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Research Article

An Approximate Solution for Flow between Two Disks Rotating about Distinct Axes at Different Speeds

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The flow of a linearly viscous fluid between two disks rotating about two distinct vertical axes is studied. An approximate analytical solution is obtained by taking into account the case of rotation with a small angular velocity difference. It is shown how the velocity components depend on the position, the Reynolds number, the eccentricity, the ratio of angular speeds of the disks, and the parameters satisfying the conditions $u = 0$ and $v = 0$ in midplane.

1. Introduction

Kármán [1] introduced an ingenious similarity transformation to study the axisymmetric flow induced by a single rotating disk. Batchelor [2] showed that this transformation can be used even when the fluid is confined between two parallel disks rotating about a common axis at different speeds. The solutions that are not axially symmetric were considered by Berker [3]. He established a one-parameter family of solutions for the flow between two disks rotating about a common axis with the same angular velocity. Later, Rajagopal [4] obtained asymmetric solutions for the flow due to porous disks rotating with equal angular velocity about a common axis. Parter and Rajagopal [5] studied Berker's problem in the case of rotation at different speeds and rigorously proved that there is a one-parameter family of solutions when the disks rotate about a common axis or distinct axes. Lai et al. [6] obtained a numerical solution for the asymmetric flows belonging to the equations established by Parter and Rajagopal [5]. Later, Lai et al. [7] presented solutions that lack symmetry for the flow in the semi-infinite interval above a single rotating

disk. Szeri et al. [8] investigated asymmetric flows above a rotating disk with uniform suction.

Flow of non-Newtonian fluids between rotating disks has also drawn attention in view of its applications in engineering practice. Maxwell and Chartoff [9] claimed that it is possible to determine the complex dynamic viscosity of a viscoelastic fluid if an instrument consisting of two parallel disks rotating with the same angular velocity about two distinct axes normal to the disks is used. In this domain, Abbott and Walters [10] obtained an exact solution for the flow of the Navier-Stokes fluid. In the case of a viscoelastic fluid, they also carried out a perturbation analysis by expanding in a power series in the distance between the axes of rotation. Rajagopal and Gupta [11] studied the possibility of existence of asymmetric solutions for the flow of a second-grade fluid between disks rotating about a common axis with the same speed. Rajagopal [12] showed that the motion represented by Berker [3] is one with constant stretch history. Rajagopal and Wineman [13] extended Berker's work [3] to the case of a special subclass of the K-BKZ. Motivated by the work of Parter and Rajagopal [5], Huilgol and Rajagopal [14] derived the equations of motion in the case of an Oldroyd-B fluid. Rajagopal [15] discussed the existence of solutions that do not possess axial symmetry for viscoelastic fluids in the case of rotation about a common axis. He also took into account the flow in an orthogonal rheometer and then discussed the flow produced by the rotations about a common axis and distinct axes when two disks have different speeds. Later, Rajagopal [16] reviewed the articles that study symmetric and asymmetric solutions for both a linearly viscous fluid and viscoelastic fluids, and discussed questions that remain unanswered. For a discussion about this subject, we also refer the reader to the book by Truesdell and Rajagopal [17].

The velocity field employed by Abbott and Walters [10] for the analysis of a viscoelastic fluid was adapted to the problem of flow between disks rotating about noncoaxial axes at different speeds by Knight [18]. Abbott and Walters considered that the components of translational velocity are related to each other. Knight took Abbott and Walters' velocity field to be a basis and obtained a full numerical solution. By assuming that the inertia effects are small, he also found an approximate analytical solution. Later, several authors took into account the perturbation procedure used by Knight and appliedit to their own problems in order to obtain approximate analytical solutions. Banerjee and Borkakati [19] studied the heat transfer characteristics of the flow when the disks are maintained at different temperatures. A. R. Rao and P. R. Rao [20] investigated the flow induced under the application of a uniform magnetic field in the axial direction. P. R. Rao and A. R. Rao [21] studied the influence of heat transfer under the application of a magnetic field. P. R. Rao and A. R. Rao [22] examined the flow between two torsionally oscillating disks with the same frequency. Rao [23] studied the flow between disks performing torsional oscillations with the same frequency in the presence of a uniform axial magnetic field.

In this paper, the flow of a linearly viscous fluid between two disks rotating with a small speed difference about distinct axes is investigated. In practice, there may be a small difference between the angular velocities even when the disks are forced to rotate with the same angular velocity. This view motivates us to examine this different-speed problem.

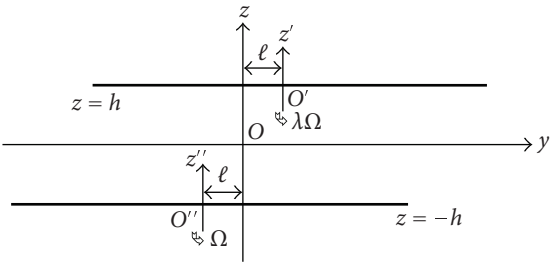

Figure 2.1. Flow geometry.

Following Parter and Rajagopal [5] and Lai et al. [6], an approximate analytical solution is obtained by employing a perturbation method. The influence of the parameters controlling the flow is carefully examined.

2. Basic equations

Let us consider an incompressible linearly viscous fluid between two disks rotating about noncoincident axes. The lower and upper disks located at $z = \mp h$ rotate with the angular velocities $\Omega_l = \Omega$ and $\Omega_u = \lambda\Omega$ about the axes through the points $O''(0, -\ell, -h)$ and $O'(0, \ell, h)$, respectively (see Figure 2.1). Thus, the appropriate boundary conditions are

$$u = -\lambda\Omega(y - \ell), \quad v = \lambda\Omega x, \quad w = 0 \quad \text{at } z = h, \tag{2.1a}$$

$$u = -\Omega(y + \ell), \quad v = \Omega x, \quad w = 0 \quad \text{at } z = -h, \tag{2.1b}$$

where u, v, w represent the velocity components along the x, y, z-directions. In the light of the above boundary conditions, we seek solutions for the velocity field of the form

$$u = \Omega x F(\zeta) - \Omega y G(\zeta) + \Omega h f(\zeta), \tag{2.2a}$$

$$v = \Omega x G(\zeta) + \Omega y F(\zeta) + \Omega h g(\zeta), \tag{2.2b}$$

$$w = \Omega h H(\zeta), \tag{2.2c}$$

where $\zeta = z/h$. Using (2.1a)-(2.1b) and (2.2a)–(2.2c), we have

$$\begin{gathered} F(1) = 0, \qquad G(1) = \lambda, \qquad H(1) = 0, \\ F(-1) = 0, \qquad G(-1) = 1, \qquad H(-1) = 0, \end{gathered} \tag{2.3}$$

$$f(1) = \delta\lambda, \qquad g(1) = 0, \qquad f(-1) = -\delta, \qquad g(-1) = 0, \tag{2.4}$$

where $\delta = \ell/h$. Substituting (2.2a)–(2.2c) into the equation of continuity and the Navier-Stokes equations, we obtain

$$2F + H' = 0, \tag{2.5}$$

$$F'' - R(F^2 - G^2 + HF') = K, \tag{2.6}$$

$$G'' - R(2FG + HG') = 0, \tag{2.7}$$

$$f'' - R(Hf' + Ff - Gg) = A, \tag{2.8}$$

$$g'' - R(Hg' + Fg + Gf) = B, \tag{2.9}$$

where $R = \Omega h^2/\nu$ is the Reynolds number, ν is the kinematic viscosity of the fluid, a prime denotes differentiation with respect to ζ, and K, A, B are the unknown constants. Equations (2.5)–(2.7) and the boundary conditions (2.3) also reflect the axially symmetric flow problem corresponding to the flow between two rotating coaxial disks. Equations (2.8)-(2.9) subject to the boundary conditions (2.4) are linear but their solutions depend on those of (2.5)–(2.7). In order to obtain a solution to (2.8)-(2.9), we need two extra conditions. For this reason, we follow Lai et al. [6] and consider that the velocity components u and v are equal to zero at a point that is defined by (x_p, y_p) in midplane. Thus, we have

$$\begin{aligned} f(0) &= -\gamma_1 F(0) + \gamma_2 G(0), \\ g(0) &= -\gamma_1 G(0) - \gamma_2 F(0), \end{aligned} \tag{2.10}$$

where $\gamma_1 = x_p/h$ and $\gamma_2 = y_p/h$.

3. Solution to the problem

As it is well known, the fluid rotates as a rigid body for the induced axisymmetric flow when two disks rotate about a common axis with the same speed. In this case, the velocity field takes the form obtained by writing $F = H = 0$, $G = 1$, and $f = g = 0$. In the case of rotation with equal angular velocity about non-coincident axes, the velocity field reduces to the form obtained for $F = H = 0$, $G = 1$, $f = f_0$, $g = g_0$, as found by Berker [24]. In the light of this knowledge, let us assume that the upper disk rotates a bit faster than the lower disk. If we define a parameter given by $\varepsilon = (\Omega_u - \Omega_l)/\Omega_l$ (i.e., $\lambda = 1 + \varepsilon$), we can expand the unknowns in terms of the parameter ε in the form

$$\begin{aligned} F(\zeta) &= \varepsilon F_1(\zeta) + O(\varepsilon^2), & G(\zeta) &= 1 + \varepsilon G_1(\zeta) + O(\varepsilon^2), \\ H(\zeta) &= \varepsilon H_1(\zeta) + O(\varepsilon^2), & K &= R + \varepsilon K_1 + O(\varepsilon^2), \\ f(\zeta) &= f_0(\zeta) + \varepsilon f_1(\zeta) + O(\varepsilon^2), & g(\zeta) &= g_0(\zeta) + \varepsilon g_1(\zeta) + O(\varepsilon^2), \\ A &= A_0 + \varepsilon A_1 + O(\varepsilon^2), & B &= B_0 + \varepsilon B_1 + O(\varepsilon^2), \end{aligned} \tag{3.1}$$

where K_1, A_0, A_1, B_0, B_1 are constants. The appropriate conditions are

$$\begin{gathered} f_0(1) = \delta, \qquad g_0(1) = 0, \qquad f_0(0) = \gamma_2, \\ g_0(0) = -\gamma_1, \qquad f_0(-1) = -\delta, \qquad g_0(-1) = 0, \end{gathered} \tag{3.2a}$$

$$\begin{gathered} F_1(1) = 0, \qquad G_1(1) = 1, \qquad H_1(1) = 0, \qquad F_1(-1) = 0, \\ G_1(-1) = 0, \qquad H_1(-1) = 0, \end{gathered} \tag{3.2b}$$

$$\begin{gathered} f_1(1) = \delta, \qquad g_1(1) = 0, \qquad f_1(0) = -\gamma_1 F_1(0) + \gamma_2 G_1(0), \\ g_1(0) = -\gamma_1 G_1(0) - \gamma_2 F_1(0), \qquad f_1(-1) = 0, \qquad g_1(-1) = 0. \end{gathered} \tag{3.2c}$$

Substituting the expressions (3.1) into (2.5)–(2.9) and equating the coefficients of different powers of ε, one obtains

$$f_0'' + Rg_0 = A_0, \tag{3.3a}$$

$$g_0'' - Rf_0 = B_0, \tag{3.3b}$$

$$F_1'' + 2RG_1 = K_1, \tag{3.4a}$$

$$G_1'' - 2RF_1 = 0, \tag{3.4b}$$

$$2F_1 + H_1' = 0, \tag{3.4c}$$

$$f_1'' + Rg_1 = R\left(H_1 f_0' + F_1 f_0 - G_1 g_0\right) + A_1, \tag{3.5a}$$

$$g_1'' - Rf_1 = R\left(H_1 g_0' + F_1 g_0 + G_1 f_0\right) + B_1. \tag{3.5b}$$

By defining $\phi_0(\zeta) = f_0(\zeta) + ig_0(\zeta)$, (3.3a)-(3.3b) and (3.2a) reduce to

$$\phi_0'' - iR\phi_0 = A_0 + iB_0, \tag{3.6a}$$

$$\phi_0(1) = \delta, \tag{3.6b}$$

$$\phi_0(0) = \gamma_2 - i\gamma_1, \tag{3.6c}$$

$$\phi_0(-1) = -\delta. \tag{3.6d}$$

The solution to (3.6a) satisfying the conditions (3.6b)–(3.6d) is

$$\phi_0(\zeta) = \delta \frac{\sinh \kappa\zeta}{\sinh \kappa} + \frac{(\gamma_2 - i\gamma_1)}{1 - \cosh \kappa}(\cosh \kappa\zeta - \cosh \kappa) \tag{3.7}$$

or

$$f_0(\zeta) = \delta \frac{P(1)P(\zeta) + Q(1)Q(\zeta)}{P^2(1) + Q^2(1)} + \frac{\gamma_2 T(\zeta) + \gamma_1 S(\zeta)}{\left[1 - D(1)\right]^2 + E^2(1)}, \tag{3.8a}$$

$$g_0(\zeta) = \delta \frac{P(1)Q(\zeta) - Q(1)P(\zeta)}{P^2(1) + Q^2(1)} + \frac{\gamma_2 S(\zeta) - \gamma_1 T(\zeta)}{\left[1 - D(1)\right]^2 + E^2(1)}, \tag{3.8b}$$

where

$$\begin{aligned}
&\kappa = \sqrt{\frac{R}{2}}(1+i), \qquad P(\zeta) = \sinh\sqrt{\frac{R}{2}}\zeta\cos\sqrt{\frac{R}{2}}\zeta,\\
&Q(\zeta) = \cosh\sqrt{\frac{R}{2}}\zeta\sin\sqrt{\frac{R}{2}}\zeta, \qquad D(\zeta) = \cosh\sqrt{\frac{R}{2}}\zeta\cos\sqrt{\frac{R}{2}}\zeta,\\
&E(\zeta) = \sinh\sqrt{\frac{R}{2}}\zeta\sin\sqrt{\frac{R}{2}}\zeta,\\
&T(\zeta) = D^2(1) - D(1) + E^2(1) + D(\zeta) - D(1)D(\zeta) - E(1)E(\zeta),\\
&S(\zeta) = E(\zeta) - E(1) + E(1)D(\zeta) - D(1)E(\zeta),\\
&A_0 = \frac{R}{[1-D(1)]^2 + E^2(1)}\{\gamma_1[D(1) - D^2(1) - E^2(1)] - \gamma_2[E(1)]\},\\
&B_0 = \frac{R}{[1-D(1)]^2 + E^2(1)}\{\gamma_1[E(1)] + \gamma_2[D(1) - D^2(1) - E^2(1)]\}.
\end{aligned} \tag{3.9}$$

Using (3.4a)-(3.4b) and (3.2b) with the definition $\varphi_1(\zeta) = F_1(\zeta) + iG_1(\zeta)$, we have

$$\varphi_1'' - 2iR\varphi_1 = K_1, \tag{3.10a}$$

$$\varphi_1(1) = i, \tag{3.10b}$$

$$\varphi_1(-1) = 0. \tag{3.10c}$$

The solution to (3.10a) subject to the boundary conditions (3.10b)-(3.10c) is

$$\varphi_1(\zeta) = \frac{i(R-K_1)}{2R\cosh c}\cosh c\zeta + \frac{i\sinh c\zeta}{2\sinh c} + \frac{iK_1}{2R}, \tag{3.11}$$

where $c = \sqrt{R}(1+i)$. Substituting the real part of the solution (3.11) into (3.4c) leads to

$$H_1(\zeta) = \frac{K_1 - R}{R\Delta_1}[Q_1(1)I_1(\zeta) - P_1(1)I_2(\zeta)] + \frac{1}{\Delta_2}[P_2(1)I_4(\zeta) - Q_2(1)I_3(\zeta)] + C_{H1}, \tag{3.12}$$

where C_{H1} is a constant and

$$\begin{aligned}
&P_1(\zeta) = \cosh\sqrt{R}\zeta\cos\sqrt{R}\zeta, \qquad P_2(\zeta) = \sinh\sqrt{R}\zeta\cos\sqrt{R}\zeta,\\
&Q_1(\zeta) = \sinh\sqrt{R}\zeta\sin\sqrt{R}\zeta, \qquad Q_2(\zeta) = \cosh\sqrt{R}\zeta\sin\sqrt{R}\zeta,\\
&\Delta_1 = P_1^2(1) + Q_1^2(1), \qquad \Delta_2 = P_2^2(1) + Q_2^2(1),\\
&I_1(\zeta) = \frac{1}{2\sqrt{R}}\left[\cosh\sqrt{R}\zeta\sin\sqrt{R}\zeta + \sinh\sqrt{R}\zeta\cos\sqrt{R}\zeta\right],\\
&I_2(\zeta) = \frac{1}{2\sqrt{R}}\left[\cosh\sqrt{R}\zeta\sin\sqrt{R}\zeta - \sinh\sqrt{R}\zeta\cos\sqrt{R}\zeta\right],\\
&I_3(\zeta) = \frac{1}{2\sqrt{R}}\left[\cosh\sqrt{R}\zeta\cos\sqrt{R}\zeta + \sinh\sqrt{R}\zeta\sin\sqrt{R}\zeta\right],\\
&I_4(\zeta) = \frac{1}{2\sqrt{R}}\left[\sinh\sqrt{R}\zeta\sin\sqrt{R}\zeta - \cosh\sqrt{R}\zeta\cos\sqrt{R}\zeta\right].
\end{aligned} \tag{3.13}$$

Since $H_1(1) = 0$, $H_1(-1) = 0$, $I_1(1) = -I_1(-1)$, $I_2(1) = -I_2(-1)$, $I_3(1) = I_3(-1)$, $I_4(1) = I_4(-1)$, we have

$$K_1 = R, \qquad C_{H1} = \frac{Q_2(1)I_3(1) - P_2(1)I_4(1)}{\Delta_2}, \tag{3.14}$$

$$F_1(\zeta) = \frac{Q_2(1)P_2(\zeta) - P_2(1)Q_2(\zeta)}{2\Delta_2}, \tag{3.15}$$

$$G_1(\zeta) = \frac{1}{2} + \frac{1}{2\Delta_2}\left[P_2(1)P_2(\zeta) + Q_2(1)Q_2(\zeta)\right], \tag{3.16}$$

$$H_1(\zeta) = \frac{P_2(1)\left[I_4(\zeta) - I_4(1)\right] - Q_2(1)\left[I_3(\zeta) - I_3(1)\right]}{\Delta_2}. \tag{3.17}$$

The functions F, G, H depicted the variation with ζ for various values of R and ε in Figure 3.1 also reflect axial symmetric flow between two disks rotating about a common axis with a small angular velocity difference.

Introducing $\phi_1(\zeta) = f_1(\zeta) + ig_1(\zeta)$ and using (3.5a)-(3.5b) with (3.2c), we have

$$\phi_1'' - iR\phi_1 = R\left(H_1\phi_0' + F_1\phi_0 + iG_1\phi_0\right) + \left(A_1 + iB_1\right), \tag{3.18a}$$

$$\phi_1(1) = \delta, \tag{3.18b}$$

$$\phi_1(0) = \frac{1}{2}\left(\gamma_2 - i\gamma_1\right), \tag{3.18c}$$

$$\phi_1(-1) = 0. \tag{3.18d}$$

For the sake of simplicity, let us rewrite $I_3(\zeta)$ and $I_4(\zeta)$ as follows:

$$I_3(\zeta) = b_1 \cosh c\zeta + b_2 \cosh d\zeta, \qquad I_4(\zeta) = -b_2 \cosh c\zeta - b_1 \cosh d\zeta, \tag{3.19}$$

where

$$b_1 = \frac{1-i}{4\sqrt{R}}, \qquad b_2 = \frac{1+i}{4\sqrt{R}}, \qquad d = \sqrt{R}(1-i). \tag{3.20}$$

Thus, with the help of the solutions (3.7) and (3.15)–(3.17), (3.18a) transforms to the following form:

$$\begin{aligned}\phi_1'' - iR\phi_1 = {} & \psi_1 \cosh e_1\zeta + \psi_2 \cosh e_2\zeta + \psi_3 \cosh e_3\zeta + \psi_3 \cosh e_4\zeta \\ & + \psi_4 \sinh e_1\zeta + \psi_5 \sinh e_2\zeta + \psi_6 \sinh e_3\zeta - \psi_6 \sinh e_4\zeta \\ & + \psi_7 \cosh \kappa\zeta + \psi_8 \sinh \kappa\zeta + \psi_9 \sinh c\zeta + \psi_{10} + \left(A_1 + iB_1\right),\end{aligned} \tag{3.21}$$

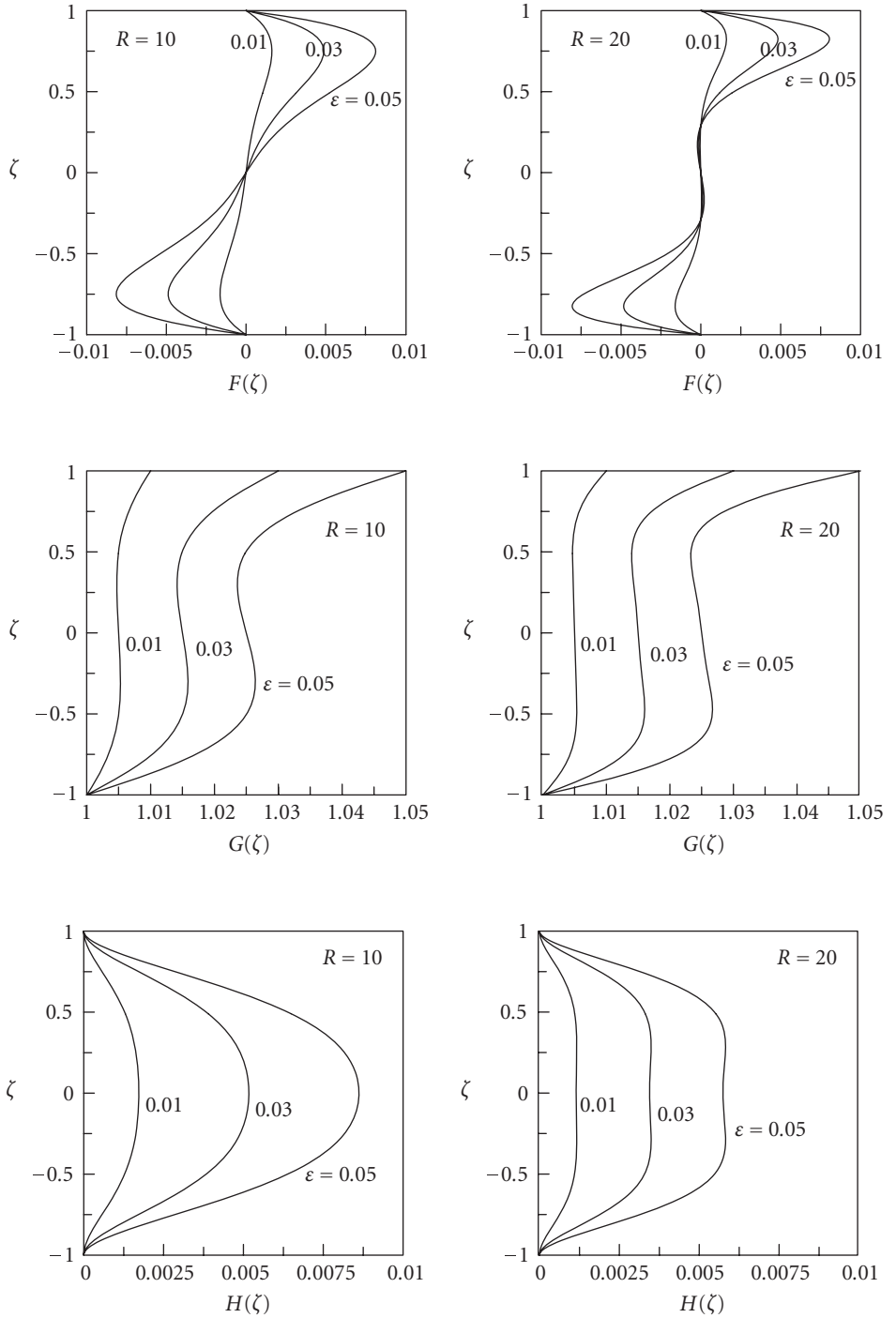

Figure 3.1. Variations of $F(\zeta)$, $G(\zeta)$, $H(\zeta)$ with ζ ($R = 10, 20$; $\varepsilon = 0.01, 0.03, 0.05$).

where

$$
\begin{aligned}
&e_1 = (\sqrt{2}+1)\sqrt{R/2}(1+i), \qquad e_2 = (\sqrt{2}-1)\sqrt{R/2}(1+i),\\
&e_3 = \lfloor(\sqrt{2}+1) - i(\sqrt{2}-1)\rfloor\sqrt{R/2}, \qquad e_4 = \lfloor(\sqrt{2}-1) - i(\sqrt{2}+1)\rfloor\sqrt{R/2},\\
&\psi_1 = \frac{R(\beta_1+q_1)}{2}, \qquad \psi_2 = \frac{R(\beta_1-q_1)}{2}, \qquad \psi_3 = \frac{R\beta_2}{2},\\
&\psi_4 = \frac{R(\beta_3+q_2)}{2}, \qquad \psi_5 = \frac{R(-\beta_3+q_2)}{2}, \qquad \psi_6 = \frac{R\beta_4}{2},\\
&\psi_7 = R(\beta_5+q_5), \qquad \psi_8 = R(\beta_6+q_4), \qquad \psi_9 = Rq_3, \qquad \psi_{10} = Rq_6,\\
&\beta_1 = -\alpha_1 b_2 + \alpha_5 b_1, \qquad \beta_2 = -\alpha_1 b_1 + \alpha_5 b_2, \qquad \beta_3 = -\alpha_2 b_2 + \alpha_6 b_1,\\
&\beta_4 = -\alpha_2 b_1 + \alpha_6 b_2, \qquad \beta_5 = \alpha_3 + \alpha_7, \qquad \beta_6 = \alpha_4 + \alpha_8,\\
&q_1 = t_1 t_3, \qquad q_2 = t_1 t_4, \qquad q_3 = -t_1 t_5, \qquad q_4 = t_2 t_3, \qquad q_5 = t_2 t_4, \qquad q_6 = -t_2 t_5,\\
&t_1 = \frac{i}{2\sinh c}, \qquad t_2 = \frac{i}{2}, \qquad t_3 = \frac{\delta}{\sinh\kappa}, \qquad t_4 = \frac{\gamma_2 - i\gamma_1}{1-\cosh\kappa}, \qquad t_5 = t_4\cosh\kappa,\\
&\alpha_1 = \frac{P_2(1)a_1}{\Delta_2}, \qquad \alpha_2 = \frac{P_2(1)a_2}{\Delta_2}, \qquad \alpha_3 = \frac{-P_2(1)a_1 I_4(1)}{\Delta_2},\\
&\alpha_4 = \frac{-P_2(1)a_2 I_4(1)}{\Delta_2}, \qquad \alpha_5 = \frac{-Q_2(1)a_1}{\Delta_2}, \qquad \alpha_6 = \frac{-Q_2(1)a_2}{\Delta_2},\\
&\alpha_7 = \frac{Q_2(1)a_1 I_3(1)}{\Delta_2}, \qquad \alpha_8 = \frac{Q_2(1)a_2 I_3(1)}{\Delta_2},\\
&a_1 = \frac{\delta\kappa}{\sinh\kappa}, \qquad a_2 = \frac{\kappa(\gamma_2 - i\gamma_1)}{1-\cosh\kappa}.
\end{aligned}
\tag{3.22}
$$

The solution to (3.21) satisfying the conditions (3.18b)–(3.18d) is

$$
\begin{aligned}
\phi_1(\zeta) = {} & \mathcal{H}_1\cosh\kappa\zeta + \mathcal{H}_2\sinh\kappa\zeta + T_1\cosh e_1\zeta + T_2\cosh e_2\zeta\\
& + T_3\cosh e_3\zeta + T_4\cosh e_4\zeta + T_5\sinh e_1\zeta + T_6\sinh e_2\zeta\\
& + T_7\sinh e_3\zeta + T_8\sinh e_4\zeta + T_9\zeta\sinh\kappa\zeta + T_{10}\zeta\cosh\kappa\zeta\\
& + T_{11}\sinh c\zeta + T_{12} - \frac{(A_1+iB_1)}{Ri},
\end{aligned}
\tag{3.23}
$$

where

$$
\begin{aligned}
&T_1 = \frac{\psi_1}{e_1^2 - Ri}, \qquad T_2 = \frac{\psi_2}{e_2^2 - Ri}, \qquad T_3 = \frac{\psi_3}{e_3^2 - Ri}, \qquad T_4 = \frac{\psi_3}{e_4^2 - Ri},\\
&T_5 = \frac{\psi_4}{e_1^2 - Ri}, \qquad T_6 = \frac{\psi_5}{e_2^2 - Ri}, \qquad T_7 = \frac{\psi_6}{e_3^2 - Ri}, \qquad T_8 = -\frac{\psi_6}{e_4^2 - Ri},\\
&T_9 = \frac{\psi_7}{2\kappa}, \qquad T_{10} = \frac{\psi_8}{2\kappa}, \qquad T_{11} = \frac{\psi_9}{c^2 - Ri}, \qquad T_{12} = \frac{\psi_{10} i}{R},\\
&\mathcal{H}_1 = \frac{1}{\cosh\kappa - 1}\left[\frac{\delta}{2} - \frac{(\gamma_2 - i\gamma_1)}{2} - \frac{(\Psi_1 + \Psi_2)}{2} + \Psi_3\right],\\
&\mathcal{H}_2 = \frac{1}{\sinh\kappa}\left[\frac{\delta}{2} - \frac{(\Psi_1 - \Psi_2)}{2}\right],\\
&A_1 + iB_1 = \frac{Ri}{2}\left[\mathcal{H}_1(\cosh\kappa + 1) + \frac{(\Psi_1 + \Psi_2)}{2} + \Psi_3 - \frac{\delta}{2} - \frac{(\gamma_2 - i\gamma_1)}{2}\right],\\
&\Psi_1 = T_1\cosh e_1 + T_2\cosh e_2 + T_3\cosh e_3 + T_4\cosh e_4 + T_5\sinh e_1 + T_6\sinh e_2\\
&\qquad + T_7\sinh e_3 + T_8\sinh e_4 + T_9\sinh\kappa + T_{10}\cosh\kappa + T_{11}\sinh c + T_{12},\\
&\Psi_2 = T_1\cosh e_1 + T_2\cosh e_2 + T_3\cosh e_3 + T_4\cosh e_4 - T_5\sinh e_1 - T_6\sinh e_2\\
&\qquad - T_7\sinh e_3 - T_8\sinh e_4 + T_9\sinh\kappa - T_{10}\cosh\kappa - T_{11}\sinh c + T_{12},\\
&\Psi_3 = T_1 + T_2 + T_3 + T_4 + T_{12}.
\end{aligned}
\tag{3.24}
$$

Figure 3.2 shows the variations of the functions f and g that represent the dimensionless x- and y-components of the translational velocity for various values of the parameters. The conditions obtained by means of the perturbation method, that is, $f(1) = \delta(1+\varepsilon)$, $f(0) = \gamma_2(1+\varepsilon/2)$, $f(-1) = -\delta$, $g(1) = 0$, $g(0) = -\gamma_1(1+\varepsilon/2)$, $g(-1) = 0$, are confirmed by Figure 3.2. It is obvious from Figure 3.2 that the influence of ε on f and g is small. When the Reynolds number R increases, the curves become flatter in the core, whereas they have a little more pronounced variation in the region near the disks.

Figures 3.3 and 3.4 illustrate the variations of the dimensionless x- and y-components of the velocity field with the position, respectively, and reveal the flow produced by the rotation of two disks with nearly the same angular velocity about distinct axes. The conditions $\overline{u}(1) = (-\overline{y} + \delta)(1+\varepsilon)$, $\overline{u}(0) = (-\overline{y} + \gamma_2)(1+\varepsilon/2)$, $\overline{u}(-1) = -(\overline{y} + \delta)$, $\overline{v}(1) = \overline{x}(1+\varepsilon)$, $\overline{v}(0) = (\overline{x} - \gamma_1)(1+\varepsilon/2)$, $\overline{v}(-1) = \overline{x}$, which are obtained by the use of the perturbation method, are clearly seen in Figures 3.3 and 3.4, where $\overline{x} = x/h$, $\overline{y} = y/h$, $\overline{u} = u/\Omega h$, $\overline{v} = v/\Omega h$.

4. Discussion and conclusions

When two disks rotate about distinct axes with the same angular velocity, the flow is a result of superposition, in each $z =$ constant plane, of a rigid body rotation with the same

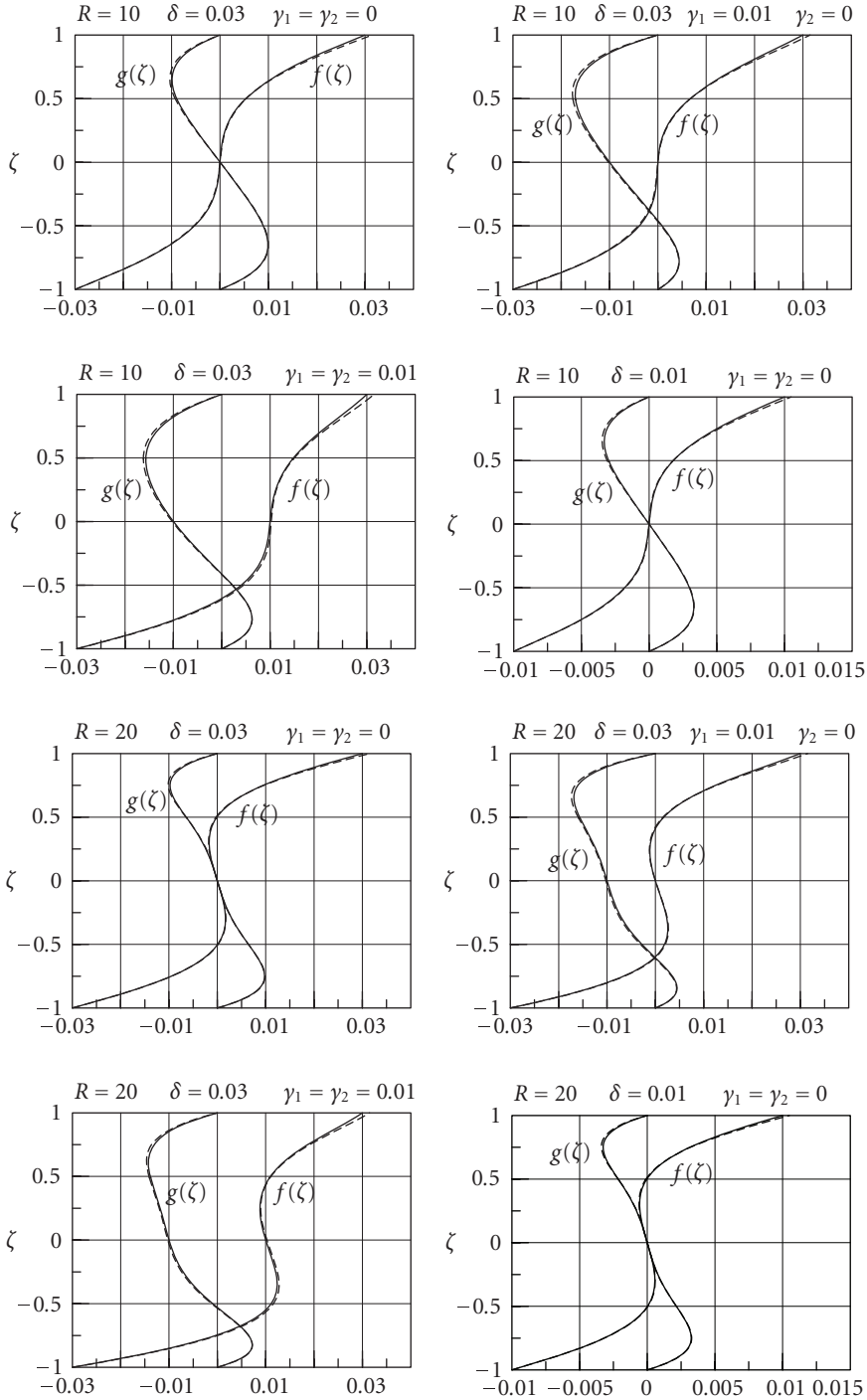

Figure 3.2. Variations of $f(\zeta)$ and $g(\zeta)$ with ζ (——— $\varepsilon = 0$; – – – – $\varepsilon = 0.05$).

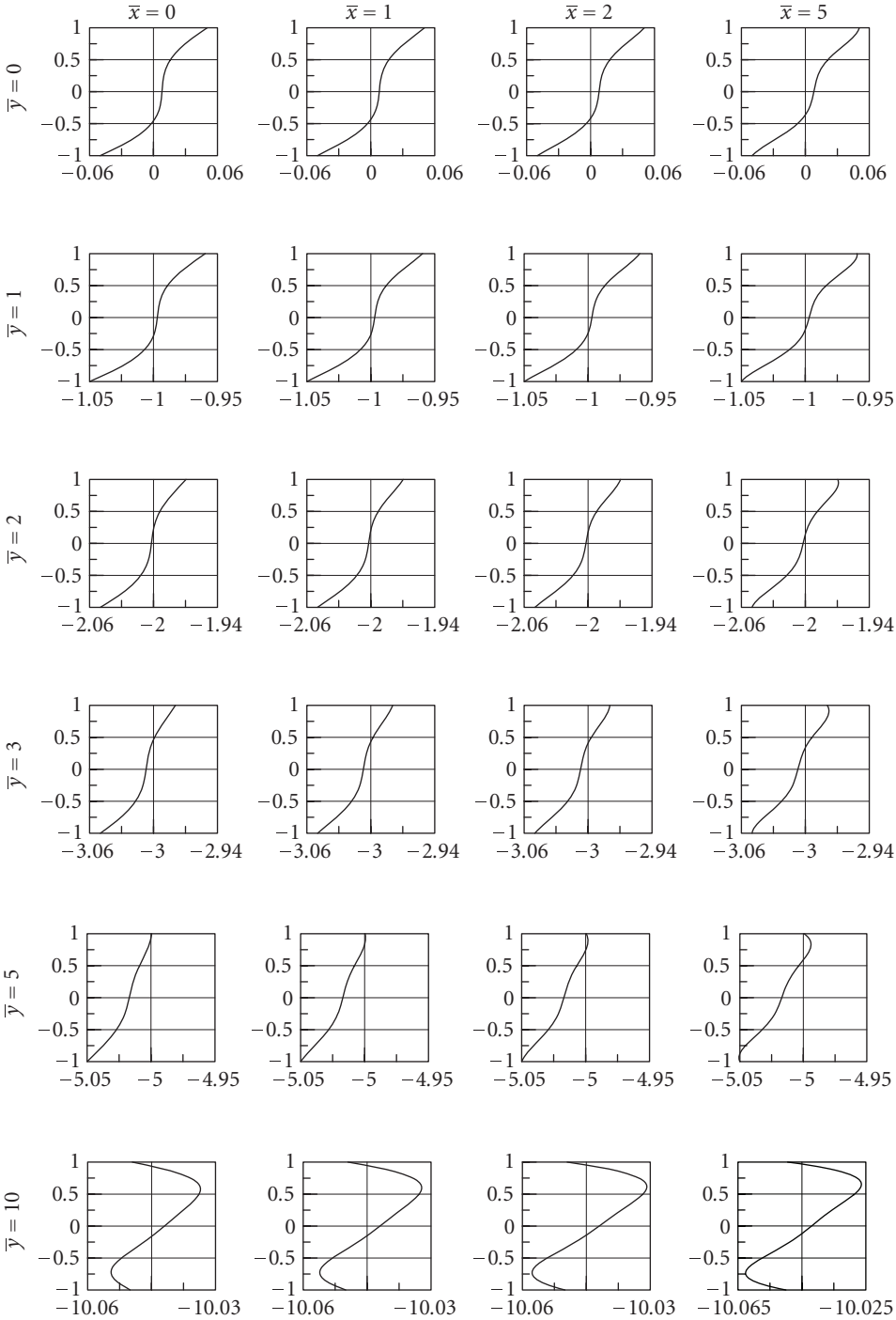

Figure 3.3. Dependence of $\overline{u}$ on $\overline{x}$, $\overline{y}$, ζ ($R = 10$; $\varepsilon = 0.01$; $\delta = 0.05$; $\gamma_1 = 0.004$; $\gamma_2 = 0.008$).

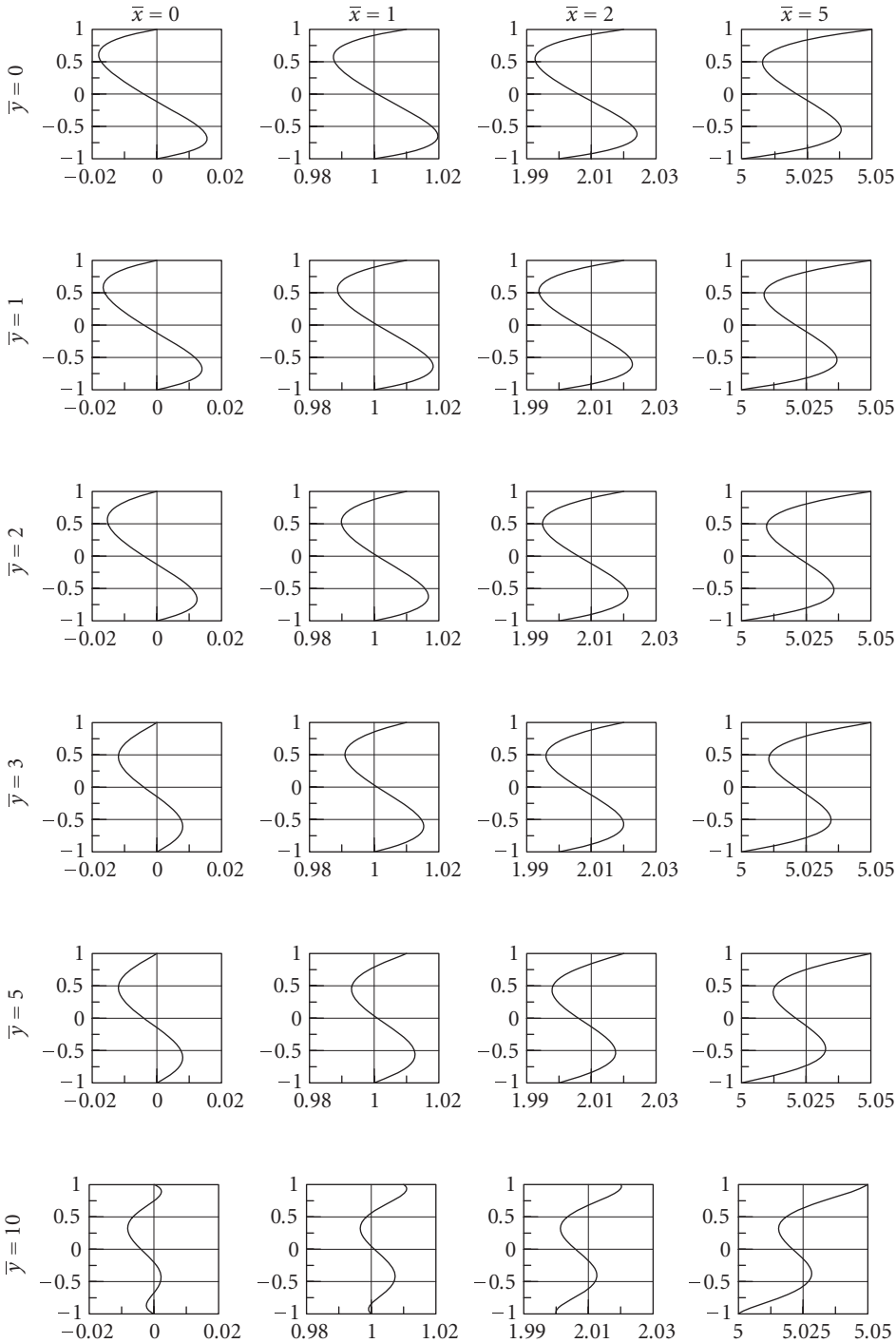

Figure 3.4. Dependence of $\overline{v}$ on $\overline{x}$, $\overline{y}$, ζ ($R = 10$; $\varepsilon = 0.01$; $\delta = 0.05$; $\gamma_1 = 0.004$; $\gamma_2 = 0.008$).

angular velocity about the vertical axis passing the origin and a rigid body translation that changes from plane to plane. In this case, it is clear that there is no flow perpendicular to the disks. If there is a difference between the angular velocities of the disks, the flow is a result of superposition, in each z = constant plane, of the Kármán flow and a rigid body translation that is different from plane to plane (see Lai et al. [6]). The rotation at different speeds causes a flow in the z-direction, which is a consequence of the Kármán flow. The equations governing the flow are the nonlinear Kármán equations and the linear equations whose coefficients include the solution to the Kármán equations. However, the boundary conditions are missing for linear equations. In order to overcome this difficulty, Lai et al. [6] proposed a parameter characterizing the stagnation points defined by $u = v = 0$ in midplane. In this paper, we follow the same way and introduce two parameters defined as $\gamma_1 = x_p/h$ and $\gamma_2 = y_p/h$, where x_p and y_p are the coordinates at which the velocity components u and v in midplane are equal to zero.

The solution to the problem is obtained by means of a perturbation analysis. From a theoretical point of view, such solutions are very practical since the effects of successive terms in the perturbation expansion decrease very rapidly. Since our perturbation analysis is valid only for small values of ε, the variation of ε is limited to a range from 0.0 to 0.05.

The effects of parameters on the velocity field are examined in detail. The conclusions which are drawn from this analysis can be summarized as follows.

(i) The dimensionless velocity components $\overline{u}$ and $\overline{v}$ are strongly dependent on $\overline{y}$ and $\overline{x}$, respectively. The effect of eccentricity is noticeable for small values of $\overline{x}$ and $\overline{y}$, but gets progressively weaker as $\overline{x}$ and $\overline{y}$ increase.

(ii) The dimensionless velocity components $\overline{u}$ and $\overline{v}$ depend strongly on γ_2 and γ_1, respectively.

(iii) Since the eccentricity is defined along the y-axis, the influence of the eccentricity parameter δ on $\overline{u}$ is readily observed, but the eccentricity has a weak effect on $\overline{v}$.

(iv) The effect of the parameter ε on the flow indicating the translational motion of rigid body is small. This effect is more pronounced in the region between midplane and the faster disk. In general, an increase in ε leads to an increase in the velocity components of the fluid.

(v) The axial velocity is the same as that produced for axisymmetric flow of the fluid between two disks rotating with different speeds; in other words, it is independent of δ, γ_1, γ_2. The fluid flows from the slower rotating disk towards the faster rotating disk. When the angular velocities are increased at the same rate, the axial velocity becomes larger. The axial velocity in the core region is nearly uniform for large Reynolds numbers. Far from the z-axis, the contribution of axial velocity to the velocity vector is insignificant.

(vi) We take into account the solutions at moderate Reynolds number where the uniqueness of von Kármán's solution is guaranteed because there are multiple solutions at high enough Reynolds number. Increasing Reynolds number R has a tendency to make the three velocity components flatter in the core region. The increase of R gives rise to the boundary layers developing on both disks.

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