Research Article

On the Plane Geometry with Generalized Absolute Value Metric

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Metric spaces are among the most important widely studied topics in mathematics. In recent years, mathematicians began to investigate using other metrics different from Euclidean metric. These metrics also find their place in the computer age in addition to their importance in geometry. In this paper, we consider the plane geometry with the generalized absolute value metric and define trigonometric functions and norm and then give a plane tiling example for engineers underlying Schwarz’s inequality in this plane.

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1. Introduction

In the late 1800’s, Minkowski [1] published a whole family of metrics providing new insight into the study of plane geometry. In recent years, mathematicians and engineers begin to investigate non-Euclidean metrics used in real-life applications. For example, Eisenberg and Khabbaz [2] discussed the applications of the taxicab metric in geometry and network theory. Similarly, Burman et al. [3] utilized the concept of $\lambda$-geometry and discussed its applications in the problem of global routing of multiterminal nets.

Our goal is to further develop the plane geometry with the generalized absolute value metric $d_g$ suggested by Kaya et al. [4], and to see how it can lead to future researches in related areas. In $\mathbb{R}^2$, the $d_g$-distances between $X = (x_1, y_1)$ and $Y = (x_2, y_2)$ are defined by

$$d_g(X, Y) = d_g((x_1, y_1), (x_2, y_2)) = a \max \{|x_1 - x_2|, |y_1 - y_2|\} + b \min \{|x_1 - x_2|, |y_1 - y_2|\}$$

for all $a, b \in \mathbb{R}$, $a \geq b \geq 0$, $a \neq 0$. According to the definition of $d_g$-distances, the shortest way between two points $X$ and $Y$ is the union of a vertical or a horizontal line segment and a line segment with the slope $\pm 2ab/(a^2 - b^2)$. 
We will show that the $d_g$-distance defined above is a metric. First, it is positive definite and symmetric, and it can be shown that $d_g$ satisfies the triangle inequality as follows. For $X = (x_1, y_1)$, $Y = (x_2, y_2)$, and $Z = (x_3, y_3)$, since

\[
|x_1 - x_2| = |x_1 - x_3 + x_3 - x_2| \leq |x_1 - x_3| + |x_3 - x_2|,
\]
\[
|y_1 - y_2| = |y_1 - y_3 + y_3 - y_2| \leq |y_1 - y_3| + |y_3 - y_2|,
\]

one obtains

\[
d_g(X, Y) = a \max \{ |x_1 - x_2|, |y_1 - y_2| \} + b \min \{ |x_1 - x_2|, |y_1 - y_2| \}
= a \max \{ |x_1 - x_3 + x_3 - x_2|, |y_1 - y_3 + y_3 - y_2| \}
+ b \min \{ |x_1 - x_3 + x_3 - x_2|, |y_1 - y_3 + y_3 - y_2| \}
\leq a \max \{ |x_1 - x_3| + |x_3 - x_2|, |y_1 - y_3| + |y_3 - y_2| \}
+ b \min \{ |x_1 - x_3| + |x_3 - x_2|, |y_1 - y_3| + |y_3 - y_2| \} = k. \tag{1.3}
\]

Let us examine the following four cases:

(i) $|x_1 - x_3| \geq |y_1 - y_3|$ and $|x_3 - x_2| \geq |y_3 - y_2|$;

(ii) $|x_1 - x_3| \geq |y_1 - y_3|$ and $|x_3 - x_2| \leq |y_3 - y_2|$;

(iii) $|x_1 - x_3| \leq |y_1 - y_3|$ and $|x_3 - x_2| \leq |y_3 - y_2|$;

(iv) $|x_1 - x_3| \leq |y_1 - y_3|$ and $|x_3 - x_2| \geq |y_3 - y_2|$.

We will only prove the first two cases since the remaining cases can be proved similarly.

If $|x_1 - x_3| \geq |y_1 - y_3|$ and $|x_3 - x_2| \geq |y_3 - y_2|$, then

\[
d_g(X, Y) \leq k = a( |x_1 - x_3| + |x_3 - x_2| ) + b(|y_1 - y_3| + |y_3 - y_2|)
= a|x_1 - x_3| + b|y_1 - y_3| + a|x_3 - x_2| + b|y_3 - y_2| \tag{1.4}
= d_g(X, Z) + d_g(Z, Y).
\]

If $|x_1 - x_3| \geq |y_1 - y_3|$ and $|x_3 - x_2| \leq |y_3 - y_2|$, then there are two possible situations.

(i) Let $|x_1 - x_3| + |x_3 - x_2| \geq |y_1 - y_3| + |y_3 - y_2|$. Then

\[
d_g(X, Y) \leq k = a( |x_1 - x_3| + |x_3 - x_2| ) + b(|y_1 - y_3| + |y_3 - y_2|). \tag{1.5}
\]

Using $|x_1 - x_3| \geq |y_1 - y_3|$ and $|x_3 - x_2| \leq |y_3 - y_2|$,

\[
d_g(X, Z) = a|x_1 - x_3| + b|y_1 - y_3|, \tag{1.6}
\]
\[
d_g(Z, Y) = a|y_3 - y_2| + b|x_3 - x_2|.
\]
Also, \(d_g(X, Y) \leq d_g(X, Z) + d_g(Y, Z)\) if and only if
\[
d_g(X, Y) \leq k = a(|x_1 - x_3| + |x_3 - x_2|) + b(|y_1 - y_3| + |y_3 - y_2|) \\
k \leq a(|x_1 - x_3| + |y_3 - y_2|) + b(|y_1 - y_3| + |x_3 - x_2|) \\
\iff a(|x_3 - x_2| - |y_3 - y_2|) + b(|y_3 - y_2| - |x_3 - x_2|) \leq 0 \quad (1.7) \\
\iff (a - b)(|x_3 - x_2| - |y_3 - y_2|) \leq 0 \iff |x_3 - x_2| \leq |y_3 - y_2|.
\]

(ii) Let \(|x_1 - x_3| + |x_3 - x_2| \leq |y_1 - y_3| + |y_3 - y_2|\). Then
\[
d_g(X, Y) \leq k = a(|y_1 - y_3| + |y_3 - y_2|) + b(|x_1 - x_3| + |x_3 - x_2|).
\] (1.8)

Using \(|x_1 - x_3| \geq |y_1 - y_3|\) and \(|x_3 - x_2| \leq |y_3 - y_2|\),
\[
d_g(X, Z) = a|x_1 - x_3| + b|y_1 - y_3|, \\
d_g(Z, Y) = a|y_3 - y_2| + b|x_3 - x_2|, \\
d_g(X, Y) \leq d_g(X, Z) + d_g(Y, Z) \iff \\
d_g(X, Y) \leq k = a(|y_1 - y_3| + |y_3 - y_2|) + b(|x_1 - x_3| + |x_3 - x_2|) \\
k \leq a(|x_1 - x_3| + |y_3 - y_2|) + b(|y_1 - y_3| + |x_3 - x_2|) \\
\iff a(|y_1 - y_3| - |x_1 - x_3|) + b(|x_1 - x_3| - |y_1 - y_3|) \leq 0 \iff (a - b)(|y_1 - y_3| - |x_1 - x_3|) \leq 0 \iff |y_1 - y_3| \leq |x_1 - x_3|.
\] (1.9)

Note that the generalized absolute value metric \(d_g\) is a generalization of the taxicab metric \(d_T\), maximum metric \(d_{\text{max}}\), and Chinese checker metric \(d_c\) in the following manner: \(d_g = d_T\) if \(a = b = 1\); \(d_g = d_{\text{max}}\) if \(a = 1\) and \(b = 0\); \(d_g = d_c\) if \(a = 1\) and \(b = \sqrt{2} - 1\), respectively. Furthermore, the unit circle in \((\mathbb{R}^2, d_g)\) is the set of points \((x, y)\) in the plane which satisfies
\[
a \max \{|x|, |y|\} + b \min \{|x|, |y|\} = 1, \quad a \geq b \geq 0, a \neq 0.
\] (1.10)

It is well known that Schwarz’s inequality is a fundamental inequality of the Euclidean plane, which states that the magnitude of the inner product of two vectors cannot exceed the product of the lengths of these two vectors in the plane. Our main goal is to show the validity of Schwarz’s inequality in \((\mathbb{R}^2, d_g)\). To achieve this goal, our presentation is organized as follows. After the definitions of trigonometric functions in \((\mathbb{R}^2, d_g)\) are given, we define the norm, prove Schwarz’s inequality, and give an area formula of a triangle in this plane (Section 3). Finally, we give a plane tiling example underlying Schwarz’s inequality in Section 4.

2. The trigonometric functions

We know that if \(X = (x, y)\) is a point on the Euclidean unit circle, then \(x = \cos \theta\) and \(y = \sin \theta\), where \(\theta\) is the angle with the positive \(x\)-axis as the initial side and the radial line passing through point \((x, y)\) as the terminal side. Now, we can define the trigonometric functions
cosine and sine of the angle \( \theta \) in \((\mathbb{R}^2, d_g)\) in the same way as the Euclidean cosine and sine functions. Let \((x, y)\) be a point on the unit circle in \((\mathbb{R}^2, d_g)\). Since the unit circle is an octagon in \((\mathbb{R}^2, d_g)\) (see [4]), trigonometric functions can be defined according to eight quadrants. If we show the cosine and sine functions by \(\cos_g \theta\) and \(\sin_g \theta\), they can be given as follows with the method in [5]:

\[
\begin{align*}
\cos_g \theta &= \begin{cases} 
\frac{\cos \theta}{a|\cos \theta| + b|\sin \theta|} & \text{if } |\tan \theta| \leq 1, \\
\frac{\cos \theta}{b|\cos \theta| + a|\sin \theta|} & \text{if } |\tan \theta| \geq 1,
\end{cases} \\
\sin_g \theta &= \begin{cases} 
\frac{\sin \theta}{a|\cos \theta| + b|\sin \theta|} & \text{if } |\tan \theta| \leq 1, \\
\frac{\sin \theta}{b|\cos \theta| + a|\sin \theta|} & \text{if } |\tan \theta| \geq 1.
\end{cases}
\end{align*}
\]

One can easily see that if \(\theta'\) is the related angle in first or second quadrant of given angle \(\theta\) in quadrant \(i\), \((\pi/4)(i - 1) \leq \theta \leq (\pi/4)i\), \(i = 1, \ldots, 8\). The relationship between \(\cos_g \theta\) and \(\cos_g \theta'\) or \(\sin_g \theta\) and \(\sin_g \theta'\) is the same as the Euclidean one.

We note that there is the nonuniform increment in arc length when the angle is incremented by a fixed amount whereas on the Euclidean unit circle. So, we want to define trigonometric functions any angle \(\theta\) between any two lines by using reference angle \(\alpha\) of \(\theta\) [6].

**Definition 2.1.** Let \(\theta\) be an angle with the reference angle \(\alpha\) which is the angle between \(\theta\) and the positive direction of the \(x\)-axis in \(d_g\)-unit circle. Then the cosine and sine functions of angle \(\theta\) with the reference angles \(\alpha\), \(g\cos \theta\), and \(g\sin \theta\) are defined by

\[
g\cos \theta := \cos_g(\alpha + \theta) \cdot \cos_g \alpha + \sin_g(\alpha + \theta) \cdot \sin_g \alpha, \\
g\sin \theta := \sin_g(\alpha + \theta) \cdot \cos_g \alpha - \cos_g(\alpha + \theta) \cdot \sin_g \alpha.
\]

One can easily see that if \(\alpha = 0\), then \(g\cos \theta = \cos_g \theta\) and \(g\sin \theta = \sin_g \theta\).

### 2.1. Variation of \(d_g\)-lengths under rotations

It is well known that all translations and rotations of the plane preserve the Euclidean distance. It is obvious that \(d_g\)-distances are invariant under all translations. Now, we want to investigate the change of \(d_g\)-lengths of a line segment after rotations as in [7].

**Theorem 2.2.** Let any two points be \(A\) and \(B\) in \((\mathbb{R}^2, d_g)\), and let the line segment \(AB\) be not parallel to the \(x\)-axis and the angle \(\alpha\) between the line segment \(AB\) and the positive direction of \(x\)-axis. If \(A' B'\) is the image of \(AB\) under the rotations with the angle \(\theta\), then

\[
d_g(A, B') = \frac{\cos_g^2 \alpha + \sin_g^2 \alpha}{\cos_g^2 (\alpha + \theta) + \sin_g^2 (\alpha + \theta)} \cdot d_g(A, B). \tag{2.3}
\]
Proof. We can translate the line segment $AB$ to the line segment $OX$ such that

$$d_g(A, B) = d_g(O, X) = k$$

(2.4)

since all the translations preserve the $d_g$-distance. $X = (k \cdot \cos \alpha, k \cdot \sin \alpha)$ and under the rotation $\theta$, $OX$ transforms to $OX'$, $X' = (k \cdot \cos(\alpha + \theta), k \cdot \sin(\alpha + \theta))$. If we calculate $d_g(O, X') = k'$ by using $d_E(O, X) = d_E(O, X')$, we obtain

$$k' = k \cdot \sqrt{\frac{\cos^2 \alpha + \sin^2 \alpha}{\cos^2(\alpha + \theta) + \sin^2(\alpha + \theta)}}.$$  

(2.5)

Corollary 2.3. If the segment $AB$ is parallel to the x-axis, then

$$d_g(A', B') = \frac{d_g(A, B)}{\sqrt{\cos^2 \theta + \sin^2 \theta}}.$$  

(2.6)

Proof. In this situation, since $\alpha = 0$, the proof is obvious.  

3. Norm in $(\mathbb{R}^2, d_g)$

In this section, we will define the norm in $(\mathbb{R}^2, d_g)$ by using the standard inner product in $\mathbb{R}^2$ as in [8].

Definition 3.1. Let $A = (x, y)$ be any vector in $(\mathbb{R}^2, d_g)$. Then the norm of $A$ is defined by

$$\|A\|_g = \sqrt{\frac{\langle A, A \rangle}{\cos^2 \alpha + \sin^2 \alpha}},$$

(3.1)

where $\alpha$ is the reference angle of $A$.

Corollary 3.2. $\|A\|_g = d_g(O, A)$.

Proof. If $A = (x, y) = (d_g(O, A) \cos \alpha, d_g(O, A) \sin \alpha)$ is used,

$$d_E(O, A) = \sqrt{\cos^2 \alpha + \sin^2 \alpha} \cdot d_g(O, A).$$

(3.2)

If we use $d_E(O, A) = \sqrt{\langle A, A \rangle}$ and Definition 3.1,

$$\|A\|_g = d_g(O, A)$$

(3.3)

is obtained.
As in Euclidean geometry, it is easily seen that the norm function \(| \cdot |_g|\) satisfies the following properties. So, the following proposition is given without proof.

**Proposition 3.3.** Let \( A = (x_1, y_1) \), \( B = (x_2, y_2) \), and \( C = (x_3, y_3) \) be any three vectors in \((\mathbb{R}^2, d_g)\) and \( r \in \mathbb{R} \). Then

(i) \(|A|_g \leq 0 \) and \(|A|_g = 0 \iff A = 0\),
(ii) \(|rA|_g = |r| \cdot |A|_g|\),
(iii) \(|A + B|_g \leq |A|_g + |B|_g|\),
(iv) \(|A - B|_g \geq |A|_g - |B|_g|\),
(v) \(|A - B|_g \leq |A - C|_g + |C - B|_g|\).

It is important that Schwarz’s inequality is valid for a metric. For example, matched filters are found in nearly every communication device, such as cell phones and television broadcasts. The detector utilizes Cauchy-Schwarz’s inequality to compare functions. When the inner product of the functions is “large,” then there are significant similarities between the two signals. This detector will help us better select which tile is appropriate for a small section of the final image. If a quantized block of the target image is mostly one color on one side but another color on the opposite side, then the matched filter detector will find a tile with similar attributes.

The following proposition shows that Schwarz’s inequality is valid for some restricted cases of \( d_g\).

**Proposition 3.4 (Schwarz’s inequality).** If \( A = (x_1, y_1) \) and \( B = (x_2, y_2) \) are any two vectors in \((\mathbb{R}^2, d_g)\) and \( b \geq 1 \), then

\[
|\langle A, B \rangle| \leq |A|_g \cdot |B|_g.
\]

**Proof.** (i) If \( A = 0 \), then \( \langle A, B \rangle = 0 \) and \( |\langle A, B \rangle| = 0 \). Furthermore, \(|A|_g = 0 \) and \(|A|_g \cdot |B|_g = 0\).

(ii) Let \( A \neq 0 \), then we know that

\[
|\langle A, B \rangle| = |x_1 x_2 + y_1 y_2| \leq |x_1| x_2 + |y_1| y_2|.
\]

\[
|A|_g \cdot |B|_g = (a \max \{|x_1|, |y_1| \} + b \min \{|x_1|, |y_1| \}) \cdot (a \max \{|x_2|, |y_2| \} + b \min \{|x_2|, |y_2| \})
\]

\[
= a^2 |x_1 x_2| + ab (|x_1 y_2| + |y_1 x_2|) + b^2 |y_1 y_2|.
\]

In this case, there are four subcases.

(1) \(|x_1| \geq |y_1|\) and \(|x_2| \geq |y_2|\).
(2) \(|x_1| \geq |y_1|\) and \(|x_2| \leq |y_2|\).
(3) \(|x_1| \leq |y_1|\) and \(|x_2| \geq |y_2|\).
(4) \(|x_1| \leq |y_1|\) and \(|x_2| \leq |y_2|\).

Here, subcase (1) is proved; the others can be similarly proved.
If the right-hand sides of (3.4) and (3.5) are observed,

\[ ab|x_1y_2| + ab|y_1x_2| + a^2|x_1x_2| + b^2|y_1y_2| \geq |x_1x_2 + y_1y_2| \]

(3.7)
is valid since \( b \geq 1, |x_1| \geq |y_1|, \) and \( |x_2| \geq |y_2| \). This completes the proof.

Now, we will give the area of the triangle \( \triangle ABC \).

**Theorem 3.5.** Let \( \triangle ABC \) be any triangle in \((\mathbb{R}^2, d_g)\) and let \( \theta \) be the angle between \( AC \) and \( BC \). Then the area \( A \) of the triangle \( \triangle ABC \) is

\[ A = \frac{1}{2} \cdot ||AC||_g \cdot ||BC||_g \cdot g \sin \theta. \]

(3.8)

**Proof.** One can take that \( C \) is the origin since all the translations preserve \( d_g \)-distance. One can easily obtain the following relation between \( g \sin \theta \) and \( \sin \theta \):

\[ g \sin \theta = \frac{||OA|| \cdot ||OB||}{||OA||_g \cdot ||OB||_g} \cdot \sin \theta. \]

(3.9)

From (3.9) and \( 2A = ||OA|| \cdot ||OB|| \cdot \sin \theta \) for \( 0 \leq \theta \leq \pi \),

\[ A = \frac{1}{2} \cdot ||AC||_g \cdot ||BC||_g \cdot g \sin \theta. \]

(3.10)

4. Application

In Latin, tessella was a small cubical piece of clay, stone, or glass used to make mosaics as mentioned in [9, 10]. The word “tessella” means “small square” (from “tessera,” square, derived from the Greek word for “four”). It also corresponds to the everyday term “tiling” which refers to the applications of tessellation, often made of glazed clay.

Certain shapes of tiles, most obviously rectangles, can be replicated to cover a surface with no gaps. These shapes are said to tessellate (from the Latin tessella, “tile”). Over the centuries, many ceramic engineers and artists have employed plane tiling in their work. They used tiles for floors and walls because they are durable, waterproof, and beautiful. If you are mathematically inclined, no doubt you see some mathematics in the tiling. There are some examples where plane tiling on floors, walls, and in paintings underlies proofs of some well-known (and some not so well-known) theorems. However, tilings provide “picture proofs” for many theorems. They also illustrate inequalities such as the arithmetic mean-geometric mean inequality. Squares and rectangles are some of the most popular tiling used by engineers and artists to tile the plane. The design of geometric shapes, which fit together to cover a surface without gaps or overlapping, has a long history. Now, we will give a plane tiling example satisfying Schwarz’s inequality in \((\mathbb{R}^2, d_g)\) as an application.

Suppose that the road is tiled with bricks of two different sizes as illustrated in **Figure 1**. Let \( A = (x_1, y_1) \) and \( B = (x_2, y_2) \) be any two vectors in \((\mathbb{R}^2, d_g)\) and the side lengths of two bricks are \( |x_1|, |x_2| \) and \( |y_1|, |y_2| \), respectively; let \( d_g \) satisfy the condition \( b \geq 1 \) such
that $C = (x_2, y_1)$ and $D = (x_1, y_2)$. Then the following tiling forms part of a proof of some special cases of Schwarz’s inequality in two dimensions.

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**References**

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