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Research Article

Asymptotic Solution of the Theory of Shells Boundary Value Problem

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This paper is dedicated to the memory of Professor J. J. Telega

Recommended by José Manoel Balthazar

This paper provides a state-of-the-art review of asymptotic methods in the theory of plates and shells. Asymptotic methods of solving problems related to theory of plates and shells have been developed by many authors. The main features of our paper are: (i) it is devoted to the fundamental principles of asymptotic approaches, and (ii) it deals with both traditional approaches, and less widely used, new approaches. The authors have paid special attention to examples and discussion of results rather than to burying the ideas in formalism, notation, and technical details.

1. Introduction

The theory of plates and shells is applied usually for technical purposes. However, a role of today's modern theory of plates and shells is certainly wider. In fact, in many important cases, the physical objects cannot be described by equations of 3D theory of elasticity. The examples can be biological membranes, liquid crystals, thin polymeric films, thin-walled objects made from materials with shape memory, as well as various nanostructure devices. Theory of plates and shells not only gives practically useful results, but outlines also a general methodology of the transition from 3D to 2D (or 1D) models. It is worth noting that development of mathematical physics in many cases has been motivated by theory of plates and shells problems, in particular we mean the problems associated with the application of variation and asymptotic methods. Note that a key (for singular asymptotics) concept of an edge effect appeared in the works of Lamb and Basset in 1890, while the concept of boundary layer occurred in fluid mechanics appeared only in 1904 [1]. The classical papers by Vishik and Lyusternik are a generalization of some results obtained

earlier by Gol'denveĭzer [2]. On the other hand, theory of plates and shells problems associated with high-technology development of materials and constructions implied development of various homogenization procedures [3–12]. The investigation of rods stability yielded a linearization procedure, whereas Koiter's approach [13] has strongly influenced today's Catastrophe theory.

Generally, asymptotic methods are applied in the field of theory of plates and shells first for transition from 3D to 2D models, and then to solve 2D problems. Our attention is focused on the latter problem. A choice of discussed and illustrated asymptotic methods is mainly motivated by authors' subjective experience. Note that in this paper we do not concern up-to-date analysis of the existing linear and nonlinear models of shells, and a reader interested at this topic is invited to follow other known works [14–18].

We also omit here purely mathematical approaches regarding theory of shells and mainly developed by the French school (see, for instance, [19–21]).

2. On the parameter of asymptotic integration

Almost always while considering any asymptotic behavior, a term "small" or "large" parameter is applied. Since this traditional meaning may lead to confusion, we further apply the term of "asymptotic integration parameters," not restricted to be necessarily small (large). Notice that any asymptotic analysis should begin with normalization of the problem, that is defining it in terms of nondimensional variables whose typical scale is of the order of one, and the relative magnitude of different physical effects is measured by nondimensional parameters or dimensionless groups [22]. In particular, in theory of plates and shells, the following parameters are often used: h/R is the ratio of shell thickness to its characteristic size, that is, radius [2, 23]; a/b is the ratio of characteristic dimensions (i.e., a plate length to its width) [24]; ω^{-1}, where ω is the dimensionless frequency of vibrations [25]; A is the dimensionless amplitude of vibrations [26]; $\varepsilon = w/h$, where w is the normal displacement (the case $\varepsilon \ll 1$ belongs to Koiter's asymptotics [13], whereas the case $\varepsilon \gg 1$ is called Pogorolev's asymptotics [27]); B_1/B_2 is the ratio of bending stiffnesses of structurally orthotropic shell or the ratio of shear rigidity to membrane rigidity [28]; a small deviation of shell shape from canonical one [29] or a changeable thickness from a constant one; the ratio of shallow shell rise H to curvature radius R, and so on.

For periodically nonhomogeneous plates and shells, small parameter is the ratio of a period of nonhomogeneity to a characteristic size of considered structure [3–12].

If it is impossible to define a suitable real physical parameter, it can be introduced to equations in a purely formal manner (artificial parameter of asymptotic integration) [30].

"Let us try to find the asymptotics of some nontrivial solutions. First of all it is necessary to guess (no better word may be chosen) in what form this asymptotics must be sought. This stage—guessing the form of the asymptotics—of course, defines formalization. Analogies, experience, physical considerations, intuition, and "just lucky" guesses are the toolkit which is used by every investigator" [31], but after the introduction of the parameters of asymptotic integration and after the choice of an asymptotic method, it is not necessary to "reinvent the wheel"—it is better to use some well-known and well-worked out approach.

3. How to find parameters of asymptotic integration

One of the most peculiar aspects of theory of plates and shells is that associated with the existence of a few parameters of asymptotic integration yielding complexity of the problem being analyzed. In general, this fact is omitted in most studies. Therefore, a domain of application of the results is not clear enough. Gol'denveĭzer [2] indicated the importance of estimation of the order of coefficients of the partial differential equations and differential operators. In this reference, the index of variation of a function has been introduced and found to be very convenient. For example [2, 24, 32, 33], one can introduce estimations for the derivatives

$$w_x \sim \varepsilon^{\alpha} w; \qquad w_y \sim \varepsilon^{\beta} w; \qquad w_t \sim \varepsilon^{\gamma} w. \tag{3.1}$$

To compare the orders of several functions, their indices of intensity are introduced in the following way:

$$w \sim \varepsilon^{\delta}; \qquad w \sim \varepsilon^{\sigma} u. \tag{3.2}$$

Parameters of asymptotic integration $\alpha, \beta, \ldots$ are chosen in a way which yields a generalization of the Newton polygon. Notice that one gets finally not only simplified boundary value problem, but also the estimation of application domains for used asymptotic simplifications.

Let us introduce some remarks. Solutions of linear boundary value problem of theory of plates and shells usually include exponential and trigonometric functions, which causes efficiency of the described technique; but, for example, the solution of corner boundary layer type can contain powers of coordinates, and in this case the indices of variations should be applied carefully. In addition, it should be noted that the described technique gives local estimations.

Although Gol'denveĭzer's monograph [2] was published long time ago, some of the results reported there have been reconsidered again in the frame of the so-called power geometry [34].

Key steps of the method will be illustrated by the example of a membrane lying on an elastic support and governed by the equation

$$\varepsilon(w_{xx} + w_{yy}) + w = 0, \tag{3.3}$$

where $w(x, y)$ is the normal displacement of membrane, and ε is the small parameter.

The parameters of asymptotic integration α, β are introduced in the following way:

$$w_x \sim \varepsilon^{\alpha} w, \quad w_y \sim \varepsilon^{\beta} w, \quad -\infty < \alpha, \beta < \infty. \tag{3.4}$$

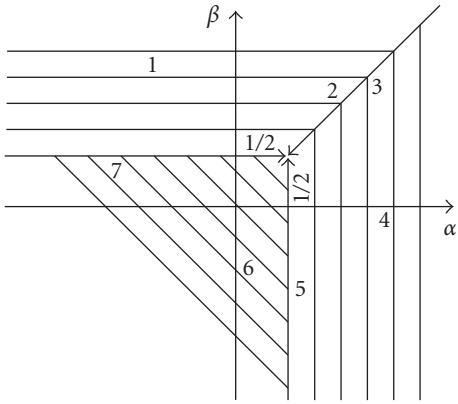

Figure 3.1. Newton polygon for (3.3).

Exponents of ε power for all terms of (3.3) follow:

$$1-2\alpha, \qquad 1-2\beta, \qquad 0. \tag{3.5}$$

Considering plane $\alpha\beta$ (see Figure 3.1), the areas corresponding to the smallest values of exponents associated with all terms of (3.3) are constructed.

Note that exponent $1-2\alpha$ is the smallest one under the choice of α and β values in area 4, exponent $1-2\beta$ in area 1, and exponent 0 in area 6 (areas 1, 4, 6 are open sets, i.e., their boundary lines are not included).

In areas 1, 4, 6 the limiting equations follow:

$$w_{yy}=0; \qquad w_{xx}=0; \qquad w=0. \tag{3.6}$$

The equations include only one term. The values of α and β associated with the equations with two terms are located on boundary lines (without point $\alpha=\beta=1/2$)

$$w_{xx}+w_{yy}=0, \qquad \varepsilon w_{xx}+w=0, \qquad \varepsilon w_{yy}+w=0. \tag{3.7}$$

Finally, for $\alpha=\beta=1/2$ in (3.3) all terms remain. Since there are no blank spaces on the $\alpha\beta$ plane, there are no other limiting systems.

Note that the occurrence of more than two parameters of the asymptotic integration results in an increase of the problem complexity. In [35, 36], effective algorithms to solve the occurring problems are introduced, whereas in [37], a generalization is proposed.

Simultaneous splitting of governing equations should be matched with an appropriate splitting of the associated boundary conditions. This complicated problem is discussed and illustrated in [2, 23, 24, 32, 33].

4. Timoshenko-type plate equations

Below, we consider an illustrative example showing the efficiency of asymptotic method [36]. According to Timoshenko, the effect of a shear deflection occurring for plate

vibration is comparable to that of rotary inertia. However, the wave front sets are predicted incorrectly due to the Timoshenko theory. On the other hand, asymptotic method shows that a transverse compression effect is comparable with effects of rotary inertia and shear deflection. Correct asymptotic theory gives a proper location of wave fronts as well as averaged characteristics of stress-strain state in the vicinity of the mentioned fronts within two-dimensional equations of the form

$$\varphi_{1xx} + a_s^2\varphi_{1yy} + e\varphi_{2xy} + cW_x - 8a_s^2(w_x + \varphi_1) - \varphi_{1tt} = 0, \tag{4.1}$$

$$\varphi_{2yy} + a_s^2\varphi_{2xx} + e\varphi_{1xy} + cW_y - 8a_s^2(w_y + \varphi_2) - \varphi_{2tt} = 0, \tag{4.2}$$

$$a_s^2(w_{xx} + w_{yy}) + e(\varphi_{1x} + \varphi_{2y}) + W - w_{tt} = 0, \tag{4.3}$$

$$W + c(\varphi_{1x} + \varphi_{2y}) + 0.5w_{tt} + \frac{1}{16}W_{tt} = 0, \tag{4.4}$$

$$M_1 = \varphi_{1x} + c\varphi_{2y} + cW, \qquad M_2 = \varphi_{2y} + c\varphi_{1x} + cW, \tag{4.5}$$

$$N = W + c\varphi_{1x} + \varphi_{2y}, \tag{4.6}$$

$$H = a_s^2(\varphi_{2x} + \varphi_{1y}), \qquad Q_1 = w_x + \varphi_1 = \beta_1, \qquad Q_2 = w_y + \varphi_2 = \beta_2, \tag{4.7}$$

where: $e = 1/(2(1-\nu))$, $c = \nu/(1-\nu)$, $a_s^2 = (1-2\nu)/(1-\nu)^2$, w is the displacement of the middle plane of the plate, φ_1, φ_2 are the rotational angles of the normal to the middle plane of the plate in the x and y directions, W is the function of changing of the plate thickness, antisymmetric with respect to the middle plane of the plate.

Compare (4.1)–(4.7) with the equations of Timoshenko plate at the shear coefficient $k^2 = 2/3$,

$$\begin{gathered}
\varphi_{1xx} + \frac{1-\nu}{2}\varphi_{1yy} + \frac{1+\nu}{2}\varphi_{2xy} - 4(1-\nu)(w_x + \varphi_1) - \frac{1}{a_1^2}\varphi_{1tt} = 0, \\
\varphi_{2yy} + \frac{1-\nu}{2}\varphi_{2xx} + \frac{1+\nu}{2}\varphi_{1xy} - 4(1-\nu)(w_y + \varphi_2) - \frac{1}{a_1^2}\varphi_{2tt} = 0, \\
w_{xx} + w_{yy} + \varphi_{1x} + \varphi_{2y} - \frac{3}{2a_s^2}w_{tt} = 0, \\
M_1 = a_1^2(\varphi_{1x} + \nu\varphi_{2y}), \qquad M_2 = a_1^2(\varphi_{2y} + \nu\varphi_{1y}), \qquad H = a_s^2(\varphi_{2x} + \varphi_{1y}), \\
Q_1 = w_x + \varphi_1 = \beta_1, \qquad Q_2 = w_y + \varphi_2 = \beta_2, \qquad a_1^2 = \frac{1-2\nu}{(1-\nu)^2}.
\end{gathered} \tag{4.8}$$

Note that (4.1)–(4.7), contrary to (4.8), govern the velocities of all waves in comparison with the 3D case.

Equations (4.8) can be obtained from (4.1)–(4.7), but using the asymptotically inconsistent procedure: the last term of (4.4) as well as function N in (4.6) should be neglected; and expression $W = -c(\varphi_{1x} + \varphi_{2y})$ should be introduced to (4.2)–(4.4).

5. Dynamic edge effect method

Due to the main idea of this approach proposed by Bolotin [25], a continuous elastic system is separated into two parts. In one of them—an interior zone—solutions may be

expressed by trigonometric functions with unknown constants. One can use exponential functions in the dynamic edge effect's zone. Then, a matching procedure permits to obtain unknown constants, and a complete solution of dynamic problem may be written in a relatively simple form. This approximate solution is very accurate for high-frequency vibrations, but even at low-frequency vibrations the error is not excessive. Dynamic edge effect method is naturally generalized for a nonlinear case [26, 32].

We should also emphasize that dynamic edge effect method works properly in connection with variation methods [26, 32]. This is due to the fact that the dynamic edge effect method gives good approximation of displacements. While finding the eigenvalues, the following general rule can be formulated: if you are looking for the eigenforms, then asymptotics should be used; if you need an eigenvalue, then the found asymptotic function can be used further by one of the variation methods.

6. Homogenization approach

Replacement of a nonhomogeneous shell by a homogeneous one with some reduced characteristics belongs to one of the most popular approximations in theory of plates and shells. We can mention structurally orthotropic theories of ribbed, corrugated, perforated plates and shells, plates and shells with many attached masses, and so forth. For many years, a design of similar simplifications depended fully on engineers' intuition, and the obtained quantities differed from each other depending on the theory used. Mathematical difficulties were caused by the occurrence of partial differential equations with rapidly changing coefficients. Beginning from the 1970s of the 20th century, the theory of homogenization of partial differential equations has been developed. It should be emphasized that a similar mathematical approach was proposed earlier in the theory of ribbed shells [12].

Using the homogenization approach, one must deal with two successively solvable problems: a local problem for periodically repeated element (cell) as well as the global homogeneous problem with some reduced parameters. As a rule, the fundamental difficulty is associated with solution of the cell problem. Although this problem can be solved numerically, an analytical solution is always highly required. The application of asymptotic methods to solve local problems allowed us to get homogenized solutions for various periodically nonhomogeneous plates and shells with correctly reduced coefficients. The areas of applicability of approximated theories are estimated, and full stress-strain states can be calculated. It is important that one can also predict boundary layers occurring in the vicinity of boundaries. Lack of this knowledge does not allow the shell stress-strain to be fully estimated. Using the homogenization procedure, one should take into account the relations between parameters of investigated structures. As an example, a deformation of a reinforced membrane governed by the following equation is analyzed:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g_1(x, y), \quad kl \le y \le (k+1)l. \tag{6.1}$$

Conditions of conjugations of the neighboring parts of membrane are:

$$\lim_{y\to kl+0} u \equiv u^{+} \equiv \lim_{y\to kl-0} u \equiv u^{-}, \tag{6.2}$$

$$\left(\frac{\partial u}{\partial y}\right)^{+} - \left(\frac{\partial u}{\partial y}\right)^{-} = d_1 \frac{\partial^2 u}{\partial x^2}, \tag{6.3}$$

$$u = 0 \quad \text{for } x = 0, H. \tag{6.4}$$

Let a characteristic period of external load be $L \gg l$, $\varepsilon = l/L \ll 1$. We introduce the variables $\eta = y/l$, $y_1 = y/L$, and the following series:

$$\begin{aligned} u = {} & u_0(x,y) + \varepsilon^{\alpha_1}\left[u_{10}(x,y) + u_1(x,y,\eta)\right] \\ & + \varepsilon^{\alpha_2}\left[u_{20}(x,y) + u_2(x,y,\eta)\right] + \cdots, \quad 0 < \alpha_1 < \alpha_2 < \cdots. \end{aligned} \tag{6.5}$$

Substituting (6.5) into (6.1)–(6.4), the following recurrent system is obtained:

$$\begin{gathered} \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} + \varepsilon^{\alpha-2}\frac{\partial^2 u_1}{\partial \eta^2} + 2\varepsilon^{\alpha-1}\frac{\partial^2 u_0}{\partial y \partial \eta} + \varepsilon^{\alpha-2}\frac{\partial^2 u_2}{\partial \eta^2} + 2\varepsilon^{\alpha-1}\frac{\partial^2 u_0}{\partial y \partial \eta} + O(\varepsilon^{\alpha}) = q(x,y), \\ \left[u_0 + \varepsilon^{\alpha}(u_{10} + u_1) + \cdots\right]^{+} = \left[u_0 + \varepsilon^{\alpha}(u_{10} + u_1) + \cdots\right]^{-}, \\ \varepsilon^{\alpha-1}\left[\left(\frac{\partial u_1}{\partial \eta}\right)^{+} - \left(\frac{\partial u_1}{\partial \eta}\right)^{+}\right] + O(\varepsilon^{\alpha}) = d\left[\frac{\partial^2 u_0}{\partial x^2} + O(\varepsilon^{\alpha})\right], \end{gathered} \tag{6.6}$$

where: $q = L^2 q_1$, $d = d_1/L$.

The character of asymptotics depends essentially on the order of magnitude of d in comparison to ε. Let us introduce the estimation $d \sim \varepsilon^{\beta}$.

Depending on the value of β, one obtains the following limiting systems:

$$0 < \alpha < 2, \quad \frac{\partial^2 u_1}{\partial \eta^2} = 0, \tag{6.7}$$

$$\alpha = 2, \quad \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} + \frac{\partial^2 u_1}{\partial \eta^2} = q(x,y), \tag{6.8}$$

$$\alpha > 2, \quad \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} = q(x,y), \tag{6.9}$$

and the following conjugation conditions:

$$\begin{aligned} & \beta < \alpha - 1, \quad \frac{\partial^2 u_0}{\partial x^2} = 0, \\ & \beta = \alpha - 1, \quad \left[\left(\frac{\partial u_1}{\partial \eta}\right)^{+} - \left(\frac{\partial u_1}{\partial \eta}\right)^{+}\right] = d\varepsilon^{1-\alpha_1}\frac{\partial^2 u_0}{\partial x^2}, \\ & \beta > \alpha - 1, \quad \left(\frac{\partial u_1}{\partial \eta}\right)^{+} = \left(\frac{\partial u_1}{\partial \eta}\right)^{+}. \end{aligned} \tag{6.10}$$

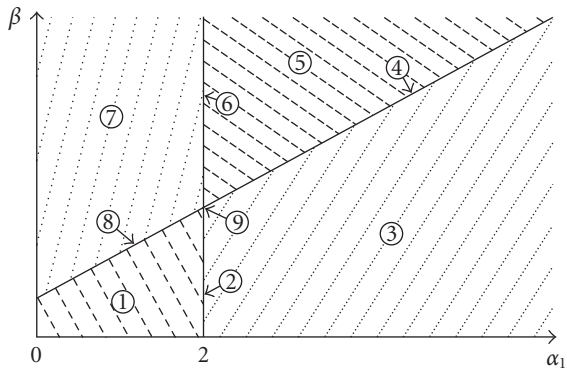

FIGURE 6.1. The plane of parameters $\beta > 0$, $\alpha > 0$.

The plane of parameters $\beta > 0$, $\alpha > 0$ is divided into nine parts (see Figure 6.1).

In zones 1–3, one has

$$\frac{\partial^2 u_1}{\partial \eta^2} = q(x, y). \tag{6.11}$$

In zones 4–6, the equation has the form of (6.9). For zones 7 and 9, the limiting systems are incorrect.

A particular role plays the case of $\alpha = 2$, $\beta = 1$ (zone 8). The corresponding limiting equation is (6.8) and

$$\begin{gathered} u^+ = u^-, \\ \left[\left(\frac{\partial u_1}{\partial \eta}\right)^+ - \left(\frac{\partial u_1}{\partial \eta}\right)^+\right] = d\varepsilon^{-1}\frac{\partial^2 u_0}{\partial x^2}. \end{gathered} \tag{6.12}$$

Solution of (6.8) can be written as follows.

$$u_1 = 0.5\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} - q(x, y)\right)\eta^2 + C_1\eta + C_2. \tag{6.13}$$

Substituting solution (6.13) into conditions (6.12) yields the homogenized boundary value problem governed by the following equations:

$$\begin{gathered} \nabla^2 u_0 + \frac{d_1}{l}\frac{\partial^2 u_0}{\partial x^2} = q(x, y), \\ u_1 = \frac{d_1(x, y)}{l}\frac{\partial^2 u_0}{\partial x^2}\eta(\eta - l). \end{gathered} \tag{6.14}$$

Observe that boundary conditions (6.4) are not satisfied. In order to construct a boundary layer u_b, the new "fast" variable $\xi = x/l$ is introduced and the following series

is applied:

$$u_n = \varepsilon^{\gamma_1} u_{11}(x,y,\xi,\eta) + \varepsilon^{\gamma_2} u_{22}(x,y,\xi,\eta) + \cdots, \quad 0 < \gamma_1 < \gamma_2 < \cdots. \tag{6.15}$$

Substituting series (6.15) into the governing boundary value problem yields the first approximation

$$\frac{\partial^2 u_{11}}{\partial \xi^2} + \frac{\partial^2 u_{11}}{\partial \eta^2} = 0, \qquad u_{11}|_{\eta=kl} = 0, \quad k = 0, \pm 1, \pm 2, \ldots. \tag{6.16}$$

Then, further construction of a boundary layer may be easily carried out using, for example, the Kantorovich procedure [38].

7. Distributional approach

Terms like $a(x/\varepsilon)$ often occur in the asymptotic problems. In order to introduce parameter ε explicitly, it is useful to apply the distributional approach [39]. As a model problem, we consider a transition from 2D ribs to 1D ones. The governing partial differential equations for bending deformation of an infinite plate on the elastic foundation, reinforced by periodic systems of ribs in two main directions, are as follows:

$$\begin{gathered} D\Delta\Delta w + Cw + D_1 F_1(x) w_{xxxx} + D_2 F_2(y) w_{yyyy} = q(x,y), \\ F_1(x) = \sum_{n=-\infty}^{\infty} \left[H(x + nl_1) - H(x + ml_1 + a) \right]; \\ F_2(y) = \sum_{n=-\infty}^{\infty} \left[H(y + nl_2) - H(y + ml_2 + a) \right], \end{gathered} \tag{7.1}$$

where H is the Heaviside function.

We suppose that the ribs are thin and choose their width a as the parameter of asymptotic integration. To introduce parameters a, b explicitly, we analyze function $f(x) = H(x) - H(x+a)$. Applying two-sided Laplace transformation, and using development into a Maclaurin series, one obtains

$$\overline{f}(p) = a + \frac{\sum_{n=1}^{\infty} (-1)^n a^{n+1} p^n}{(n+1)!}, \tag{7.2}$$

where $\overline{f}(p)$ is the Laplace transform of $f(x)$ $(x \to p)$.

The inverse Laplace transform leads to the following series:

$$f(x) = a\delta(x) + \frac{\sum_{n=1}^{\infty} (-1)^n a^{n+1} \delta^{(n)}(x)}{(n+1)!}, \tag{7.3}$$

where $\delta(x)$ is the Dirac function.

Functions $F_1(x)$ and $F_2(y)$ can be expanded in a similar way. As a result, we obtain the following equations:

$$D\Delta\Delta w + Cw + D_1\Phi_1(x)w_{xxxx} + D_2\Phi_2(y)w_{yyyy} = q(x,y),$$

$$\begin{aligned}\Phi_1(x) &= \Phi_{10}(x) + \Phi_{11}(x) + \Phi_{12}(x)\\ &= \sum_{n=-\infty}^{\infty} a\delta(x+nl_1) - 0.5\sum_{n=-\infty}^{\infty} a^2\delta'(x+nl_1) + \sum_{n=-\infty}^{\infty}\sum_{k=2}^{\infty}(-1)^k a^{k+1}\delta^{(n)}(x+nl_1),\\ \Phi_2(y) &= \Phi_{20}(y) + \Phi_{21}(y) + \Phi_{22}(y)\\ &= \sum_{n=-\infty}^{\infty} a\delta(y+nl_2) - 0.5\sum_{n=-\infty}^{\infty} a^2\delta'(y+nl_2) + \sum_{n=-\infty}^{\infty}\sum_{k=2}^{\infty}(-1)^k a^{k+1}\delta^{(n)}(y+nl_2).\end{aligned} \tag{7.4}$$

A solution to the equation can be sought in the form of the following series:

$$w = w_0 + \sum_{i=0}^{\infty} a^i w_i. \tag{7.5}$$

In the zero-order approximation, one gets a plate with 1D ribs governed by the following PDE:

$$D\Delta\Delta w_0 + Cw + D_1\Phi_{10}(x)w_{0xxxx} + D_2\Phi_{20}(y)w_{0yyyy} = q(x,y). \tag{7.6}$$

Note that an influence of the ribs width appears in the next approximations.

8. Real and asymptotic errors

Accuracy of asymptotic methods is usually estimated by an asymptotic error, that is, owing to the order of estimation of the last omitted term. However, a researcher engaged in theory of plates and shells is more interested in a real rather than asymptotic error. It may happen that in order to increase real accuracy of the obtained solution, one has to omit the asymptotic character of constructed solutions. Some methods for decreasing the real error of constructed approximate solutions follow.

(1) Asymptotically accurate semimembrane theory of cylindrical shells can be developed using the condition of absence of shear and torsion deformations in the shell middle surface. However, the condition of absence of shear deformations is realized with less accuracy than for torsion deformation. Although, a theory constructed on the basis of only rotary deformation absence is asymptotically inaccurate, practically it gives more accurate results [40–42].

(2) Donnell-Mushtari-Marguerre equations are good approximation of complete system of nonlinear dynamical shell equations except a case, when vibrations

form in circumferential direction can be modeled as $\cos(2\pi y/R)$. Shkutin [43] proposed a slight modification of the Donnell-Mushtari-Marguerre equations for overcoming this drawback, which is exhibited by the following equations:

$$\begin{gathered}\frac{D}{h}\nabla_1^4 W = W_{xx}\Phi_{yy} + W_{yy}\Phi_{xx} - 2W_{xy}\Phi_{xy} + \frac{1}{R}\Phi_{xx} - \rho h W_{tt} = 0,\\ \frac{1}{E}\nabla_1^4\Phi + \frac{1}{R}W_{xx} + W_{xx}W_{yy} - \left(W_{xy}\right)^2 = 0,\end{gathered} \tag{8.1}$$

where:

$$\nabla_1^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^2}{\partial y^2}\left(\frac{\partial^2}{\partial y^2} + \frac{1}{R^2}\right). \tag{8.2}$$

(3) Owing to the asymptotic splitting of the boundary value problems, a fundamental error is introduced by simplification of the boundary conditions. In many cases, one may analytically obtain a general solution of edge effect equations. Using this solution, it is possible to exclude exactly the terms of the edge effect solution from the boundary conditions and hence avoid splitting of the boundary conditions [44].

(4) The method of composite equations is devoted to constructing uniformly suitable solutions on the basis of various limiting cases [45]. A fundamental idea of the method can be formulated in the following way. First, the components of the governing equations are detected, which, when neglected, lead to nonhomogeneity in a zero-order approximation. Second, the mentioned components are defined in a relatively simple way (they must include essential properties in the nonhomogeneous states). Matching of the limiting relations leads to uniformly suitable equations. In the theory of plates and shells, a composite equation of the stress-strain fundamental state has been obtained, unifying the semimembrane and membrane theories and a plane plate deformation. A simple edge effect and bending of the plate are included in a composite equation of the edge effect type. The obtained composite equations are of the fourth order because of a longitudinal variable and are applicable in the whole range of different loadings [26, 32, 46].

(5) For a posteriori error estimation of the asymptotic solutions, singular version of the Kantorovich theorem [46] can be successfully used [47].

9. Beyond the series locality

The principal shortcoming of asymptotic methods is the local nature of solutions yielded by them. Problems of elimination of the expansion locality, evaluation of the convergence domain, and construction of uniformly suitable solutions are highly expected.

There are many approaches to these problems [26, 30, 32, 48–50]: the method of analytic continuation, Borel summation procedure, Euler transformation, and Domb-Sykes diagram [45]. As a rule, they need a significant number of the expansion components.

Not diminishing the merits of the mentioned techniques, let us, however, note that in practice, only a few of the first components of the expansion of perturbations are usually known. Lately, the situation has indeed changed a little due to computer application. It may happen that a number of terms of asymptotic series can be increased without any serious problems. For instance, computing improvement terms with respect to an eigenvalue are usually successfully defined by eigenvalues and eigenfunctions. The knowledge of the nth eigenfunction allows us to define $2n+1$ eigenvalues [51]. However, until now, there are usually 3–5 components available in a perturbation series, and exactly from this segment of the series, we have to extract all available information. To this end, the method of Padé approximants may be very useful. Let us now define Padé approximants. Let

$$\begin{aligned} F(\varepsilon) &= \sum_{i=0}^{\infty} C_i \varepsilon^i, \\ F_{mn}(\varepsilon) &= \frac{\sum_{i=0}^{m} a_i \varepsilon^i}{\sum_{i=0}^{m} b_i \varepsilon^i}, \end{aligned} \tag{9.1}$$

where the coefficients a_i, b_i are determined from the following condition: the first $(m+n)$ components of the expansion of the rational function $F_{mn}(\varepsilon)$ in a Maclaurin series coincide with the first $(m+n+1)$ components of the series $F(\varepsilon)$. $F_{mn}(\varepsilon)$ is the Padé approximation of the function $F(\varepsilon)$.

Padé approximants perform meromorphic continuation of the function given in the form of the power series. If the Padé approximants sequence converges to a given function, then the roots of its denominators tend to singular points.

A wide application of the Padé approximants is observed due to its suitable properties. Among others, we should mention the effect of error autocorrection: even very significant errors in the coefficients of Padé approximants do not affect the accuracy of the approximation [52, 53]. This is because the errors in the numerator and the denominator of Padé approximants compensate each other. In other words, the errors in the coefficients of the Padé approximants are not distributed in an arbitrary way, but form the coefficients of a new approximant to the approximated function.

Padé approximants can be used for a heuristic evaluation of the domain of applicability of a perturbation series. The ε values, up to which the difference between calculations according to the truncated perturbation series and its diagonal Padé approximants do not exceed a given value (e.g., 5%), can be considered as a limiting value for applicability of the perturbation series.

10. Homotopy perturbation technique

Dorodnitzyn [54] proposed a method of introducing the parameter ε into the input boundary value problem in such a way that for $\varepsilon = 0$ the simplified problem is obtained, whereas for $\varepsilon = 1$ the input problem is governed. Then, the perturbation method can be used. Now, this approach is known as a homotopy perturbation technique [49, 55].

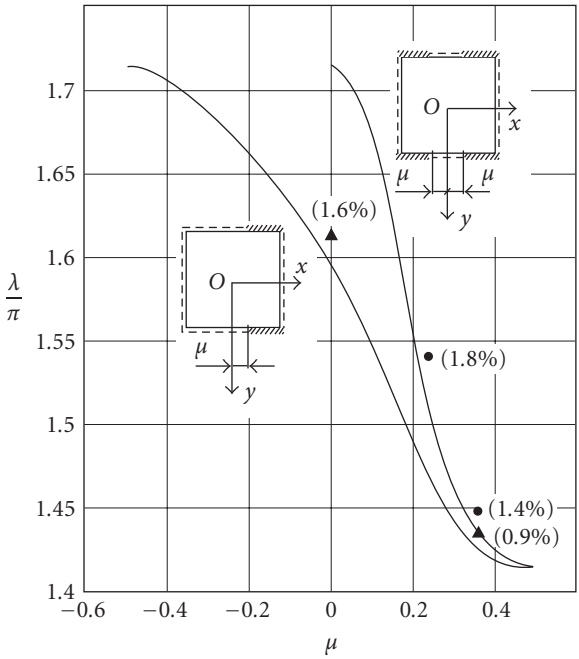

FIGURE 10.1. Relationship between the vibration frequency λ and the clamped segment length.

Unfortunately, perturbation series for $\varepsilon = 1$ usually diverges. In order to overcome this difficulty, the Padé approximants can be used effectively [26, 30, 32, 56].

Let us focus on the application of the homotopy perturbation method [26, 30, 32] when solving mixed BVP of the vibration of a rectangular plate ($-0.5k \le x \le 0.5k$, $-0.5 \le y \le 0.5$), simply supported at $x = \pm 0.5k$, and having mixed boundary conditions of the "clamped-simple supported" type, symmetrical to the y axis or the sides $y = \pm 0.5k$ (Figure 10.1). The governing equation is

$$\nabla^4 w - \lambda w = 0. \tag{10.1}$$

The boundary conditions after introducing a homotopy parameter have the following form:

$$\begin{gathered} w = 0, \quad w_{xx} = 0 \quad \text{for } x = \pm 0.5k, \\ w = 0, \quad w_{yy} = \overline{H}(x)\varepsilon(w_{yy} \pm w_y) \quad \text{for } y = \pm 0.5, \end{gathered} \tag{10.2}$$

where $\overline{H}(x) = -H(x-\mu) + H(-x-\mu)$.

Substituting w and λ in the form of ε-series

$$w = w_0 + \varepsilon w_1 + \cdots, \qquad \lambda = \lambda_0 + \varepsilon\lambda_1 + \cdots, \tag{10.3}$$

and after applying the usual perturbation procedure to the boundary value problem (10.1), (10.2), one obtains

$$\lambda_0 = \pi^4\psi^2, \qquad \lambda_1 = 4\pi^2 n^2 \gamma_{mm},$$
$$\lambda_2 = 4\pi^2 n^2 \gamma_{mm}\left\{1 - \frac{\gamma_{mm}}{\pi^2\psi}\left[\frac{\pi\alpha}{2}\operatorname{cth}^{(-1)^m}\left(\frac{\pi\alpha}{2}\right) + \frac{n^2}{\psi} - \frac{3}{2}\right]\right\}$$
$$- \frac{2n^2}{\psi}\sum_{\substack{\{i=1,3,5,\ldots\}\\ \{i=2,4,6,\ldots\}}}^{\infty} \gamma_{im}\left[\alpha_i \operatorname{cth}^{(-1)^i}\left(\frac{\alpha_i}{2}\right) + \begin{Bmatrix} -\phi_i \operatorname{cth}^{(-1)^i}(\phi_i/2) \\ \beta_i \operatorname{cth}^{(-1)^i}(\beta_i/2) \end{Bmatrix}\right], \quad \begin{Bmatrix} i^2 > m^2 + n^2 k \\ i^2 < m^2 + n^2 k \end{Bmatrix}, \tag{10.4}$$

where

$$\psi = n^2 + \frac{m^2}{k^2}, \qquad \alpha = \sqrt{2\frac{m^2}{k^2} + n^2}, \qquad \alpha_i = \sqrt{\frac{i^2 + m^2}{k^2} + n^2}, \qquad \beta_i = \pi\sqrt{\frac{m^2 - i^2}{k^2} + n^2},$$

$$\gamma_{im} = \begin{cases} 2(0.5 - \mu) + \dfrac{(-1)^m}{\pi m}\sin(2\pi\mu m), & \text{for } i = m \\ \dfrac{4}{\pi}\dfrac{1}{(m^2 - i^2)}\left[\begin{Bmatrix} i \\ m \end{Bmatrix}\sin(\pi\mu i)\cos(\pi\mu m) - \begin{Bmatrix} m \\ i \end{Bmatrix}\sin(\pi\mu m)\cos(\pi\mu i)\right], & \text{for } i \neq m, \end{cases} \tag{10.5}$$

and $\sum'$ is the sum without the component $i = m$.

Truncated perturbation series for $\mu = 0$ (both sides $y = \pm 0.5$ are completely clamped) for the square plate gives $(1.4783\pi)^4$. Padé approximants are

$$\lambda_p(\varepsilon) = \frac{a_0 + a_1\varepsilon}{1 + b_1\varepsilon}, \quad a_0 = \lambda_0,\ a_1 = \lambda_1 + b_1\lambda_0,\ b_1 = \frac{-\lambda_2}{\lambda_1}, \tag{10.6}$$

and for $\varepsilon = 1$ one obtains $\lambda_p = (1.7081\pi)^4$, while numerical value $\lambda = (1.7050\pi)^4$. Figure 10.1 presents the relation of λ versus μ and some experimental data (dots and triangles).

11. Theories of higher-order approximations

In order to increase approximation accuracy, the terms of higher order may remain in the input equations, but such an approach can increase the order of the approximate partial differential equation. This problem can be overcome by Padé approximants. Let us consider vibrations of a stretched beam modeled by the following equations:

$$w_{tt} - w_{\xi\xi} + \varepsilon w_{\xi\xi\xi\xi} = 0, \tag{11.1}$$
$$w = w_{\xi\xi} = 0 \quad \text{for } \xi = 0, 1. \tag{11.2}$$

Note that one may obtain a string-type model from (11.1) for $\varepsilon = 0$, namely,

$$w_{tt} - w_{\xi\xi} = 0, \tag{11.3}$$
$$w = 0 \quad \text{for } \xi = 0, 1. \tag{11.4}$$

In (11.1), instead of the differential operator $-\partial^2/\partial\xi^2 + \varepsilon\partial^4/\partial\xi^4$, one can use the following approximation:

$$\frac{-\partial^2}{\partial\xi^2} + \frac{\varepsilon\partial^4}{\partial\xi^4} \approx \frac{-\partial^2/\partial\xi^2}{(1+\varepsilon\partial^2/\partial\xi^2)}. \tag{11.5}$$

Finally one obtains

$$\left(1+\varepsilon\frac{\partial^2}{\partial\xi^2}\right)w_{tt} - w_{\xi\xi} = 0. \tag{11.6}$$

The associated boundary conditions have the form (11.4). Observe that if the model (11.3), (11.4) approximates eigenvalues of the initial problem up to the order of ε, then model (11.6), (11.4) includes second-order approximation of ε^2 preserving the equation order with respect to the spatial coordinates.

12. Matching of limiting asymptotics

It happens often that solutions related to two limiting values of a certain parameter can be easily constructed. In this case, one can define a solution valid for all parameter values with a help of two-point Padé approximants [26, 30, 32]. Let

$$F(\varepsilon) = \begin{cases} \sum_{i=0}^{\infty} a_i\varepsilon^i & \text{when } \varepsilon \longrightarrow 0, \\ \sum_{i=0}^{\infty} b_i\varepsilon^{-i} & \text{when } \varepsilon \longrightarrow A. \end{cases} \tag{12.1}$$

The two-point Padé approximation is represented by the following rational function:

$$F(\varepsilon) = \frac{\sum_{k=0}^{m} a_k\varepsilon^k}{\sum_{k=0}^{n} b_k\varepsilon^k}, \tag{12.2}$$

where $k+1$ $(k = 0,1,\ldots,n+m+1)$ are the coefficients of a Taylor expansion if $\varepsilon \to 0$, and $m+n+1-k$ are the coefficients of a Laurent series, and for $\varepsilon \to A$ they coincide with the corresponding coefficients of the series (12.1).

As an example, we consider the problem of nonlinear deformation of a sphere. The solution

$$\begin{gathered} Q = 0.42\varepsilon + 0.3\varepsilon^3 + 0(\varepsilon^5), \\ \varepsilon = 2(w/h)\sqrt{3\sqrt{1-\nu^2}}, \qquad Q = \frac{0.5qR^2 3\sqrt{1-\nu^2}}{Eh^2} \end{gathered} \tag{12.3}$$

has been obtained by means of the asymptotic methods for a closed sphere subjected to the uniform external pressure q [27]. In the above, w is the amplitude of post-buckling axially symmetric equilibrium form.

In the region of small displacements, the Koiter approach [13] holds, and hence

$$Q = 1 + 0(\varepsilon^{-4}). \tag{12.4}$$

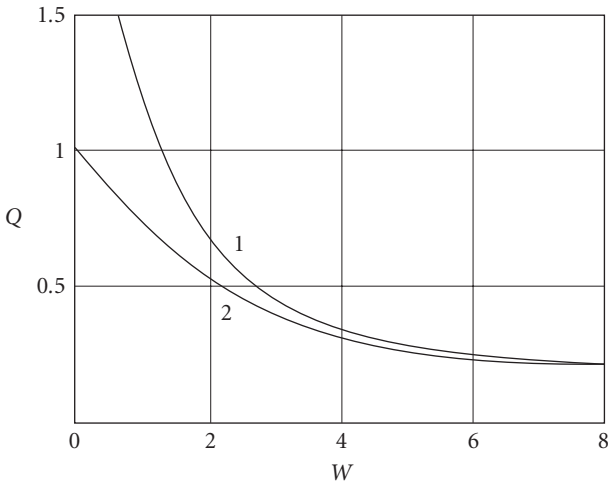

Figure 12.1. Matching of quasilinear and essentially nonlinear asymptotics.

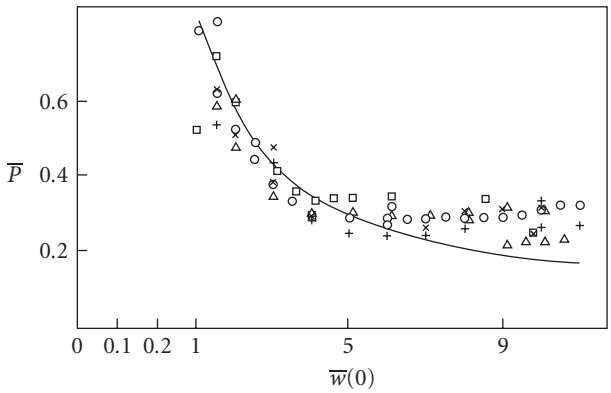

Figure 12.2. Comparison of two-point Padé approximation solution (solid line) with experimental results.

By matching expansions (12.3) and (12.4) with the two-point Padé approximation, one obtains the following solution [27]:

$$Q = \frac{A}{A + 2.19}, \quad A = \varepsilon^4 + 0.082\varepsilon^3 + 0.386\varepsilon^2 + 0.92\varepsilon. \tag{12.5}$$

Curves 1 and 2 in Figure 12.1 correspond to solutions (12.3), (12.5), respectively. Accuracy of solution (12.5) is confirmed by comparison with the precise numerical solution.

In Figure 12.2, results of comparison of experimental data for post-buckling equilibrium states of shallow elliptic parabolic-shaped shells under external pressure [57] with the solution based on two-point Padé approximation [27] are shown, where $\overline{w} = w/h$; $\overline{P} = (0.5qR_1R_2 3\sqrt{1-\nu^2})/Eh^2$.

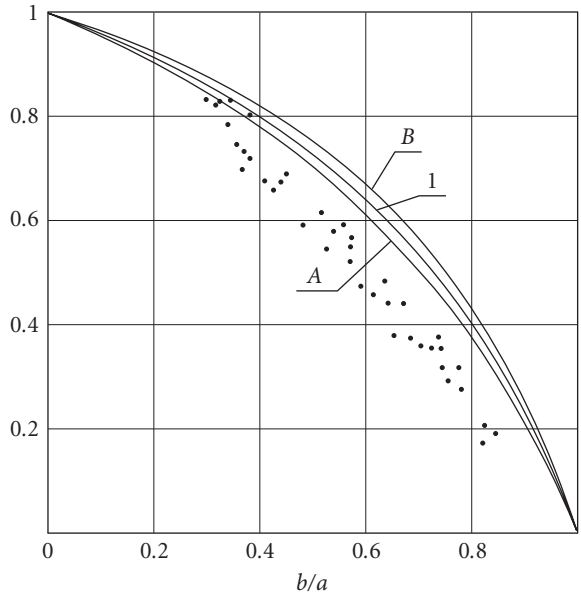

Figure 12.3. Homogenized coefficients of perforated plates.

The second example is associated with homogenization of a rectangular plate with circular perforations. Analytical solutions for small and large holes were obtained [12] by using the AM perturbation of the domain and boundary form. Coefficients A and B of the homogenized equation

$$A(W_{xxxx} + W_{yyyy}) + 2BW_{xxyy} = q(x, y) \tag{12.6}$$

are yielded by the following expressions (for $\nu = 0.3$):

$$A = \frac{1-\lambda}{1-0.5785\lambda}, \qquad B = \frac{1-\lambda}{1-0.6701\lambda}, \tag{12.7}$$

where $\lambda = b/a$ (b is the diameter of the hole, a is the length of the square cell side).

Figure 12.3 shows the numerical results for A and B.

The values of coefficients are compared with both theoretical results obtained by means of the two-period elliptic functions (curve 1 in Figure 12.3) and experimental results (points in Figure 12.3).

Evidently, the two-point Padé approximants are not a panacea. As a rule, one of the limit expansions ($\varepsilon \to 0$ or $\varepsilon \to A$) contains logarithmic or exponential terms. In this case, one can use the method of asymptotically equivalent functions. Suppose that we have a perturbation approach in powers of ε for $\varepsilon \to 0$ and asymptotic expansions $F(\varepsilon)$ containing logarithm for $\varepsilon \to A$. By definition, an asymptotically equivalent function is the ratio with unknown coefficients a_i, b_i, containing both powers of ε and function $F(\varepsilon)$. The coefficients a, b are chosen in such a way, that the expansion of a ratio in powers of

ε matches the corresponding perturbation expansion and the asymptotic behavior of the ratio for $\varepsilon \to \infty$ coincides with $F(\varepsilon)$.

13. Nonlinear problems

Although asymptotic techniques regarding linear problems of theory of plates and shells are relatively good developed, there are still many unsolved tasks related to nonlinear problems. In particular, nonlinear systems with distributed parameters exhibit various internal resonance between modes. It may happen that neglection of higher modes may yield also erroneous results [58]. In [59], the asymptotic method has been proposed, where all modes of vibrations can be approximately applied. In order to show main features of the proposed approach, let us consider free vibrations of a membrane attached to a nonlinear foundation. The governing equation follows:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \beta_1 u - \varepsilon\beta_2 u^3 = 0, \tag{13.1}$$

where ε is a nondimensional parameter ($\varepsilon \ll 1$).

The boundary conditions have the following form:

$$u|_{x=0,l_1} = u|_{y=0,l_2} = 0. \tag{13.2}$$

A being sought periodic solution satisfies the following periodicity requirement:

$$u(t) = u(t+T), \tag{13.3}$$

where $T = 2\pi/\omega$ is a period, and ω is a natural vibration frequency.

We are going to find natural frequencies of vibrations associated with such fundamental modes that in the associated linear case (for $\varepsilon = 0$) only half waves appear in both x- and y-directions. Now we proceed in a usual way, using Lindstedt-Poincaré procedure [60]. Namely, we scale time

$$\tau = \omega t, \tag{13.4}$$

and the following series are introduced:

$$\begin{aligned} u &= u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots, \\ \omega &= \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots. \end{aligned} \tag{13.5}$$

Substitution of relations (13.5) into (13.1), (13.2), (13.3) and comparison of the terms standing by the same powers of ε yields the following set of linear boundary value problems:

$$\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} - \omega_0^2 \frac{\partial^2 u_0}{\partial \tau^2} - \beta_1 u_0 = 0, \tag{13.6}$$

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} - \omega_1^2 \frac{\partial^2 u_1}{\partial \tau_1^2} - \beta_1 u_1 = 2\omega_0\omega_1 \frac{\partial^2 u_0}{\partial \tau^2} + \beta_2 u_0^3. \tag{13.7}$$

Both boundary conditions (13.2) and periodicity relations (13.3) are cast to the following form:

$$u_i|_{x=0,l_1} = u_i|_{y=0,l_2} = 0;$$
$$u_i(\tau) = u_i(\tau + 2\pi). \tag{13.8}$$

A solution to (13.6) has the following form

$$u_{0,0} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{m,n} \sin\left(\frac{\omega_{m,n}^{\text{lin}}}{\omega_{1,0}}\tau\right) \sin\left(\frac{\pi m}{l_1}x\right) \sin\left(\frac{\pi n}{l_2}y\right), \tag{13.9}$$

where $A_{1,1}$ is the amplitude of the principal mode; $A_{m,n}$, $m,n = 1,2,3,\ldots$, $(m,n) \neq (1,1)$ are the amplitudes of the successive modes: $\omega_{m,n}^{\text{lin}} = \sqrt{(\pi^2 m^2/l_1^2) + (\pi^2 n^2/l_2^2) + \beta_1}$, $m,n = 1,2,3,\ldots$ are the natural frequencies of the linear system, and $\omega_0 = \omega_{1,1}^{\text{lin}}$.

Next approximation regarding ε is found owing to solution of the boundary value problem governed by (13.7), (13.8). In order to cancel the secular terms, the coefficient standing by the terms of the form $\sin((\omega_{m,n}^{\text{lin}}/\omega_0)\tau)\sin((\pi m/l_1)x)\sin((\pi n/l_2)y)$, $m,n = 1,2,3,\ldots$, occurred in the right-hand side of (13.7) are assigned to zero. This approach yields the following infinite system of algebraic equations:

$$\frac{2A_{m,n}\omega_1}{\beta_2\omega_0}\left(\omega_{m,n}^{\text{lin}}\right)^2 = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{p=1}^{\infty}\sum_{s=1}^{\infty} C_{m,n}^{(ijklps)} A_{i,j}A_{k,l}A_{p,s}, \quad m,n = 1,2,3,\ldots. \tag{13.10}$$

The coefficients $C_{m,n}^{(ijklps)}$ are found via substitution of relations (13.9) into the right-hand side of (13.7).

Note that system (13.10) can be solved via a reduction method. However, owing to introduction of many equations the essential difficulties regarding efficient computation may appear. Besides, the mentioned approach does not include higher modes interaction. In order to omit the mentioned problem, one may introduce a new parameter μ such that for $\mu = 0$ the studied system is essentially simplified. Then, the solution as a series regarding that parameter is constructed and finally $\mu = 1$ is assumed.

In our study in the right-hand side of each (m,n)th equation of system (13.10), a parameter μ is introduced before the terms $A_{i,j}A_{r,l}A_{p,s}$ for which the following condition holds: $(i > m) \cup (k > m) \cup (p > m) \cup (j > n) \cup (l > n) \cup (s > n)$. Note that now system (13.10) takes "triangle" form for $\mu = 0$, whereas for $\mu = 1$ the system takes the initial form. The solution is sought further in the following series form:

$$\omega_1 = \omega^{(0)} + \mu\omega^{(1)} + \mu^2\omega^{(2)} + \cdots,$$
$$A_{m,n} = A_{m,n}^{(0)} + \mu A_{m,n}^{(1)} + \mu^2 A_{m,n}^{(2)} + \cdots, \quad m,n = 1,2,3,\ldots, \ (m,n) \neq (1,1), \tag{13.11}$$

and then one assumes $\mu = 1$. Finally, let us emphasize that the mentioned approach allows to contain in systems (13.10) arbitrary number of equations.

14. Merits and demerits of the asymptotic methods

Advantages of asymptotic methods follow.

(1) Essentially simplified solutions, which in many cases can be obtained in an analytical way.

(2) Asymptotic methods are easily matched with other approaches, that is, numerical, variational ones, and so forth. Owing to the introduced simplification of the input boundary value problem and separation of the associated peculiarities of the considered problem, one may effectively apply numerical approaches. Asymptotic methods allow us to exhibit the structure of solution and the type of approximating functions in the Bubnov-Galerkin, Rayleigh-Ritz, Trefftz and Kantorovich approaches. Owing to the construction of zero order solution, it can be applied as a starting solution for other iteration processes like the Newton-Kantorovich method.

(3) Asymptotic methods are strictly associated with a physical aspect of the analyzed problem allowing for it easier understanding.

(4) Asymptotic methods allow us to explain mathematical and physical bases of approximated engineering methods, increasing their accuracy and reliability of obtained results.

(5) Asymptotic methods give a possibility of a unified approach to various different problems exhibiting their common aspects and internal unity.

However, the main drawback of asymptotic methods is generated by insufficiently accurate results of low approximations, since a construction of successive approximations is not always easy. Also an accuracy of the estimation of asymptotic methods and intervals of their applicability in many cases causes serious difficulties.

15. Concluding remarks

Many important methods like WKB [24] or matched asymptotic expansion [61] are omitted in our review. Other interesting problems such as junction of plates and shells with 1D and 3D bodies or junction of two shells [62, 63], solutions of shell problems in singular domains [60, 64], have not been considered either.

In addition, an application of the asymptotic methods in the localization problems [24, 65] are also omitted in our paper.

One can also add to this list the problems of bonding, which arise in laminated plates and shells and which attracted many researches in the recent past [66, 67].

Important results related to the so-called first-order accuracy problems in theory of plates and shells have been reported by Gol'denveĭzer et al. [68] and Nazarov [69]. They show, among others, that inclusion into consideration of 3D boundary layers may improve accuracy order of the being modeled systems.

In our paper, we are mainly focused on linear problems. On the other hand there is no doubt that the development regarding application of asymptotic methods devoted to analysis of nonlinear problems plays a key role in nonlinear problems of both dynamics and stability of continuous systems [70, 71].

It is expected that further development of asymptotic methods is associated with combined numerical-analytical approaches and includes them in standard codes. This is important because an accurate numerical computation of shells with arbitrarily small thickness is impossible in practice. Standard finite-element codes usually fail to give accurate results for $h/R \sim 0.01$ or 0.001.

Nowadays, in order to compute thin-walled structures, the standard finite-element codes are used. It seems that asymptotic information is rather rarely applied. On the other hand, asymptotic methods belong to fundamental ones during the construction of mathematical models of physical processes [22, 72]. "Design of computational or experimental schemes without the guidance of asymptotic information is wasteful at best, dangerous at worst, because of possible failure to identify crucial (stiff) features of the process and their localization in coordinate and parameter space. Moreover, all experience suggests that asymptotic solutions are useful numerically far beyond their nominal range of validity, and can often be used directly, at least at a preliminary product design stage, for example, saving the need for accurate computation until the final design stage, where many variables have been restricted to narrow ranges" [72].

Finally, there are many books and papers devoted to the considered problems, and therefore only some of them are cited. However, a reader may find additional references in [24, 30, 32, 33, 50, 60, 65, 73–76] to extend knowledge associated with asymptotic approaches to plates and shells modeling.

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