1). So we consider the initial value problem

$$\begin{aligned} \dot{z} &= F(z) + \varepsilon g(z, t, \varepsilon), \\ \dot{t} &= 1, \\ z(\tilde{s}_+(\alpha, \varepsilon)) &= \left(\xi(\alpha, \varepsilon), \frac{1}{2} \right), \quad t(\tilde{s}_+(\alpha, \varepsilon)) = \tilde{s}_+(\alpha, \varepsilon) + \alpha, \end{aligned} \quad (3.8)$$

that is the initial value problem

$$\begin{aligned} \dot{z} &= F(z) + \varepsilon g(z, \tilde{s}_+(\alpha, \varepsilon) + s + \alpha, \varepsilon), \\ z(0) &= \left(\xi(\alpha, \varepsilon), \frac{1}{2} \right), \quad s \geq 0. \end{aligned} \quad (3.9)$$

We note $\tilde{\gamma}_+(0) = (\xi(\alpha, 0), 1/2)$. By taking

$$z(s) = \tilde{\gamma}_+(s) + \varepsilon w(s) \quad (3.10)$$

in (3.9), we get

$$\begin{aligned} \dot{w} &= DF(\tilde{\gamma}_+(s))w + \frac{1}{\varepsilon} \{F(\tilde{\gamma}_+(s) + \varepsilon w) - F(\tilde{\gamma}_+(s)) - DF(\tilde{\gamma}_+(s))\varepsilon w\} \\ &\quad + g(\tilde{\gamma}_+(s) + \varepsilon w, \tilde{s}_+(\alpha, \varepsilon) + s + \alpha, \varepsilon), \quad s \geq 0, \\ w(0) &= (\tilde{\psi}_+(\alpha, \varepsilon), 0) = \Psi_+(\alpha, \varepsilon), \end{aligned} \quad (3.11)$$

where

$$\tilde{\psi}_+(\alpha, \varepsilon) = (\xi(\alpha, \varepsilon) - \xi(\alpha, 0))/\varepsilon. \quad (3.12)$$

Now we can repeat the above arguments of (2.22) to solve (3.11). So we again take

$$\Gamma = \dot{\tilde{\gamma}}_+(0)^\perp = (\dot{\tilde{\gamma}}_{+2}(0), -\dot{\tilde{\gamma}}_{+1}(0)). \quad (3.13)$$

The statement of Lemma 2.2 changes as follows.

LEMMA 3.1. *Problem*

$$\begin{aligned} \dot{w} &= DF(\tilde{\gamma}_+(s))w + h, \\ w(0) &= u \end{aligned} \quad (3.14)$$

has a bounded solution w on $\mathbb{R}_+$ for a $h \in C_b(\mathbb{R}_+, \mathbb{R}^2)$ if and only if

$$\int_0^{+\infty} (h(s), \tilde{\mu}(s)) ds + (\dot{\tilde{\gamma}}_+(0)^\perp, u) = 0. \quad (3.15)$$

Moreover, if condition (3.15) holds, then problem (3.14) has a unique bounded solution $w = \tilde{w}(u, h)$ on $\mathbb{R}_+$. Furthermore, there is a constant $\tilde{c} > 0$ such that

$$\|\tilde{w}(u, h)\| \leq \tilde{c}(\|h\| + |u|). \quad (3.16)$$

Here, $\tilde{\mu}$ is a bounded solution on $\mathbb{R}_+$ of the adjoint linear equation

$$\dot{w} = -DF(\tilde{\gamma}_+(s))^* w \quad (3.17)$$

with $w(0) = \Gamma$.

Condition (3.15) yields that instead of projection S from Section 2, we take a mapping $\tilde{S}: \mathbb{R}^2 \times Y \rightarrow Y$ defined by

$$\tilde{S}(u)h = h - \int_0^{+\infty} \left[\left(h(\theta), \frac{\tilde{\mu}(\theta)}{\|\tilde{\mu}\|_2^2} \right) d\theta \right] \tilde{\mu} - (\dot{\tilde{\gamma}}_+(0)^\perp, u) \frac{\tilde{\mu}}{\|\tilde{\mu}\|_2^2}. \quad (3.18)$$

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Then we have

$$\int_0^{+\infty} (\tilde{S}(u)h(s), \tilde{\mu}(s))ds + (\dot{\tilde{\gamma}}_+(0)^\perp, u) = 0. \quad (3.19)$$

So we split (3.11) as follows:

$$\begin{aligned} \dot{w} = DF(\tilde{\gamma}_+(s))w + \tilde{S}(\Psi_+(\alpha, \varepsilon)) \left[\frac{1}{\varepsilon} \{F(\tilde{\gamma}_+(s) + \varepsilon w) - F(\tilde{\gamma}_+(s)) - DF(\tilde{\gamma}_+(s))\varepsilon w\} \right. \\ \left. + g(\tilde{\gamma}_+(s) + \varepsilon w, \tilde{s}_+(\alpha, \varepsilon) + s + \alpha, \varepsilon) \right], \end{aligned} \quad (3.20)$$

$$w(0) = \Psi_+(\alpha, \varepsilon),$$

$$\begin{aligned} \int_0^{+\infty} \left(\frac{1}{\varepsilon} \{F(\tilde{\gamma}_+(s) + \varepsilon w) - F(\tilde{\gamma}_+(s)) - DF(\tilde{\gamma}_+(s))\varepsilon w\} \right. \\ \left. + g(\tilde{\gamma}_+(s) + \varepsilon w, \tilde{s}_+(\alpha, \varepsilon) + s + \alpha, \varepsilon), \tilde{\mu}(s) \right) ds \\ + (\dot{\tilde{\gamma}}_+(0)^\perp, \Psi_+(\alpha, \varepsilon)) = 0. \end{aligned} \quad (3.21)$$

By using Lemma 3.1, we can solve (3.20) to obtain its solution

$$w = \tilde{w}(\alpha, \varepsilon, s). \quad (3.22)$$

Then by inserting it into (3.21), we arrive at a bifurcation equation

$$\begin{aligned} \tilde{B}(\alpha, \varepsilon) = \int_0^{+\infty} \left(\frac{1}{\varepsilon} \{F(\tilde{\gamma}_+(s) + \varepsilon \tilde{w}(\alpha, \varepsilon, s)) - F(\tilde{\gamma}_+(s)) - DF(\tilde{\gamma}_+(s))\varepsilon \tilde{w}(\alpha, \varepsilon, s)\} \right. \\ \left. + g(\tilde{\gamma}_+(s) + \varepsilon \tilde{w}(\alpha, \varepsilon, s), \tilde{s}_+(\alpha, \varepsilon) + s + \alpha, \varepsilon), \tilde{\mu}(s) \right) ds \\ + (\dot{\tilde{\gamma}}_+(0)^\perp, \Psi_+(\alpha, \varepsilon)) = 0. \end{aligned} \quad (3.23)$$

We have

$$\tilde{M}(\alpha) = \tilde{B}(\alpha, 0) = \int_0^{+\infty} (g(\gamma_+(s), a_+ + s + \alpha, 0), \tilde{\mu}(s)) + \dot{\gamma}_{+2}(0)\tilde{\psi}_+(\alpha, 0) = 0, \quad (3.24)$$

where we use that $\tilde{s}_+(\alpha, 0) = a_+$ and

$$\frac{1}{\varepsilon} \{F(\tilde{\gamma}_+(s) + \varepsilon \tilde{w}(\alpha, \varepsilon, s)) - F(\tilde{\gamma}_+(s)) - DF(\tilde{\gamma}_+(s))\varepsilon \tilde{w}(\alpha, \varepsilon, s)\} = O(\varepsilon). \quad (3.25)$$

Any simple root α_0 of $\tilde{M}(\alpha)$ gives the solvability of $\tilde{B}(\alpha, \varepsilon) = 0$ with respect to $\alpha = \tilde{\alpha}(\varepsilon)$ for any ε small with $\tilde{\alpha}(0) = \alpha_0$.

Furthermore, from (3.6), we get

$$f_{-2}(\eta_+(a_+))\partial_\varepsilon \tilde{s}_+(\alpha, 0) + \partial_\varepsilon \tilde{\gamma}(a_+, \alpha, 0) = 0, \quad (3.26)$$

while (3.7) and (3.12) give

$$\tilde{\psi}_+(\alpha, 0) = \partial_\varepsilon \xi(\alpha, 0) = f_{-1}(\eta_+(a_+))\partial_\varepsilon \tilde{s}_+(\alpha, 0) + \partial_\varepsilon \tilde{x}(a_+, \alpha, 0), \quad (3.27)$$

which altogether imply

$$\tilde{\psi}_+(\alpha, 0) = -f_{-1}(\eta_+(a_+)) \frac{\partial_\varepsilon \tilde{y}(a_+, \alpha, 0)}{f_{-2}(\eta_+(a_+))} + \partial_\varepsilon \tilde{x}(a_+, \alpha, 0). \quad (3.28)$$

Next, we derive from (3.4) for

$$w(s) = \partial_\varepsilon \tilde{z}(a_+, \alpha, 0) = (\partial_\varepsilon \tilde{x}(a_+, \alpha, 0), \partial_\varepsilon \tilde{y}(a_+, \alpha, 0)) \quad (3.29)$$

the linear variational initial value problem

$$\begin{aligned} \dot{w} &= Df_-(\eta_+(s))w + g(\eta_+(s), s + \alpha, 0), \\ w(0) &= (\partial_\varepsilon \tilde{\varphi}_+(\alpha, 0), 0). \end{aligned} \quad (3.30)$$

But (3.2) implies

$$\partial_x f_{-2}(\eta_+(0)) \partial_\varepsilon \tilde{\varphi}_+(\alpha, 0) + g_2(\eta_+(0), \alpha, 0) = 0. \quad (3.31)$$

So instead of (3.30), we consider the linear initial value problem

$$\begin{aligned} \dot{w} &= Df_-(\eta_+(s))w + g(\eta_+(s), s + \alpha, 0), \\ w(0) &= \left(-\frac{g_2(\eta_+(0), \alpha, 0)}{\partial_x f_{-2}(\eta_+(0))}, 0 \right). \end{aligned} \quad (3.32)$$

Summarizing we arrive at the following result.

THEOREM 3.2. *Let function $\tilde{M}$ be given by (3.24) along with formulas (3.28), (3.29), and (3.32). If there is a simple root of $\tilde{M}$, then homoclinic solution $\tilde{y}$ bifurcates to a bounded solution on $\mathbb{R}$ of (3.1) with $\varepsilon \neq 0$ small.*

4. Example

We present in this section an illustrative example. Let a_+ be the unique (positive) solution of the equation

$$e^{a_+}(1 - a_+) = \frac{1}{2}. \quad (4.1)$$

We note that $a_+ \sim 0.768039$. Then we set

$$a = e^{a_+}(2 - a_+) \sim 2.65554. \quad (4.2)$$

In this section, we consider system (3.1) with

$$\begin{aligned} f_+(z) &= \begin{cases} \dot{x} = y, \\ \dot{y} = x - 3y, \end{cases} & f_-(z) &= \begin{cases} \dot{x} = y, \\ \dot{y} = 2y - x, \end{cases} \\ F(z) &= \begin{cases} \dot{x} = -2ay, \\ \dot{y} = -\frac{1}{2a}x, \end{cases} & g(x, t, \varepsilon) &= \cos t \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned} \quad (4.3)$$

It is not difficult to see that now we have

$$\begin{aligned} \eta_+(s) &= \begin{cases} e^s(2-s), \\ e^s(1-s), \end{cases} & \tilde{\gamma}_+(s) &= \begin{cases} e^{-s}a, \\ e^{-s}/2, \end{cases} \\ \tilde{\gamma}_-(s) &= \begin{cases} -e^s a, \\ e^s/2, \end{cases} & \eta_-(s) &= \begin{cases} -ae^{s-a_-} + \left(\frac{1}{2} + a\right)e^{s-a_-}(s-a_-), \\ \frac{1}{2}e^{s-a_-} + \left(\frac{1}{2} + a\right)e^{s-a_-}(s-a_-), \end{cases} \end{aligned} \quad (4.4)$$

where $a_- \sim -0.122043$ is the unique (negative) solution of the equation

$$e^{a_-} + \left(\frac{1}{2} + a\right)a_- = \frac{1}{2}. \quad (4.5)$$

We note that $(a, 1/2) = \eta_+(a_+)$ and system (3.32) has now the form

$$\begin{aligned} \dot{w} &= \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} w + \cos(s + \alpha) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ w(0) &= (\cos \alpha, 0). \end{aligned} \quad (4.6)$$

After some computations, function (3.24) has now the form

$$\begin{aligned} \widetilde{M}(\alpha) &= \frac{a}{2} (\cos(a_+ + \alpha) - \sin(a_+ + \alpha)), \\ &+ \frac{1}{4(1-a)} w_2(a_+) - \frac{1}{2} w_1(a_+), \end{aligned} \quad (4.7)$$

where $w(s) = (w_1(s), w_2(s))$ solves (4.6), that is, we have

$$\begin{aligned} w_1(a_+) &= \left(\cos \alpha + \frac{1}{2} \sin \alpha \right) e^{a_+} - \frac{1}{2} (\cos \alpha + \sin \alpha) e^{a_+} a_+ - \frac{1}{2} \sin(a_+ + \alpha), \\ w_2(a_+) &= \frac{\cos \alpha}{2} e^{a_+} - \frac{1}{2} (\cos \alpha + \sin \alpha) e^{a_+} a_+ - \frac{1}{2} \cos(a_+ + \alpha). \end{aligned} \quad (4.8)$$

Then, (4.7) takes the form

$$\widetilde{M}(\alpha) = -0.441052 \cos \alpha - 1.7501 \sin \alpha. \quad (4.9)$$

Function (4.9) has two different simple roots over the period 2π . By applying Theorem 3.2, we get the existence of two bounded solutions of (3.1) with (4.3) near to $\tilde{\gamma}$, which is homoclinic to a small hyperbolic 2π -periodic solution of (3.1) with (4.3).

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