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Research Article

A Wavelet Galerkin Finite-Element Method for the Biot Wave Equation in the Fluid-Saturated Porous Medium

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A wavelet Galerkin finite-element method is proposed by combining the wavelet analysis with traditional finite-element method to analyze wave propagation phenomena in fluid-saturated porous medium. The scaling functions of Daubechies wavelets are considered as the interpolation basis functions to replace the polynomial functions, and then the wavelet element is constructed. In order to overcome the integral difficulty for lacking of the explicit expression for the Daubechies wavelets, a kind of characteristic function is introduced. The recursive expression of calculating the function values of Daubechies wavelets on the fraction nodes is deduced, and the rapid wavelet transform between the wavelet coefficient space and the wave field displacement space is constructed. The results of numerical simulation demonstrate that the method is effective.

1. Introduction

The fluid-saturated porous medium is modeled as a two-phase system consisting of a solid and a fluid phase. It is assumed that the solid phase is homogenous, isotropic, elastic frame and the fluid phase is viscous, compressible, and filled with the pore space of solid frame. Compared with the single-phase medium theory, fluid-saturated porous medium theory can describe the formation underground more precisely and the fluid-saturated porous medium elastic wave equation can bring more lithology information than ever. For these reasons, fluid-saturated porous medium theory can be used widely in geophysics exploration and engineering surveying.

In 1956, a theory was developed for the propagation of stress waves in a porous elastic solid containing compressible viscous fluid by Biot [1, 2]. Biot described the Second-Kind P wave in fluid-saturated porous medium firstly. Since then, many researchers paid their attention to the propagation characters of elastic wave in saturated porous medium and obtained many achievements [3, 4]. Complicated equations given in Biot dynamic theory can be solved by analytical methods with some simple boundary conditions. Most dynamic problems in fluid-saturated porous medium are solved using numerical methods, especially using finite-element method. Ghaboussi and Wilson [5] first proposed a multidimensional finite element numerical scheme to solve the linear coupled governing equations. Prevose [6] proposed an efficient finite element procedure to analyze wave propagation phenomena in fluid saturated porous medium and presented some numerical results which demonstrate the versatility of the proposed procedure. Simon et al. [7, 8] presented an analytical solution for a transient analysis of a one-dimensional column of a fluid saturated porous elastic solid and presented a comparison of this exact closed-form solution with finite-element method for several transient problems in porous media. Yazdchi et al. [9, 10] combined the finite element method with the boundary element method and the infinite element method, constructed the finite-infinite element method and the finite-boundary element method to deal with the two-phase model in lateral extensive field and obtained better result. Zhao et al. [11] proposed an explicit finite element method for Biot dynamic formulation in fluid-saturated porous medium. It does not need to assemble a global stiffness matrix and solve a set of linear equations in each time step by using the decoupling-technique. For the problem of local high gradient, finite element method improves the calculation precision by employing the higher-order polynomial or the denser mesh. However, the increment of polynomial order and mesh knots inevitably needs more computational work. Meanwhile, the condition of numerical dissipation will limit the frequency range that can be obtained. To overcome these disadvantages, wavelet analysis is introduced to the finite-element method in this paper. As a new method, the development of wavelet analysis is recent fairly in many fields. Its desirable advantages are the multiresolution analysis property and various basis functions for structure analysis. According to different requirement, the corresponding scaling functions and wavelet functions can be adopted to improve the numerical calculation precision. Especially, those wavelets with compactly supported property and orthogonality, such as Daubechies wavelets, can play an important role in many problems [12]. Because of the compactly supported property, if the Daubechies wavelets are considered as the interpolation functions of the finite element method, the coefficient matrices obtained are sparse matrices and their condition number can be proved independent of the dimension [13]. Moreover, a new method could be provided because of the existence of various basis functions, which can increase the resolution without changing mesh.

In this paper, the wavelet Galerkin finite element method is applied to the direct simulation of the wave equation in the fluid-saturated porous medium. The scaling functions of Daubechies wavelets are considered as the interpolation basis functions instead of the polynomial functions and the wavelet element is constructed. Because a kind of characteristic function is introduced, the integral difficulty for lacking of the explicit expression for the Daubechies wavelets is solved. Based on the recursive expression of calculating the function values of Daubechies wavelets on the fraction nodes, the rapid wavelet transform between the wavelet coefficient space and the wave field displacement space is constructed and reduces the computational cost. The results of numerical simulation demonstrate the method is effective.

2. Wavelet Galerkin Finite-Element Method

2.1. Wavelet Galerkin Finite-Element Method

For purpose of constructing the wavelet Galerkin finite element method, we consider a typical boundary value problem:

$$L(u,v) = f, \tag{2.1}$$

$$B(u,v)|_{\partial\Omega} = g, \tag{2.2}$$

where $L(\cdot,\cdot)$ is differential operator, $B(\cdot,\cdot)$ is boundary operator, u, v are the unknown functions in the solving domain Ω, and $\partial\Omega$ is the boundary.

Supposing u, v are the exact solutions of (2.1) and (2.2), then one gets

$$L(u,v) - f \equiv 0, \tag{2.3}$$

and if $L(u,v)$ and f are continuous, (2.3) is equal to

$$\int_{\Omega} (L(u,v) - f)\phi_k \mathrm{d}x\,\mathrm{d}y = 0. \tag{2.4}$$

In fact, because of the derivation of one-dimensional wavelet basis element facilitates a straightforward discussion of multidimensional tensor product wavelet basis element and multiresolution analysis property of wavelet function [12], the functions u, v can be assumed to consist of a superposition of scaling functions at j level and wavelet functions at the same and higher levels:

$$u(x,y) = U_1(x)U_2(y), \tag{2.5}$$

$$v(x,y) = V_1(x)V_2(y), \tag{2.6}$$

where

$$\begin{aligned}
U_1(x) &= \sum_k a_{j,k}\phi_{jk}(x) + \sum_{i\geq j,k} a_{i,k}\psi_{ik}(x),\\
U_2(y) &= \sum_k b_{j,k}\phi_{jk}(y) + \sum_{i\geq j,k} b_{i,k}\psi_{ik}(y),\\
V_1(x) &= \sum_k c_{j,k}\phi_{jk}(x) + \sum_{i\geq j,k} c_{i,k}\psi_{ik}(x),\\
V_2(y) &= \sum_k d_{j,k}\phi_{jk}(y) + \sum_{i\geq j,k} d_{i,k}\psi_{ik}(y).
\end{aligned} \tag{2.7}$$

Upon substituting (2.4) and (2.5) into (2.3), we can obtain an equation system of wavelet coefficients, whose coefficient matrix consists of the following integrals:

$$\begin{gathered}\int_{\Omega}\phi_{jq}\psi_{ir}, \quad \int_{\Omega}\phi_{jq}\phi_{jr}, \quad \int_{\Omega}\psi_{iq}\psi_{lr}, \\ \int_{\Omega}\phi_{jq}^{(m)}\psi_{ir}^{(n)}, \quad \int_{\Omega}\phi_{jq}^{(m)}\phi_{jr}^{(n)}, \quad \int_{\Omega}\psi_{iq}^{(m)}\psi_{lr}^{(n)}.\end{gathered} \tag{2.8}$$

In conventional finite element method, these integrals would be calculated by Gauss quadrature formulae. However, it is not feasible for most wavelet functions. In many cases, there is no explicit expression for the function, in this paper, we choose the Daubechies wavelet as the basis function, and they cannot be integrated numerically due to their unusual smoothness characteristics. Moreover, the wavelet function is defined in terms of scaling function, so these integrals can be rewritten in terms of scaling function alone.

Define the connection coefficients [14–16]:

$$\begin{aligned}\Gamma_{p,r}^{0,0} &= \int_{\Omega}\phi(x-p)\phi(x-r)\mathrm{d}x, \\ \Gamma_{p,r}^{m,n} &= \int_{\Omega}\frac{\mathrm{d}^{(m)}\phi}{\mathrm{d}x^{m}}(x-p)\frac{\mathrm{d}^{(n)}\phi}{\mathrm{d}x^{n}}(x-r)\mathrm{d}x, \\ &= \int_{\Omega}\phi^{(m)}(x-p)\phi^{(n)}(x-r)\mathrm{d}x.\end{aligned} \tag{2.9}$$

Once these integrals can be calculated, all the integrals in (2.8) can be obtained and eventually construct the stiffness matrix and load matrix of wavelet Galerkin finite element method.

2.2. The Calculation of Wavelet Connection Coefficients

From what has been discussed earlier, the quality matrix, stiffness matrix, and the load matrix are composed of the integral values of Daubechies wavelets. However, it is well known that Daubechies wavelets have no explicit expression. In order to solve this problem, a kind of characteristic function is introduced:

$$\chi_{[0,1]}(x) = \begin{cases} 1 & 0 \le x \le 1, \\ 0 & \text{otherwise}. \end{cases} \tag{2.10}$$

Set $\xi = 2x$ then

$$\chi_{[0,1]}\left(\frac{\xi}{2}\right) = \begin{cases} 1 & 0 \le \xi \le 2, \\ 0 & \text{otherwise}. \end{cases} \tag{2.11}$$

So the trivial two-scale equation of characteristic function is obtained:

$$\chi_{[0,1]}\left(\frac{\xi}{2}\right) = \chi_{[0,1]}(\xi) + \chi_{[1,2]}(\xi) = \chi_{[0,1]}(\xi) + \chi_{[0,1]}(\xi - 1). \tag{2.12}$$

Set

$$\tau_{k,s}^{0,0} = \int_R \chi_{[0,1]}(x)\phi(x-k)\phi(x-s)\mathrm{d}x. \tag{2.13}$$

Substituting $\phi(x) = \sum_k a_k\phi(2x-k)$ into (2.13), one obtains

$$\begin{aligned}
\tau_{k,s}^{0,0} &= \int_R \chi_{[0,1]}(x)\sum_l a_l\phi(2(x-k)-l)\sum_m a_m\phi(2(x-s)-m)\mathrm{d}x \\
&= \int_R \chi_{[0,1]}(x)\sum_l\sum_m a_l a_m\phi(2x-2k-l)\phi(2x-2s-m)\mathrm{d}x \\
&= \frac{1}{2}\int_R \chi_{[0,1]}\left(\frac{\xi}{2}\right)\sum_l\sum_m a_l a_m\phi(\xi-2k-l)\phi(\xi-2s-m)\mathrm{d}\xi \\
&= \frac{1}{2}\sum_l\sum_m a_l a_m\int_R \left(\chi_{[0,1]}(\xi)+\chi_{[0,1]}(\xi-1)\right)\phi(\xi-2k-l)\phi(\xi-2s-m)\mathrm{d}\xi \\
&= \frac{1}{2}\sum_l\sum_m a_l a_m\int_R \chi_{[0,1]}(\xi)\phi(\xi-2k-l)\phi(\xi-2s-m)\mathrm{d}\xi \\
&\quad + \frac{1}{2}\sum_l\sum_m a_l a_m\int_R \chi_{[0,1]}(\xi-1)\phi(\xi-2k-l)\phi(\xi-2s-m)\mathrm{d}\xi \\
&= \frac{1}{2}\sum_l\sum_m a_l a_m\tau_{2k+l,2s+m}^{0,0} + \frac{1}{2}\sum_l\sum_m a_l a_m\tau_{2k+l-1,2s+m-1}^{0,0} \\
&= \frac{1}{2}\sum_p\sum_r \left(a_{p-2k}a_{r-2s} + a_{p-2k+1}a_{r-2s+1}\right)\tau_{p,r}^{0,0}.
\end{aligned} \tag{2.14}$$

It is not difficult to show that we will require the solution of an eigenvalue problem having the form

$$\tau_{k,s}^{0,0} = \frac{1}{2}A'\tau_{p,r}^{0,0\prime}, \qquad 2N-1 \le k, s \le 0, \qquad 2N-1 \le p, r \le 0, \tag{2.15}$$

where A' is a $(2N-1)\times(2N-1)$ partitioned matrix, each submatrix is also a $(2N-1)\times(2N-1)$ matrix, in which $a'_{sr} = a_{p-2k}a_{r-2s} + a_{p-2k+1}a_{r-2s+1}$.

Considering the requirement of numerical simulation set

$$1 \le i = s+8 \le 8, \qquad 1 \le j = r+8 \le 8, \qquad 1 \le m = k+8 \le 8, \qquad 1 \le n = p+8 \le 8, \tag{2.16}$$

then A'_{kp} is changed to A'_{mn}, in which $\tilde{a}'_{ij} = a_{n-2m+8}a_{j-2i+8} + a_{n-2m+9}a_{j-2i+9}$.

However, the eigenvalue problem does not uniquely define the solution, it is essential to introduce an additional condition to define the solution uniquely.

It is well known that the Daubechies wavelets satisfy

$$1 = \sum_k \phi(x-k). \tag{2.17}$$

By multiplying (2.17) by itself, and subsequently multiplying the product by the characteristic function $\chi_{[0,1]}(x)$, one obtains

$$\begin{aligned} 1 &= \sum_k \sum_l \phi(x-k)\phi(x-l), \\ \chi_{[0,1]}(x) &= \sum_k \sum_l \chi_{[0,1]}(x)\phi(x-k)\phi(x-l). \end{aligned} \tag{2.18}$$

Now, a single integration yields a first normalization condition:

$$1 = \sum_k \sum_l \tau_{k,l}^{0,0}. \tag{2.19}$$

So, the unique solution of the eigenvalue problem is defined.

The same step can be followed to calculate

$$\tau_{k,s}^{m,n} = \int_R \chi_{[0,1]}(x)\phi^{(m)}(x-k)\phi^{(n)}(x-s)\mathrm{d}x. \tag{2.20}$$

Substituting $\phi^{(m)}(x) = 2^m \sum_l a_l \phi^{(m)}(2x-k)$ and $\phi^{(n)}(x) = 2^n \sum_q a_q \phi^{(n)}(2x-k)$ into (2.20), one gets

$$\begin{aligned} \tau_{k,s}^{m,n} &= \int_R \chi_{[0,1]}(x) 2^{m+n} \sum_l a_l \phi^{(m)}(2(x-k)-l) \sum_q a_q \phi^{(n)}\big(2(x-s)-q\big)\mathrm{d}x \\ &= 2^{m+n} \int_R \chi_{[0,1]}(x) \sum_l \sum_q a_l a_q \phi^{(m)}(2x-2k-l)\phi^{(n)}\big(2x-2s-q\big)\mathrm{d}x \\ &= 2^{m+n-1} \int_R \chi_{[0,1]}\left(\frac{\xi}{2}\right) \sum_l \sum_q a_l a_q \phi^{(m)}(\xi-2k-l)\phi^{(n)}\big(\xi-2s-q\big)\mathrm{d}\xi \\ &= 2^{m+n-1} \sum_l \sum_q a_l a_q \int_R \big(\chi_{[0,1]}(\xi) + \chi_{[0,1]}(\xi-1)\big)\phi^{(m)}(\xi-2k-l)\phi^{(n)}\big(\xi-2s-q\big)\mathrm{d}\xi \end{aligned}$$

$$
\begin{aligned}
&= 2^{m+n-1}\sum_{l}\sum_{q} a_l a_q \int_R \chi_{[0,1]}(\xi)\phi^{(m)}(\xi-2k-l)\phi^{(n)}(\xi-2s-q)\mathrm{d}\xi \\
&\quad + 2^{m+n-1}\sum_{l}\sum_{q} a_l a_q \int_R \chi_{[0,1]}(\xi-1)\phi^{(m)}(\xi-2k-l)\phi^{(n)}(\xi-2s-q)\mathrm{d}\xi \\
&= 2^{m+n-1}\sum_{l}\sum_{q} a_l a_q \tau^{m,n}_{2k+l,2s+q} + 2^{m+n-1}\sum_{l}\sum_{q} a_l a_q \tau^{m,n}_{2k+l-1,2s+q-1} \\
&= 2^{m+n-1}\sum_{p}\sum_{r}\left(a_{p-2k}a_{r-2s} + a_{p-2k+1}a_{r-2s+1}\right)\tau^{m,n}_{p,r},
\end{aligned}
\tag{2.21}
$$

namely,

$$\tau^{m,n}_{k,s} = 2^{m+n-1}A'\tau^{m,n}_{p,r}. \tag{2.22}$$

The polynomial reproducing property is employed to construct the additional condition:

$$x^m = \sum_k p_k \phi(x-k), \tag{2.23}$$

$$x^n = \sum_l p_l \phi(x-l), \tag{2.24}$$

Explicit form for calculating the coefficients $p_k p_l$ can be found in [17].

By differentiating (2.23) m times, one obtains

$$m! = \sum_k p_k \phi^{(m)}(x-k). \tag{2.25}$$

By differentiating (2.24) n times, one gets

$$n! = \sum_l p_l \phi^{(n)}(x-l). \tag{2.26}$$

However (2.25) can be multiplied by (2.26), and subsequently multiplying the product by the characteristic function $\chi_{[0,1]}(x)$,

$$m!n!\chi_{[0,1]}(x) = \sum_k\sum_l \chi_{[0,1]}(x)p_k p_l \phi^{(m)}(x-k)\phi^{(n)}(x-l). \tag{2.27}$$

By integrating (2.27), one obtains the additional condition.

$$m!n! = \sum_k\sum_l p_k p_l \tau^{m,n}_{k,l}. \tag{2.28}$$

Then, the unique solution of the eigenvalue problem is defined.

3. Wavelet Galerkin Finite-Element Solution of 1D Elastic Wave Equation in Fluid-Saturated Porous Medium

From the Biot theory, the 1D differential equation governing wave propagation in the fluid-saturated porous medium, without fluid viscosity, can be expressed as

$$\begin{gathered}\frac{\partial}{\partial x}\left((\lambda+2\mu+\alpha M)\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial x}\left(\alpha M\frac{\partial \omega}{\partial x}\right)=\rho\ddot{u}+\rho_f\ddot{\omega}-f_1,\\ \frac{\partial}{\partial x}\left(\alpha M\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial x}\left(M\frac{\partial \omega}{\partial x}\right)=\rho_f\ddot{u}+m\ddot{\omega}-f_2,\end{gathered}\tag{3.1}$$

where u is the solid displacement and ω is the relative fluid to solid displacement. β is the porosity, $\rho=(1-\beta)\rho_s+\beta\rho_f$ is the bulk density of solid-fluid mixture, and ρ_s and ρ_f are the densities of solid and fluid, respectively. Also t is time and λ_b,μ are the Lame coefficients, $\lambda=\lambda_b+\alpha^2M$, where α is the effective stress parameter and M is the compressibility of pore fluid. $\alpha=1-K_b/K_s$, $M=K_s/[\alpha+\beta(K_s/K_f-1)]$ where K_s,K_f,K_b are the bulk change modulus of the solid, fluid, and skeleton, respectively. Moreover $K_b=\lambda_b+2\mu/3$, $m=\rho_f/\beta$, Finally f is seismic focus, and $f_1=(1-\beta)f$, $f_2=\beta(2\beta-1)f$.

Multiplying both sides of the fluid-saturated porous medium wave equation by the Daubechies wavelets basis function $\phi_{jk}(x)=2^{j/2}\phi(2^jx-k)$, and integrating them at $[0,L]$, we can get

$$\begin{gathered}\int_0^L\left(\frac{\partial}{\partial x}\left((\lambda+2\mu+\alpha M)\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial x}\left(\alpha M\frac{\partial \omega}{\partial x}\right)\right)\phi_{jk}(x)\mathrm{d}x=\int_0^L(\rho\ddot{u}+\rho_f\ddot{\omega}-f_1)\phi_{jk}(x)\mathrm{d}x\\ \int_0^L\left(\frac{\partial}{\partial x}\left(\alpha M\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial x}\left(M\frac{\partial \omega}{\partial x}\right)\right)\phi_{jk}(x)\mathrm{d}x=\int_0^L(\rho_f\ddot{u}+m\ddot{\omega}-f_2)\phi_{jk}(x)\mathrm{d}x.\end{gathered}\tag{3.2}$$

By using integration by part

$$\begin{gathered}(\lambda+2\mu+\alpha M)\frac{\partial u}{\partial x}\phi_{jk}(x)\Big|_0^L-\int_0^L(\lambda+2\mu+\alpha M)\frac{\partial u}{\partial x}\frac{\partial\phi_{jk}(x)}{\partial x}\mathrm{d}x+\alpha M\frac{\partial\omega}{\partial x}\phi_{jk}(x)\Big|_0^L\\ -\int_0^L\alpha M\frac{\partial\omega}{\partial x}\frac{\partial\phi_{jk}(x)}{\partial x}\mathrm{d}x=\int_0^L(\rho\ddot{u}+\rho_f\ddot{\omega}-f_1)\phi_{jk}(x)\mathrm{d}x,\end{gathered}\tag{3.3}$$

$$\begin{gathered}\alpha M\frac{\partial u}{\partial x}\phi_{jk}(x)\Big|_0^L-\int_0^L\alpha M\frac{\partial u}{\partial x}\frac{\partial\phi_{jk}(x)}{\partial x}\mathrm{d}x+M\frac{\partial\omega}{\partial x}\phi_{jk}(x)\Big|_0^L-\int_0^L M\frac{\partial\omega}{\partial x}\frac{\partial\phi_{jk}(x)}{\partial x}\mathrm{d}x\\ =\int_0^L(\rho_f\ddot{u}+m\ddot{\omega}-f_2)\phi_{jk}(x)\mathrm{d}x.\end{gathered}\tag{3.4}$$

Set

$$u(x,t) = \sum_{l=2-2N-2^jL}^{0} a_l(t)\phi_{jl}(x),$$
$$\omega(x,t) = \sum_{l=2-2N-2^jL}^{0} b_l(t)\phi_{jl}(x). \tag{3.5}$$

Upon substituting (3.5) into (3.3) and (3.4), one gets

$$\begin{aligned}
&(\lambda+2\mu+\alpha M)\frac{\partial u}{\partial x}\phi_{jk}(x)\Big|_0^L + \alpha M\frac{\partial \omega}{\partial x}\phi_{jk}(x)\Big|_0^L \\
&\quad - \int_0^L \left((\lambda+2\mu+\alpha M)\sum_{l=2-2N-2^jL}^{0} a_l(t)\frac{\partial\phi_{jl}(x)}{\partial x}\frac{\partial\phi_{jk}(x)}{\partial x} + \alpha M\sum_{l=2-2N-2^jL}^{0} b_l(t)\frac{\partial\phi_{jl}(x)}{\partial x}\frac{\partial\phi_{jk}(x)}{\partial x}\right)\mathrm{d}x \\
&\quad = \int_0^L \left(\rho\sum_{l=2-2N-2^jL}^{0} a_l''(t)\phi_{jl}(x) + \rho_f\sum_{l=2-2N-2^jL}^{0} b_l''(t)\phi_{jl}(x) - f_1\right)\phi_{jk}(x)\mathrm{d}x, \\
&\alpha M\frac{\partial u}{\partial x}\phi_{jk}(x)\Big|_0^L + M\frac{\partial \omega}{\partial x}\phi_{jk}(x)\Big|_0^L \\
&\quad - \int_0^L \left(\alpha M\sum_{l=2-2N-2^jL}^{0} a_l(t)\frac{\partial\phi_{jl}(x)}{\partial x}\frac{\partial\phi_{jk}(x)}{\partial x} + M\sum_{l=2-2N-2^jL}^{0} b_l(t)\frac{\partial\phi_{jl}(x)}{\partial x}\frac{\partial\phi_{jk}(x)}{\partial x}\right)\mathrm{d}x \\
&\quad = \int_0^L \left(\rho_f\sum_{l=2-2N-2^jL}^{0} a_l''(t)\phi_{jl}(x) + m\sum_{l=2-2N-2^jL}^{0} b_l''(t)\phi_{jl}(x) - f_2\right)\phi_{jk}(x)\mathrm{d}x.
\end{aligned} \tag{3.6}$$

By rearranging, (3.6) and become

$$\begin{aligned}
&(\lambda+2\mu+\alpha M)\frac{\partial u}{\partial x}\phi_{jk}(x)\Big|_0^L + \alpha M\frac{\partial \omega}{\partial x}\phi_{jk}(x)\Big|_0^L \\
&\quad - \left((\lambda+2\mu+\alpha M)\sum_{l=2-2N-2^jL}^{0} a_l(t) + \alpha M\sum_{l=2-2N-2^jL}^{0} b_l(t)\right)\int_0^L \frac{\partial\phi_{jl}(x)}{\partial x}\frac{\partial\phi_{jk}(x)}{\partial x}\mathrm{d}x \\
&\quad = \left(\rho\sum_{l=2-2N-2^jL}^{0} a_l''(t) + \rho_f\sum_{l=2-2N-2^jL}^{0} b_l''(t)\right)\int_0^L \phi_{jl}(x)\phi_{jk}(x)\mathrm{d}x - f_1\int_0^L \phi_{jk}(x)\mathrm{d}x, \\
&\alpha M\frac{\partial u}{\partial x}\phi_{jk}(x)\Big|_0^L + M\frac{\partial \omega}{\partial x}\phi_{jk}(x)\Big|_0^L \\
&\quad - \left(\alpha M\sum_{l=2-2N-2^jL}^{0} a_l(t) + M\sum_{l=2-2N-2^jL}^{0} b_l(t)\right)\int_0^L \frac{\partial\phi_{jl}(x)}{\partial x}\frac{\partial\phi_{jk}(x)}{\partial x}\mathrm{d}x \\
&\quad = \left(\rho_f\sum_{l=2-2N-2^jL}^{0} a_l''(t) + m\sum_{l=2-2N-2^jL}^{0} b_l''(t)\right)\int_0^L \phi_{jl}(x)\phi_{jk}(x)\mathrm{d}x - f_2\int_0^L \phi_{jk}(x)\mathrm{d}x.
\end{aligned} \tag{3.7}$$

If select $L = 1$, $j = 0$, (3.5) become

$$u(x,t) = \sum_{l=1-2N}^{0} a_l(t)\phi(x-l)$$
$$\omega(x,t) = \sum_{l=1-2N}^{0} b_l(t)\phi(x-l) \tag{3.8}$$

Set

$$\mathbf{A} = (a_{1-2N}, a_{2-2N} \cdots a_0) \qquad \mathbf{B} = (b_{1-2N}, b_{2-2N} \cdots b_0),$$
$$\mathbf{R} = (\mathbf{A}, \mathbf{B})^T = (a_{1-2N}, a_{2-2N} \cdots a_0, b_{1-2N}, b_{2-2N} \cdots b_0)^T. \tag{3.9}$$

Then, (3.7) can be changed into an equation system of coefficient **R:**

$$\overline{\mathbf{M}}\ddot{\mathbf{R}} + \mathbf{P}\mathbf{R} = \mathbf{F} + \mathbf{Q}, \tag{3.10}$$

where

$$\overline{\mathbf{M}} = \begin{pmatrix} \rho\mathbf{E} & \rho_f\mathbf{E} \\ \rho_f\mathbf{E} & m\mathbf{E} \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} -(\lambda + 2\mu + \alpha M)\mathbf{G} & -\alpha M\mathbf{G} \\ -\alpha M\mathbf{G} & -M\mathbf{G} \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} \mathbf{F_1} \\ \mathbf{F_2} \end{pmatrix},$$

$$\mathbf{E} = \begin{pmatrix} \int_0^1 \mathfrak{A}\mathfrak{A}\mathrm{d}x & \int_0^1 \mathfrak{B}\mathfrak{A}\mathrm{d}x & \cdots & \int_0^1 \phi(x)\mathfrak{A}\mathrm{d}x \\ \int_0^1 \mathfrak{A}\mathfrak{B}\mathrm{d}x & \int_0^1 \mathfrak{B}\mathfrak{B}\mathrm{d}x & \cdots & \int_0^1 \phi(x)\mathfrak{B}\mathrm{d}x \\ \vdots & \vdots & \vdots & \vdots \\ \int_0^1 \mathfrak{A}\phi(x)dx & \int_0^1 \mathfrak{B}\phi(x)dx & \cdots & \int_0^1 \phi(x)\phi(x)\mathrm{d}x \end{pmatrix},$$

$$\mathbf{G} = \begin{pmatrix} \int_0^1 \mathfrak{C}\mathfrak{C}\mathrm{d}x & \int_0^1 \mathfrak{D}\mathfrak{C}\mathrm{d}x & \cdots & \int_0^1 \phi'(x)\mathfrak{C}\mathrm{d}x \\ \int_0^1 \mathfrak{C}\mathfrak{D}\mathrm{d}x & \int_0^1 \mathfrak{D}\mathfrak{D}\mathrm{d}x & \cdots & \int_0^1 \phi'(x)\mathfrak{D}\mathrm{d}x \\ \vdots & \vdots & \vdots & \vdots \\ \int_0^1 \mathfrak{C}\phi'(x)\mathrm{d}x & \int_0^1 \mathfrak{D}\phi'(x)\mathrm{d}x & \cdots & \int_0^1 \phi'(x)\phi'(x)\mathrm{d}x \end{pmatrix},$$

$$\mathbf{F_1} = f_1\left(\int_0^1 \mathfrak{A}\mathrm{d}x, \int_0^1 \mathfrak{B}\mathrm{d}x, \ldots, \int_0^1 \phi(x)\mathrm{d}x\right)^T,$$

$$\mathbf{F_2} = f_2\left(\int_0^1 \mathfrak{A}\mathrm{d}x, \int_0^1 \mathfrak{B}\mathrm{d}x, \ldots, \int_0^1 \phi(x)\mathrm{d}x\right)^T,$$

$$\mathbf{Q} = (\mathbf{Q_1}, \mathbf{Q_2})^T,$$

$$\mathbf{Q}_1 = \left((\lambda + 2\mu + \alpha M)\frac{\partial u}{\partial x} + \alpha M \frac{\partial \omega}{\partial x}\right)(\mathfrak{A}, \mathfrak{B} \cdots \phi(x))^T\Big|_0^1,$$

$$\mathbf{Q}_2 = \left(\alpha M \frac{\partial u}{\partial x} + M \frac{\partial \omega}{\partial x}\right)(\mathfrak{A}, \mathfrak{B} \cdots \phi(x))^T\Big|_0^1, \tag{3.11}$$

where $\mathfrak{A}$ denote $\phi(x-1+2N)$, $\mathfrak{B}$ denote $\phi(x-2+2N)$, $\mathfrak{C}$ denote $\phi'(x-1+2N)$ and $\mathfrak{D}$ denote $\phi'(x-2+2N)$.

Using the second-order center difference to approximate the two derivatives in (3.10), we can obtain

$$\overline{\mathbf{M}}\frac{\mathbf{R}^{n+1} - 2\mathbf{R}^n + \mathbf{R}^{n-1}}{(\Delta t)^2} + \mathbf{P}\mathbf{R}^n = \mathbf{F} + \mathbf{Q}. \tag{3.12}$$

Arranging (3.12), we have

$$\overline{\mathbf{M}}\mathbf{R}^{n+1} = \left(2\overline{\mathbf{M}} - (\Delta t)^2\mathbf{P}\right)\mathbf{R}^n - \overline{\mathbf{M}}\mathbf{R}^{n-1} + (\Delta t)^2\mathbf{F} + (\Delta t)^2\mathbf{Q}, \tag{3.13}$$

given the initial conditions:

$$a_k(0) = b_k(0) = 0, \qquad a_k(1) = b_k(1) = 0. \tag{3.14}$$

So, we can obtain the wavelet coefficients at each time level by solving (3.13) and (3.14) with some boundary conditions, and then substitute the wavelet coefficients into (3.8), the wave field displacements can be obtained.

4. Rapid Wavelet Transform

In order to obtain the wave field displacements conveniently and quickly, the fast wavelet transform between the wavelet coefficients space and the wave field displacements space is constructed as follows:

$$U = \Phi P, \tag{4.1}$$

U is the wave field displacement vector, P is the wavelet coefficient vector, Φ is the wavelet transform matrix.

For the sake of simplicity, take the DB2 wavelet as the example. There are 7 nodes in solution field:

$$U = \left(u\left(\frac{1}{8}\right), u\left(\frac{1}{4}\right), u\left(\frac{3}{8}\right), u\left(\frac{1}{2}\right), u\left(\frac{5}{8}\right), u\left(\frac{3}{4}\right), u\left(\frac{7}{8}\right)\right)^T, \quad P = \left(p_{-2}, p_{-1}, p_0\right)^T,$$

$$\Phi = \begin{pmatrix} \phi\left(\frac{1}{8}+2\right) & \phi\left(\frac{1}{8}+1\right) & \phi\left(\frac{1}{8}\right) \\ \phi\left(\frac{1}{4}+2\right) & \phi\left(\frac{1}{4}+1\right) & \phi\left(\frac{1}{4}\right) \\ \phi\left(\frac{3}{8}+2\right) & \phi\left(\frac{3}{8}+1\right) & \phi\left(\frac{3}{8}\right) \\ \phi\left(\frac{1}{2}+2\right) & \phi\left(\frac{1}{2}+1\right) & \phi\left(\frac{1}{2}\right) \\ \phi\left(\frac{5}{8}+2\right) & \phi\left(\frac{5}{8}+1\right) & \phi\left(\frac{5}{8}\right) \\ \phi\left(\frac{3}{4}+2\right) & \phi\left(\frac{3}{4}+1\right) & \phi\left(\frac{3}{4}\right) \\ \phi\left(\frac{7}{8}+2\right) & \phi\left(\frac{7}{8}+1\right) & \phi\left(\frac{7}{8}\right) \end{pmatrix}. \tag{4.2}$$

It is important for constructing the fast wavelet transform to solve the function values of the Daubechies wavelets on the fraction nodes. So, the recursive expression of calculating the function values of Daubechies wavelets on the fraction nodes is deduced to save the computational cost.

$$\Phi\left(\frac{2^n i + p}{2^n}\right) = \begin{cases} A\Phi\left(\dfrac{2^{n-1} i + q}{2^{n-1}}\right) & \text{if } 2 \cdot \dfrac{p}{2^n} \le 1 \\ B\Phi\left(\dfrac{2^{n-1} i + q}{2^{n-1}}\right) & \text{if } 2 \cdot \dfrac{p}{2^n} > 1 \end{cases} \tag{4.3}$$

in which $i = 0, 1, \ldots 2N - 2$, $p = 1 : 2 : 2^n - 1$, $q = p \bmod 2^{n-1}$, n controls the mesh partition.

5. Numerical Simulation

To verify the correctness and accuracy of the wavelet Galerkin finite element method, two examples are given to compare the results obtained by this method with an analytical solution. An one-dimensional column of length l as sketched in Figure 1 is considered. It is assumed that the side walls and the bottom are rigid, frictionless, and impermeable. At top,

the stress σ_y and the pressure p are prescribed. The boundary conditions are

$$\begin{aligned} u|_{y=0} &= \omega|_{y=0} = 0, \\ \sigma|_{y=l} &= -P_0 f(t), \\ p|_{y=l} &= 0. \end{aligned} \tag{5.1}$$

For this model, if the permeability tends to infinity, that is, $\kappa \to \infty$, the analytical solutions in time domain are [18]

$$\begin{aligned} u_y = \frac{P_0}{E(d_1\lambda_2 - d_2\lambda_1)} \sum_{n=0}^{\infty} (-1)^{-n} \{ & d_2 [(t - \lambda_1(l(2n+1) - y)) H(t - \lambda_1(l(2n+1) - y)) \\ & - (t - \lambda_1(l(2n+1) + y)) H(t - \lambda_1(l(2n+1) + y))] \\ & - d_1 [(t - \lambda_2(l(2n+1) - y)) H(t - \lambda_2(l(2n+1) - y)) \\ & - (t - \lambda_2(l(2n+1) + y)) H(t - \lambda_2(l(2n+1) + y))] \}, \end{aligned} \tag{5.2}$$

$$\begin{aligned} p = \frac{P_0 d_1 d_2}{E(d_1\lambda_2 - d_2\lambda_1)} \sum_{n=0}^{\infty} (-1)^{-n} [& H(t - \lambda_1(l(2n+1) - y)) + H(t - \lambda_1(l(2n+1) + y)) \\ & - H(t - \lambda_2(l(2n+1) - y)) + H(t - \lambda_2(l(2n+1) + y))], \end{aligned} \tag{5.3}$$

where E is Young modulus, assuming a Heaviside step function as temporal behavior, that is, $f(t) = H(t)$, and together with vanishing initial conditions:

$$\begin{aligned} d_i &= \frac{E\lambda_i^2 - (\rho - \rho_f)}{(\alpha - Q)\lambda_i} \quad (i = 1, 2), \\ Q &= \frac{\beta^2 \rho_f}{\rho_\alpha + \beta\rho_f} \quad (\kappa \to \infty), \\ \rho_\alpha &= 0.66\beta\rho_f. \end{aligned} \tag{5.4}$$

However λ_i are the characteristic roots of following characteristic equation

$$E\frac{Q}{\rho_f}\lambda^4 - \left(E - \frac{\beta^2}{M} + (\rho - Q\rho_f)\frac{Q}{\rho_f} + (\alpha - Q)^2\right)\lambda^2 + \frac{\beta^2(\rho - Q\rho_f)}{M} = 0. \tag{5.5}$$

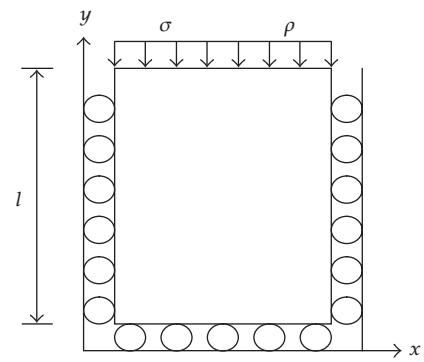

Figure 1: Model of fluid saturated porous medium.

Table 1: The parameters of fluid saturated porous medium.

	$K_b(\mathrm{P}a)$	$G(\mathrm{P}a)$	$\rho(\mathrm{kg/m^3})$	β	$K_s(\mathrm{P}a)$	$\rho_f(\mathrm{kg/m^3})$	$K_f(\mathrm{P}a)$
rock	8.0×10^9	6.0×10^9	2548	0.19	3.6×10^{10}	1000	3.3×10^9
soil	2.1×10^8	9.8×10^7	1884	0.48	1.1×10^{10}	1000	3.3×10^9
sediment	3.7×10^7	2.2×10^7	1396	0.76	3.6×10^{10}	1000	2.3×10^9

Supposing

$$A = E\frac{Q}{\rho_f}, \qquad B = E - \frac{\beta^2}{M} + (\rho - Q\rho_f)\frac{Q}{\rho_f} + (\alpha - Q)^2,$$
$$C = \frac{\beta^2(\rho - Q\rho_f)}{M}, \tag{5.6}$$

one gets

$$\lambda_1 = -\lambda_3 = \sqrt{\frac{B + \sqrt{B^2 - 4AC}}{2A}},$$
$$\lambda_2 = -\lambda_4 = \sqrt{\frac{B - \sqrt{B^2 - 4AC}}{2A}}. \tag{5.7}$$

In the first example, the length of column is chosen as $l = 1000\,\mathrm{m}$, and three very different materials, a rock (Berea sandstone), a soil (coarse sand), and a sediment (mud) are chosen. The material data are given in Table 1. In Figures 2, 3, 4, we record the pressure $p(t, y = 995\,\mathrm{m})$, five meters behind the excitation ($y = l = 1000\,\mathrm{m}$). The numerical results (plotted with dot) are compared with the analytical solution (5.3), shown as solid lines in Figures 2, 3, 4. In the second example, the length of column is chosen as $l = 10\,\mathrm{m}$. We choose a material-soil, Figures 5, 6 demonstrate the numerical results—the displacements $u_y(t, y = 5\,\mathrm{m})$ and the pressure $p(t, y = 5\,\mathrm{m})$. All the figures show that the numerical solutions are perfectly close to the analytical solutions, so the method developed in this paper has a very high degree of calculating accuracy.

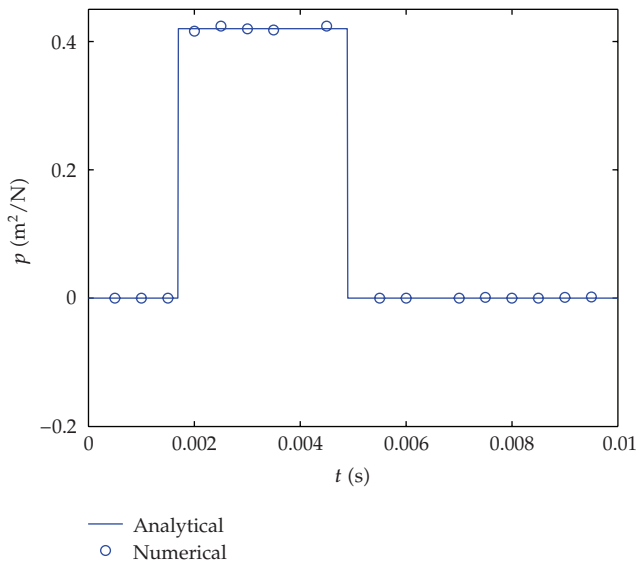

Figure 2: The pressure of rock ($l = 1000\,\mathrm{m}$, $y = 995\,\mathrm{m}$).

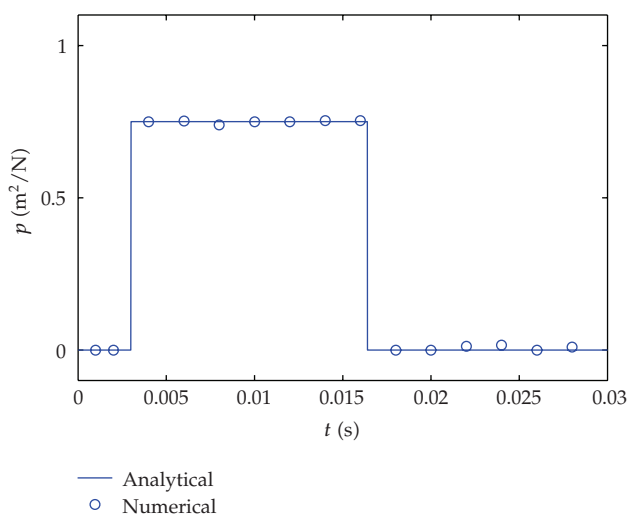

Figure 3: The pressure of soil ($l = 1000\,\mathrm{m}$, $y = 995\,\mathrm{m}$).

6. Conclusion

In this article, the wavelet Galerkin finite element method is constructed by combining the finite element method with wavelet analysis, and is applied to the numerical simulation of the fluid-saturated porous medium elastic wave equation. For the beautiful and deep mathematic properties of Daubechies wavelets, such as the compactly supported property and vanishing moment property, the wavelet Galerkin finite element method has the

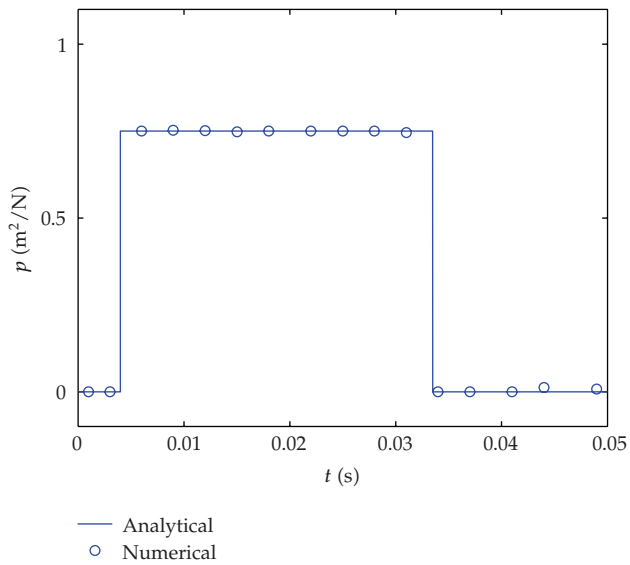

Figure 4: The pressure of sediment ($l = 1000\,\mathrm{m}$, $y = 995\,\mathrm{m}$).

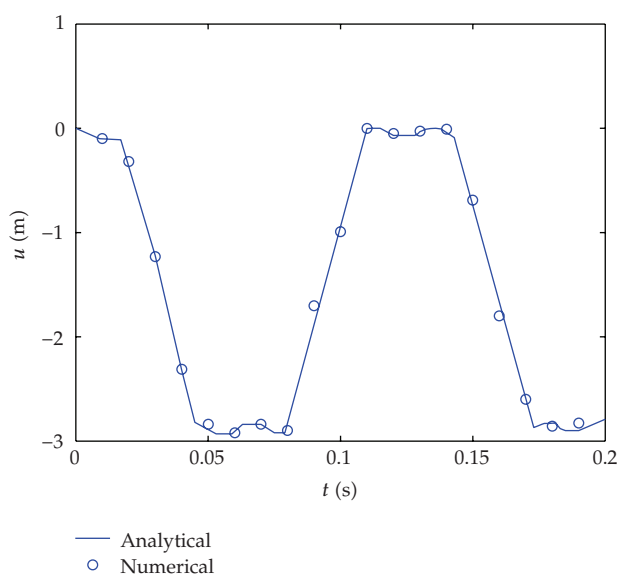

Figure 5: The displacement of soil ($l = 1000\,\mathrm{m}$, $y = 5\,\mathrm{m}$).

feature of quick iterative rate and high numerical precision. Moreover, contrasts to h- or p-based FEM, a new refine algorithm can be presented because of the multi-resolution property of the wavelet analysis. The algorithm can increase the numerical precision by adopting various wavelet basis functions or various wavelet spaces, without refining the mesh.

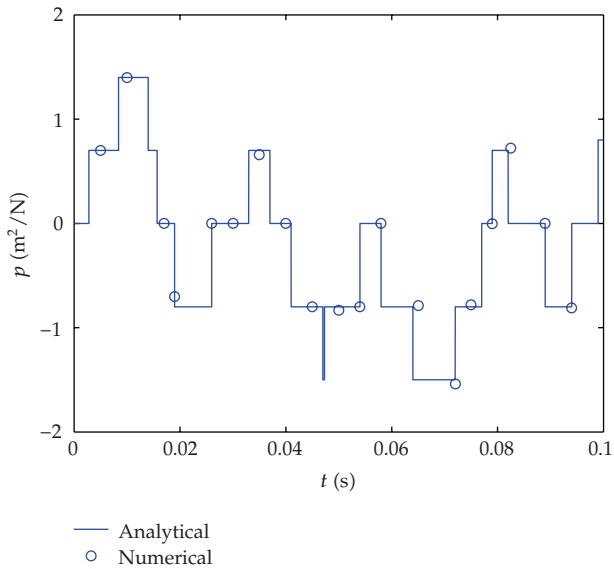

Figure 6: The pressure of soil ($l = 1000\,\mathrm{m}$, $y = 5\,\mathrm{m}$).

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