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Research Article

New Poisson's Type Integral Formula for Thermoelastic Half-Space

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A new Green's function and a new Poisson's type integral formula for a boundary value problem (BVP) in thermoelasticity for a half-space with mixed boundary conditions are derived. The thermoelastic displacements are generated by a heat source, applied in the inner points of the half-space and by temperature, and prescribed on its boundary. All results are obtained in closed forms that are formulated in a special theorem. A closed form solution for a particular BVP of thermoelasticity for a half-space also is included. The main difficulties to obtain these results are in deriving of functions of influence of a unit concentrated force onto elastic volume dilatation $\Theta^{(k)}$ and, also, in calculating of a volume integral of the product of function $\Theta^{(k)}$ and Green's function in heat conduction. Using the proposed approach, it is possible to extend the obtained results not only for any canonical Cartesian domain, but also for any orthogonal one.

1. Introduction

The main objective of this paper is to prove a theorem (Section 2.2) about deriving a Poisson's type integral formula for a thermoelastic half-space with the homogeneous locally mixed mechanical boundary conditions and with the nonhomogeneous Dirichlet's boundary condition for temperature. To prove this theorem we need some equations for Green's integral formula in stationary thermoelasticity (Section 1.1) and thermoelastic influence functions (Sections 1.2 and 1.3) for a solid body, suggested and published by the first-named author earlier. Needed special cases of these equations for a half-space are given in Section 2.1. An example of application of the obtained new Poisson's type integral formula is given in Section 3.

1.1. General Green's Integral Formula in Stationary Thermoelasticity

The Green's function plays the leading role in finding the solutions in integrals for boundary value problems (BVPs) in different fields of mathematical physics. The theory of thermoelasticity, which is a synthesis of the theory of heat conduction and elasticity theory, is one of such fields. By now, a number of theories of thermoelasticity have been developed and described in classical scientific literature [1–6]. The most developed theory, which is widely used in practical calculations, is the theory of thermal stresses, that is, the theory of uncoupled thermoelasticity, where the temperature field does not depend on the field of elastic displacements. According to this theory the formulation of a BVP consists in nonhomogeneous Lame's equations

$$\mu\nabla^2 u_k(\xi) + (\lambda+\mu)\theta_{,k}(\xi) - \gamma T_{,k}(\xi) = 0 \tag{1.1}$$

with respective mechanical boundary conditions, where λ, μ are Lame's constants of elasticity; $\gamma = \alpha_t(2\mu + 3\lambda)$ is the thermoelastic constant; α_t is the coefficient of the linear thermal expansion. The temperature field T in (1.1) has to be determined from the BVP in heat conduction that consists in the Poisson's equation

$$\nabla^2 T(\xi) = -a^{-1}F(\xi) \quad \xi \in V, \tag{1.2}$$

with the respective boundary conditions for heat actions: temperature $T(y)$, heat flux $a(\partial T(y)/\partial n_y)$, or certain law of heat exchange between exterior medium and surface of the body $\alpha T(y) + a[\partial T(y)/\partial n_y]$, where F is the heat source; a is thermal conductivity; α is the coefficient of convective heat conductivity. To solve the BVP of stationary thermoelasticity in (1.1) and (1.2) using traditional methods, at the first stage we need to solve the BVP of heat conduction in (1.2) with the given boundary conditions and to find the temperature field. At the second stage we need to solve BVP of thermoelasticity in (1.1) with the already known temperature field and with the given mechanical boundary conditions. On the base of the influence functions introduced in the works [7–15] the following new general Green's type integral formula for determining the fields of displacements, described by BVP in (1.1)-(1.2), was suggested:

$$\begin{aligned} u_k(\xi) = {} & a^{-1}\int_V F(x)U_k(x,\xi)dV(x) - \int_{\Gamma_D} T(y)\left[\frac{\partial U_k(y,\xi)}{\partial n_y}\right]d\Gamma_D(y) \\ & + \int_{\Gamma_N}\left[\frac{\partial T(y)}{\partial n_y}\right]U_k(y,\xi)d\Gamma_N(y) + \int_{\Gamma_M}\left[T(y) + \frac{(a\alpha^{-1})\partial T(y)}{\partial n_y}\right]U_k(y,\xi)d\Gamma_M(y), \end{aligned} \tag{1.3}$$

where Γ_D, Γ_N, and Γ_M denote the surfaces on which the boundary conditions of Dirichlet's, Neumann's, or mixed type are prescribed. One of the advantages of this formula is that the desired thermoelastic displacements u_k are determined in the integral form directly via the prescribed inner heat source and other thermal data, given on the boundary.

1.2. Main Thermoelastic Influence Functions

Thus, the introduced functions of influence of a unit heat source on thermoelastic displacements $U_k(x,\xi)$ in (1.3) are determined by the following general integral formula [11]:

$$U_k(x,\xi) = \gamma \int_V G(x,z)\Theta^{(k)}(z,\xi)dV(z), \quad x,z,\xi \in V. \tag{1.4}$$

This formula represents the Mayzel's formula, when the temperature field (the Green's function G in heat conduction problem) is generated by an inner unit point heat source. The volume dilatation $\Theta^{(k)}$ in (1.4) has to be determined from the following Lame set of equations:

$$\mu\nabla^2 U_i^{(k)}(x,\xi) + (\lambda+\mu)\Theta_{,i}^{(k)}(x,\xi) = -\delta_{ik}\delta(x-\xi), \quad i,k = 1,2,3, \tag{1.5}$$

with the respective homogeneous mechanical boundary conditions. In (1.5) $\delta(x-\xi)$ is the Dirac's function and δ_{ik} is the Kronecker's symbol. The Green's function G in (1.4) has to be determined from the following equation in heat conduction:

$$\nabla_x^2 G(x,\xi) = -\delta(x-\xi), \quad x,\xi \in V, \tag{1.6}$$

with the respective homogeneous boundary conditions for heat actions. A generalization of influence function and Green's integral formula on the BVP in the classical theory of stationary thermoelasticity [1–4] was given for the first time in the papers [7–10]. The considered influence functions $U_k(x,\xi)$ have the physical sense as displacements at an inner point of observation $x \equiv (x_1,x_2,x_3)$, generated by a unit heat source, applied at an inner point $\xi \equiv (\xi_1,\xi_2,\xi_3)$, and described by the Dirac's function. According to (1.4) they are determined by a convolution over the body V of two influence functions. The first influence function is the Green's function G for the BVP in heat conduction. The other functions $\Theta^{(k)}$ are functions of influence of inner concentrated body forces on elastic volume dilatation.

Finally, the influence functions $U_k(x,\xi)$ are functions of double influence [7–15], which take in consideration both physical phenomena (heat conduction and elasticity) in a solid body.

(1) *Over the coordinates of the point of observation* $x \equiv (x_1,x_2,x_3)$ *for thermoelastic displacements, they satisfy the equations of the BVP for determining Green's functions in the theory of heat conduction* (1.6). *The only difference is that the unit heat source is replaced by the function of influence of the unit concentrated forces on the volume dilatation*

$$\nabla_x^2 U_k(x,\xi) = -\gamma\ \Theta^{(k)}(x,\xi) \tag{1.7}$$

with the respective homogeneous boundary conditions for heat actions.

(2) *Over the coordinates of the point of application* $\xi \equiv (\xi_1, \xi_2, \xi_3)$ *of the unit point heat source, they satisfy the BVP for determining components of the Green's matrix* (1.5). *The only difference is that the unit concentrated body forces are replaced with the derivatives of Green's functions of the heat conduction problem*

$$\mu \nabla_{\xi}^{2} U_k(x,\xi) + (\lambda + \mu)\Theta_{,\xi_k}(x,\xi) - \gamma G_{,\xi_k}(x,\xi) = 0 \tag{1.8}$$

with the respective homogeneous mechanical boundary conditions.

1.3. Other Thermoelastic Influence Functions

Take note that all influence functions in (1.3) are determined on the boundary independently or using the respective limits from the main influence functions $U_k(x,\xi)$:

(a) the formula for influence functions of a unit point heat flux $a(\partial T(y)/\partial n_y) = \delta(x-y)$ on the surface Γ_N on thermoelastic displacements is

$$U_k(y,\xi) = \lim_{x\to y} \gamma \int_V G(x,z)\Theta^{(q)}(z,\xi)dV(z) = \lim_{x\to y} U_k(x,\xi), \quad x,\xi,z \in V,\ y \in \Gamma_N, \tag{1.9}$$

(b) the formula for influence functions of a unit point temperature $T(y) = -\delta(x-y)$ on the surface Γ_D on thermoelastic displacements is,

$$\frac{\partial U_k(y,\xi)}{\partial n_y} = \lim_{x\to y} \gamma \int_V \frac{\partial G(x,z)}{\partial n_x}\Theta^{(q)}(z,\xi)dV(z), \quad x,\xi,z \in V = \lim_{x\to y} \frac{\partial U_k(x,\xi)}{\partial n_x},\ y \in \Gamma_D, \tag{1.10}$$

(c) the formula for influence functions of unit point heat exchange of the body with exterior medium $\alpha T(y) + a\lfloor \partial T(y)/\partial n_y \rfloor = \delta(x-y)$ through the surface Γ_M on thermoelastic displacements is,

$$U_k(y,\xi) = \lim_{x\to y} \gamma \int_V G(x,z)\Theta^{(k)}(z,\xi)dV(z), \quad x,\xi,z \in V = \lim_{x\to y} U_k(x,\xi),\ y \in \Gamma_M. \tag{1.11}$$

The formula in (1.3) can be treated also as a generalization of Mayzel's formula [4–6] for those cases when the temperature field satisfies the BVP of heat conduction. Temperature field in this case is caused by the inner heat source and by the prescribed on the boundary temperature, heat flux, or certain law of heat exchange between exterior medium and surface of the body.

The advantage of the proposed integral formula in (1.3) is that it allows us to unite the two-staged process of solving the BVP in the theory of thermoelasticity (the first stage comprises finding temperature fields, and the second stage comprises finding thermoelastic displacements) in one stage. Also, the advantage of the integral formula in (1.3) in comparison with the well-known Mayzel's integral formula is that the thermoelastic displacements are determined directly via given heat actions. Besides, for any concrete

type of BVP we can obtain all possible solutions for different laws describing the above-mentioned heat actions. The main difficulties for practical realization of the integral formula in (1.3) and (1.4) are to derive the functions of influence of a unit concentrated force on elastic volume dilatation $\Theta^{(k)}$ and of the Green's functions in heat conduction G. In addition we need to compute some volume integrals of the product of the above-mentioned functions. These difficulties, especially deriving functions $\Theta^{(k)}$, were overcome successfully for Cartesian canonical domains [11]. For cylindrical and spherical domains only general integral representations for $\Theta^{(k)}$ and Green's matrices were proposed [16, 17].

2. Elastic Response of a Half-Space to a Unit Point Heat Source

In this section we give a theorem for determining the thermoelastic displacements for a half-space in the form of volume and surface integrals, which is a particular case of the general integral formula in (1.3). To do this, first, on the basis of the theory described above we construct the functions of influence of the inner unit point heat source on the thermoelastic displacements $U_k(x,\xi)$. At the second step we have to calculate (on the basis of the main influence functions $U_k(x,\xi)$) the other influence functions $\partial U_k(y,\xi)/\partial n_y$ and to write the Poisson's type integral formulas for respective BVP of thermoelasticity for half-space. At the last step it is necessary to prove that the obtained influence functions and Poisson's type integral formulas satisfy the respective BVP.

2.1. Main Influence Functions and Green's Integral Formula for a Thermoelastic Half-space

To obtain the functions of influence of an inner unit point heat source $F(\xi) = \delta(x-\xi)$ on thermoelastic displacements for a half-space we will use the general formula in (1.4) rewritten as follows:

$$U_k(x,\xi) = \gamma \int_0^\infty \iint_{-\infty}^\infty G(x,z)\ \Theta^{(k)}(z,\xi) dz_1 dz_2 dz_3, \quad |U_k(x,\xi)| < \infty. \tag{2.1}$$

Before starting to formulate the theorem we have to give the following Poisson's type integral formula for half-space $V(0 \le x_1 < \infty;\ \ -\infty < x_2, x_3 < \infty)$:

$$u_k(\xi) = a^{-1} \int_0^\infty \iint_{-\infty}^\infty F(x) U_k(x,\xi) dx_1 dx_2 dx_3 - \iint_{-\infty}^\infty T(y) \frac{\partial U_k(y,\xi)}{\partial n_{y1}} dy_2 dy_3, \quad |u_k(\xi)| < \infty, \tag{2.2}$$

where the thermoelastic displacements $u_k(\xi)$ are generated by the inner heat source $F(x)$ and given temperature $T(y)$ on the boundary plane $\Gamma(y_1 = 0;\ \ -\infty < y_2, y_3 < \infty)$. The formula in (2.2) is obtained from the suggested general integral formula from (1.3), wherein it should be taken into account that on the all boundary plane Γonly Dirichlet's condition (temperature) is given. Note that when we check if the boundary conditions for thermoelastic stresses

$\sigma_{ij}(x,\xi)$ with respect to coordinates of the point $\xi \equiv (\xi_1,\xi_2,\xi_3)$ are satisfied, we have to use the Duhamel-Neumann law

$$\sigma_{ij} = \mu(U_{i,j} + U_{j,i}) + \delta_{ij}(\lambda U_{j,j} - \gamma G), \quad i,j = 1,2,3. \tag{2.3}$$

2.2. A Theorem about Deriving a Poisson's Integral Formula for Thermoelastic Half-Space

Theorem 2.1. *Let the field of displacements $u_k(\xi)$ at inner points $\xi \equiv (\xi_1,\xi_2,\xi_3)$ of the elastic half-space $V(0 \le x_1 < \infty;\ -\infty < x_2, x_3 < \infty)$ be determined by nonhomogeneous Lame equations*

$$\mu\nabla^2 u_k(\xi) + (\lambda+\mu)\theta_{,k}(\xi) - \gamma T_{,k}(\xi) = 0, \tag{2.4}$$

and at the points $y \equiv (0, y_2, y_3)$ of its boundary plane $\Gamma(y_1 = 0;\ -\infty < y_2, y_3 < \infty)$ the following homogeneous mixed mechanical conditions are given:

$$\begin{gathered}\sigma_{11}(\xi_1 = 0,\ \xi_2 = y_{2,},\ \xi_3 = y_3) = 0, \qquad u_2(\xi_1 = 0,\ \xi_2 = y_{2,},\ \xi_3 = y_3) = 0,\\ u_3(\xi_1 = 0,\ \xi_2 = y_{2,},\ \xi_3 = y_3) = 0,\end{gathered} \tag{2.5}$$

where σ_{11} are normal stresses which are determined by the Duhamel-Neumann law (2.3).

Let also the temperature field $T(\xi)$ in (2.4)*, generated by the inner heat source $F(\xi)$ and temperature $T(y) \in \Gamma$ (Dirichlet's boundary condition), satisfy the following BVP in heat conduction:*

$$\nabla^2 T(\xi) = -a^{-1}F(\xi), \quad \xi \in V, \quad T(\xi_1 = 0,\ \xi_2 = y_{2,},\ \xi_3 = y_3) = T(y), \quad y \equiv (0, y_2, y_3) \in \Gamma. \tag{2.6}$$

If the inner heat source and boundary temperature satisfies the conditions

$$\int_0^\infty \iint_{-\infty}^{\infty} |F(x)| dx_1 dx_2 dx_3 < \infty, \qquad \iint_{-\infty}^{\infty} |T(y)| dy_2 dy_3 < \infty, \tag{2.7}$$

then the solution of this BVP in (2.4)–(2.7) *of thermoelasticity for searched displacements $u_k(\xi)$ for half-space exists, and it can be presented by the following Poisson's type integral formula, written in the matrix form,*

$$\mathbf{u}(\xi) = \frac{1}{a}\int_0^\infty \iint_{-\infty}^{\infty} F(x)\mathbf{U}(\mathbf{x},\xi) dx_1 dx_2 dx_3 - \iint_{-\infty}^{\infty} T(y)\mathbf{Q}(\mathbf{y},\xi) dy_2 dy_3, \quad y \equiv (0, y_2, y_3), \tag{2.8}$$

where $|\mathbf{u}(\xi)| < \infty$ everywhere and vanish at infinity $\lim_{\xi_1\to\infty} u_k(\xi) \to 0$, $\lim_{|\xi_2|\to\infty} u_k(\xi) \to 0$, $\lim_{|\xi_3|\to\infty} u_k(\xi) \to 0$.

The matrices $\mathbf{U}(\mathbf{x},\xi)$ *and* $\mathbf{Q}(\mathbf{y},\xi) = \partial\mathbf{U}(y,\xi)/\partial n_{y1}$ *of influence of an inner unit point heat source* $F(\xi) = \delta(x-\xi)$ *and of a unit point temperature* $T(y) = \delta(y-\xi)$ *on the* Γ *onto thermoelastic displacements, also the matrix* $\mathbf{u}(\xi)$ *of searched displacements in* (2.8), *are determined as follows:*

$$\mathbf{U}(\mathbf{x},\xi) = \begin{pmatrix} U_1(x,\xi) \\ U_2(x,\xi) \\ U_3(x,\xi) \end{pmatrix} = \frac{\gamma}{8\pi(\lambda+2\mu)} \begin{pmatrix} -\left(\dfrac{x_1+\xi_1}{R_1} + \dfrac{x_1-\xi_1}{R}\right) \\ (x_2-\xi_2)\left(\dfrac{1}{R_1} - \dfrac{1}{R}\right) \\ (x_3-\xi_3)\left(\dfrac{1}{R_1} - \dfrac{1}{R}\right) \end{pmatrix},$$

$$\mathbf{Q}(\mathbf{y},\xi) = \begin{pmatrix} Q_1(y,\xi) \\ Q_2(y,\xi) \\ Q_3(y,\xi) \end{pmatrix} = \frac{\gamma}{(\lambda+2\mu)R} \begin{pmatrix} 1 - \dfrac{\xi_1^2}{R^2} \\ (y_2-\xi_2)\dfrac{\xi_1}{R^2} \\ (y_3-\xi_3)\dfrac{\xi_1}{R^2} \end{pmatrix}, \qquad \mathbf{u}(\xi) = \begin{pmatrix} u_1(\xi) \\ u_2(\xi) \\ u_3(\xi) \end{pmatrix}, \tag{2.9}$$

where $R = R(x,\xi) = |x-\xi|$; $R_1(x,\xi) = |x-\xi_1^*|$; $\xi \equiv (\xi_1,\xi_2,\xi_3)$; $\xi_1^* \equiv (-\xi_1,\xi_2,\xi_3)$; $x \equiv (x_1,x_2,x_3)$ *in the matrix* $\mathbf{U}(\mathbf{x},\xi)$, *and* $R = R(y,\xi) = |y-\xi|$; $R_1(y,\xi) = |y-\xi_1^*|$; $\xi \equiv (\xi_1,\xi_2,\xi_3)$; $\xi_1^* \equiv (-\xi_1,\xi_2,\xi_3)$; $y \equiv (0,y_2,y_3)$ *in the matrix* $\mathbf{Q}(\mathbf{y},\xi)$.

Proof. First, well-known Green's function G for Poisson's equation for a half-space is rewritten, and then, in the Subsection 2.2.1, we derive the volume dilatation $\Theta^{(k)}(x,\xi)$. In Subsection 2.2.2 it is shown how to derive thermoelastic influence functions $U_k(x,\xi)$. Checking the correctitude of the derived already thermoelastic influence functions $U_k(x,\xi)$ is given in Subsection 2.2.3. Finally, in Subsection 2.2.4, on the base of the functions $U_k(x,\xi)$, the Poisson's type integral formula for stated BVP of thermoelasticity is derived and checked.

To obtain the matrix $\mathbf{U}(\mathbf{x},\xi)$ in (2.9) for the BVP in (2.4)–(2.6) we will use the integral formula in (2.1). The functions $G(x,\xi)$ and $\Theta^{(k)}(x,\xi)$ in this equation are the Green's functions of Dirichlet problem in heat conduction and, respectively, the influence functions of a unit concentrated body force $\delta_{ik}\delta(x-\xi)$ onto volume dilatation in theory of elasticity for the half-space V. So, to get the Green's function $G(x,\xi)$ we have to solve the BVP, which consists of the equation in the heat conduction with the homogeneous boundary conditions similar to those in (2.6):

$$\nabla_x^2 G(x,\xi) = -\delta(x-\xi), \quad x,\xi \in V, \quad G = 0, \quad y_1 = 0, \quad -\infty < y_2, y_3 < \infty. \tag{2.10}$$

In the presented paper we recall this Green's function from the handbook [11] (see problem and answer 15.P.1):

$$G(x,\xi) = \frac{1}{4\pi}\left(R^{-1} - R_1^{-1}\right), \qquad R = \sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2 + (x_3-\xi_3)^2},$$
$$R_1 = \sqrt{(x_1+\xi_1)^2 + (x_2-\xi_2)^2 + (x_3-\xi_3)^2}. \tag{2.11}$$

This Green's function is well known and is presented also in the books [18, 19]. □

2.2.1. Deriving the Volume Dilatation $\Theta^{(k)}(x,\xi)$

To get the influence functions $\Theta^{(k)}(x,\xi)$, usually, we have to solve the following BVP, which consists of Lame's equations in theory of elasticity and homogeneous boundary conditions as in (2.5)

$$\begin{gathered}\mu\nabla^2 U_i^{(k)}(x,\xi) + (\lambda+\mu)\Theta_{,i}^{(k)}(x,\xi) = -\delta_{ik}\delta(x-\xi), \quad x,\xi \in V, \\ \sigma_{11}^{(k)}(y,\xi) = U_2^{(k)}(y,\xi) = U_3^{(k)}(y,\xi) = 0, \quad y_1 = 0, \quad -\infty < y_2, y_3 < \infty\end{gathered} \tag{2.12}$$

and then, on the base of displacements $U_i^{(k)}(x,\xi)$, to compute volume dilatation

$$\Theta^{(k)}(x,\xi) = U_{j,j}^{(k)}(x,\xi). \tag{2.13}$$

But as it will be shown below, in the case of the boundary condition in (2.12), we can derive the volume dilatation $\Theta^{(k)}(x,\xi)$ using the equation

$$\nabla^2\Theta^{(k)}(x,\xi) = -\frac{1}{\lambda+2\mu}\frac{\partial}{\partial x_k}\delta(x-\xi) \tag{2.14}$$

only, and its integral representation via respective Green's function $G_\Theta(x,\xi)$:

$$\Theta^{(k)}(x,\xi) = -\frac{1}{\lambda+2\mu}\frac{\partial}{\partial \xi_k}G_\Theta(x,\xi) + \int_\Gamma \left[\frac{\partial\Theta^{(k)}(y,\xi)}{\partial n_\Gamma} - \Theta^{(k)}(y,\xi)\frac{\partial}{\partial n_\Gamma}\right]G_\Theta(y,x)\,d\Gamma(y). \tag{2.15}$$

To show this, first prove that the boundary conditions in (2.12) lead to the zero volume dilatation on the boundary plane $\Gamma(y_1 = 0;\ -\infty < y_2, y_3 < \infty)$:

$$\Theta^{(k)}(y,\xi) = 0. \tag{2.16}$$

To do this we write the Hooke's law for the normal stresses $\sigma_{11}^{(k)}$ in the form

$$\sigma_{11}^{(k)} = 2\mu U_{1,1}^{(k)} + \lambda\Theta^{(k)} = (\lambda+2\mu)\Theta^{(k)} - 2\mu\left(U_{2,2}^{(k)} + U_{3,3}^{(k)}\right), \quad i,j = 1,2,3. \tag{2.17}$$

Next we know that the boundary conditions

$$U_2^{(k)} = U_3^{(k)} = 0 \tag{2.18}$$

lead to the following zero tangential derivatives on the boundary plane $\Gamma(y_1 = 0;\ -\infty < y_2, y_3 < \infty)$:

$$U_{2,2}^{(k)} = U_{3,3}^{(k)} = 0. \tag{2.19}$$

Finally, substituting equalities (2.19) and boundary conditions $\sigma_{11}^{(k)} = 0$ in the Hooke's law (2.17) we come to the zero volume dilatation on the boundary Γ in (2.16).

So, as for the BVP for $\Theta^{(k)}$, we have (2.14) and (2.16), then the respective BVP for Green's function G_Θ for considered half-space can be written in the form

$$\nabla_x^2 G_\Theta(x,\xi) = -\delta(x-\xi), \quad x,\xi \in V, \quad G_\Theta = 0, \quad y_1 = 0, \quad -\infty < y_2, y_3 < \infty, \tag{2.20}$$

which coincides with the BVP in (2.10). This fact leads to the conclusion that $G_\Theta \equiv G$, and, as a result, the Green's function G_Θ is determined by expressions in (2.11).

Finally, if we introduce the expressions $G_\Theta \equiv G$ in (2.10), (2.11) and the equality $\Theta^{(k)}(y,\xi) = 0$ in (2.16) into the representation (2.15), taking into account the equality

$$\left[\frac{\partial \Theta^{(k)}(y,\xi)}{\partial n_\Gamma} - \Theta^{(k)}(y,\xi)\frac{\partial}{\partial n_\Gamma}\right] G_\Theta(y,x) = 0, \tag{2.21}$$

we obtain the searched volume dilatation of the elastic BVP in (2.12) for half-space, written in the form

$$\Theta^{(k)}(x,\xi) = -\frac{1}{4\pi(\lambda+2\mu)}\frac{\partial}{\partial \xi_k}\left(R^{-1} - R_1^{-1}\right). \tag{2.22}$$

2.2.2. Deriving the Thermoelastic Influence Functions $U_k(x,\xi)$

Now we have both functions: $G(x,\xi)$ and $\Theta^{(k)}(x,\xi)$ needed for deriving the thermoelastic influence functions $U_k(x,\xi)$, using (2.1). So, substituting functions $G(x,\xi)$ and $\Theta^{(k)}(x,\xi)$ from (2.11) and (2.22) in (2.1), then, calculating the volume integral in the special way (see appendix), we will get the following expression for functions $U_k(x,\xi)$:

$$\begin{aligned} U_k(x,\xi) &= \gamma \int_0^\infty \iint_{-\infty}^\infty G(x,z)\Theta^{(k)}(z,\xi)\,dz_1 dz_2 dz_3 \\ &= -\gamma \int_0^\infty \iint_{-\infty}^\infty \frac{1}{4\pi}\left(R^{-1} - R_1^{-1}\right)\frac{1}{4\pi(\lambda+2\mu)}\frac{\partial}{\partial \xi_k}\left(R^{-1} - R_1^{-1}\right) dz_1 dz_2 dz_3, \\ U_k(x,\xi) &= \frac{\gamma(\lambda+2\mu)^{-1}}{8\pi}\frac{\partial}{\partial \xi_k}[R(x,\xi) - R_1(x,\xi)], \end{aligned} \tag{2.23}$$

that being presented in the matrix form coincide with the matrix $\mathbf{U}(\mathbf{x},\xi)$ in (2.9). From the expressions $U_k(x,\xi)$ in (2.23) we can see that $|U_k(x,\xi)| < \infty$ and that these displacements vanish at infinity: $\lim_{\xi_1\to\infty} U_k(x,\xi) \to 0$, $\lim_{|\xi_2|\to\infty} U_k(x,\xi) \to 0$, and $\lim_{|\xi_3|\to\infty} U_k(x,\xi) \to 0$.

At the next step of the proof of the theorem we have to check the correctness of the functions $U_k(x,\xi)$. To do this we have to use the fact that these functions satisfy both physical phenomena: elasticity in (1.5) and heat conduction in (1.4). So, they must satisfy the following

BVP of thermoelasticity in respect to the coordinates of the point of application $\xi \equiv (\xi_1, \xi_2, \xi_3)$ of the inner unit heat source $F(\xi) = \delta(x - \xi)$:

$$\begin{gathered}\mu \nabla_{\xi}^{2} U_k(x,\xi) + (\lambda + \mu)\Theta_{,\xi_k}(x,\xi) - \gamma G_{,\xi_k}(x,\xi) = 0, \\ \sigma_{11} = U_2 = U_3 = 0, \quad \xi_1 = 0, \ -\infty < \xi_2, \xi_3 < \infty,\end{gathered} \tag{2.24}$$

which follows from the integral formula in (2.31) and boundary conditions in (2.12) (see also (1.5)). Also, the functions $U_k(x,\xi)$ have to satisfy the following fictive BVP in heat conduction with respect to the coordinates of the point of observation $x \equiv (x_1, x_2, x_3)$:

$$\nabla_{x}^{2} U_k(x,\xi) = -\gamma\ \Theta^{(k)}(x,\xi), \quad x,\xi \in V, \quad U_k(y,\xi) = 0, \quad y \equiv (0, y_2, y_3) \in \Gamma, \tag{2.25}$$

which follows from the integral formula in (2.23) and boundary conditions in (2.10) (see also (1.4)). In addition, the functions $U_k(x,\xi)$ in (2.23) have to vanish at infinity. In (2.25) the functions $\Theta^{(k)}(x,\xi)$ are determined by expression in (2.22), but in (2.24) the functions $\Theta(x,\xi)$ are determined on the base of displacements $U_k(x,\xi)$ in (2.23), using the rule $\Theta(x,\xi) = U_{k,k}(x,\xi)$, where derivatives are taken with respect to coordinates of the point $\xi \equiv (\xi_1, \xi_2, \xi_3)$.

2.2.3. Checking the Correctitude of the Thermoelastic Influence Functions $U_k(x,\xi)$

To check (2.24) and (2.25) we need to compute the following values:

$$\begin{gathered}\nabla_{\xi}^{2} R(x,\xi) = \nabla_{x}^{2} R(x,\xi) = 2R^{-1}(x,\xi), \qquad \nabla_{\xi}^{2} R_1(x,\xi) = \nabla_{x}^{2} R_1(x,\xi) = 2R_1^{-1}(x,\xi), \\ \nabla_{\xi}^{2} R^{-1}(x,\xi) = \nabla_{\xi}^{2} R_1^{-1}(x,\xi) = \nabla_{x}^{2} R^{-1}(x,\xi) = \nabla_{x}^{2} R_1^{-1}(x,\xi) = 0.\end{gathered} \tag{2.26}$$

Check first (2.24). So, from (2.23), (2.24), and (2.26) it follows that

$$\begin{gathered}\mu \nabla_{\xi}^{2} U_k(x,\xi) = \frac{\gamma\mu}{4\pi(\lambda + 2\mu)} \frac{\partial}{\partial \xi_k}\left[R^{-1}(x,\xi) - R_1^{-1}(x,\xi)\right], \\ (\lambda + \mu)\ \Theta_{,\xi_k}(x,\xi) = \frac{\gamma(\lambda + \mu)}{4\pi(\lambda + 2\mu)} \frac{\partial}{\partial \xi_k}\left[R^{-1}(x,\xi) - R_1^{-1}(x,\xi)\right].\end{gathered} \tag{2.27}$$

From (2.27) and (2.11) it follows that

$$\mu \nabla_{\xi}^{2} U_k(x,\xi) + (\lambda + \mu)\ \Theta_{,k}(x,\xi) = \frac{\gamma}{4\pi} \frac{\partial}{\partial \xi_k}\left[R^{-1}(x,\xi) - R_1^{-1}(x,\xi)\right] = \gamma G_{,k}(x,\xi), \tag{2.28}$$

where the expression for Poisson's function $G(x,\xi)$ in (2.11) was used. So, due to (2.28), we come to the conclusion that the functions in (2.23) verify (2.24) at inner points. To check boundary conditions in (2.24) we need to calculate the derivatives in the expressions for

displacements $U_k(x,\xi)$ in (2.23) and calculate the values of the normal stresses for $\xi_1 = 0$. So, using the Duhamel-Neumann law (2.3) for normal stresses σ_{11}

$$\sigma_{11} = 2\mu U_{1,1} + (\lambda\Theta - \gamma G), \tag{2.29}$$

where the displacements U_1, the volume dilatation Θ are determined on the base of expressions in (2.23), and the Green's function G is determined by (2.11), we will obtain

$$2\mu U_{1,1}(x,\xi)\big|_{\xi_1=0} = \gamma\frac{\mu(\lambda+2\mu)^{-1}}{4\pi}\left[\frac{1}{R(x,\xi)} - \frac{1}{R_1(x,\xi)} - \frac{(x_1-\xi_1)^2}{R(x,\xi)} + \frac{(x_1+\xi_1)^2}{R_1(x,\xi)}\right]\Bigg|_{\xi_1=0} = 0,$$

$$\lambda\Theta^{(k)}(x,\xi)\Big|_{\xi_1=0} = \frac{\lambda\gamma}{4\pi(\lambda+2\mu)}\left(R^{-1} - R_1^{-1}\right)\Bigg|_{\xi_1=0} = 0, \qquad G(x,\xi)|_{\xi_1=0} = \frac{1}{4\pi}\left(R^{-1} - R_1^{-1}\right)\Bigg|_{\xi_1=0} = 0. \tag{2.30}$$

Substituting expressions in (2.30) in (2.29) we obtain $\sigma_{11}|_{\xi_1=0} = 0$, so the first boundary condition in (2.24) is satisfied. Using the expressions in (2.23) for U_2 and U_3, and we obtain

$$U_k(x,\xi)|_{\xi_1=0} = \gamma\frac{(\lambda+2\mu)^{-1}}{8\pi}\frac{\partial}{\partial\xi_k}[R(x,\xi) - R_1(x,\xi)]\Bigg|_{\xi_1=0} = 0, \quad k = 2,3. \tag{2.31}$$

So, as we can see from (2.31) and from $\sigma_{11}|_{\xi_1=0} = 0$, the BVP in (2.24) is satisfied. Now we can confirm that the thermoelastic influence functions $U_k(x,\xi)$ satisfy BVP of thermoelasticity in (2.24). Next we have to prove that the influence functions in (2.23) satisfy with respect to the coordinates of the point of observation $x \equiv (x_1, x_2, x_3)$ the fictive BVP in heat conduction, described by (2.25). Indeed, using (2.22), (2.26) and (2.23), we will obtain

$$\nabla_x^2 U_k(x,\xi) = \frac{\gamma}{4\pi(\lambda+2\mu)}\frac{\partial}{\partial\xi_k}\left(R^{-1}(x,\xi) - R_1^{-1}(x,\xi)\right) = -\gamma\Theta^{(k)}(x,\xi), \tag{2.32}$$

which means that at inner points (2.25) is satisfied. Also, taking into account expressions in (2.23) for displacements $U_k(x,\xi)$ we come to the conclusion that the boundary conditions in (2.25) on the boundary plane are also satisfied, which means $U_k(y,\xi) = 0$; $y \equiv (0, y_2, y_3) \in \Gamma$.

2.2.4. Deriving and Checking the Poisson's Type Integral Formula

The next step is to calculate the other influence functions in (1.10) $\partial U_k(y,\xi)/\partial n_{y1}$ and to get Poisson's type integral formula. So, to get the Poisson's type integral formula we have to use formula (2.2) for the half-space. The influence functions $U_k(x,\xi)$ in this formula are determined by (2.23). The functions $\partial U_k(y,\xi)/\partial n_{y1}$, as functions of influence of unit point

temperature, given on the boundary plane, according to the formula in (1.10) have to be determined on the basis of the functions $U_k(x,\xi)$ in (2.23) as following

$$\begin{aligned}
\frac{\partial U_k(y,\xi)}{\partial n_{y1}} &= -\lim_{x\to y}\frac{\partial U_k(x,\xi)}{\partial x_1} = -\lim_{x\to y}\frac{\gamma(\lambda+2\mu)^{-1}}{8\pi}\frac{\partial^2}{\partial x_1\partial\xi_k}[R(x,\xi)-R_1(x,\xi)] \\
&= \lim_{x\to y}\frac{\gamma(\lambda+2\mu)^{-1}}{4\pi}\frac{\partial^2}{\partial\xi_1\partial\xi_k}R(x,\xi), \\
\frac{\partial U_k(y,\xi)}{\partial n_{y1}} &= \frac{\gamma(\lambda+2\mu)^{-1}}{4\pi}\frac{\partial^2}{\partial\xi_1\partial\xi_k}R(y,\xi).
\end{aligned} \tag{2.33}$$

As we see, the expressions for $\partial U_k(y,\xi)/\partial n_{y1}$ in (2.33) vanish at infinity.

Introducing the influence functions in (2.23) and (2.33) into the formula in (2.2) we obtain the following Poisson's type integral formula:

$$\begin{aligned}
U_k(\xi) &= a^{-1}\int_0^\infty\iint_{-\infty}^{\infty}F(x)U_k(x,\xi)dx_1dx_2dx_3 - \iint_{-\infty}^{\infty}T(y)\frac{\partial U_k(y,\xi)}{\partial n_{y1}}dy_2dy_3 \\
&= \frac{\gamma}{8\pi(\lambda+2\mu)}\left[a^{-1}\int_0^\infty\iint_{-\infty}^{\infty}F(x)\frac{\partial}{\partial\xi_k}[R(x,\xi)-R_1(x,\xi)]dx_1dx_2dx_3,\right. \\
&\qquad \left. -2\iint_{-\infty}^{\infty}T(y)\frac{\partial^2}{\partial\xi_1\partial\xi_k}R(y,\xi)dy_2dy_3\right], \quad |u_k(\xi)|<\infty.
\end{aligned} \tag{2.34}$$

Finally, to notice that integrals with unbounded intervals will exist, which means that the displacements $|u_k(\xi)|<\infty$, when the following conditions are satisfied:

$$\int_0^\infty\iint_{-\infty}^{\infty}|F(x)|dx_1dx_2dx_3<\infty, \qquad \iint_{-\infty}^{\infty}|T(y)|dy_2dy_3<\infty, \tag{2.35}$$

because, as was mentioned above, the kernels in (2.34) vanish at infinity. The conditions in (2.35) will be satisfied in case the functions $F(x)$ and $T(y)$ are given on the bounded domains.

If in (2.23) and (2.33) we calculate the derivatives, and, if we present the obtained results in matrix form, then we can be sure that the influence matrices $\mathbf{U}(x,\xi)$, $\partial\mathbf{U}(\mathbf{y},\xi)/\partial\mathbf{n}_{\mathbf{y}_1} = \mathbf{Q}(\mathbf{y},\xi)$, and Poisson's type integral formula in (2.34) coincide with the results in (2.8) and (2.9).

The obtained displacements, described by Poisson's type integral formula (2.34), must satisfy the stated in the theorem BVP of thermoelasticity, described by (2.4) and (2.5). To check this we substitute the integral formula from (2.34) into (2.4):

$$\begin{aligned}
\mu\nabla^2u_k(\xi)+(\lambda+\mu)\theta_{,k}(\xi) &= a^{-1}\int_0^\infty\iint_{-\infty}^{\infty}F(x)\left[\mu\nabla_\xi^2U_k(x,\xi)+(\lambda+\mu)\Theta_{,k}(x,\xi)\right]dx_1dx_2dx_3 \\
&\quad -\iint_{-\infty}^{\infty}T(y)\frac{\partial}{\partial n_{y_1}}\left[\mu\nabla_\xi^2U_k(y,\xi)+(\lambda+\mu)\Theta_{,k}(y,\xi)\right]dy_2dy_3.
\end{aligned} \tag{2.36}$$

Taking into account the already proved equality in (2.28) and the Poisson's type integral formula for BVP in heat conduction, described by (2.6),

$$T(\xi) = \int_0^\infty \iint_{-\infty}^{\infty} F(x)G(x,\xi)dx_1dx_2dx_3 - \iint_{-\infty}^{\infty} T(y)\frac{\partial}{\partial n_{y_1}}G(y,\xi)\ dy_2dy_3, \tag{2.37}$$

the following is obtained:

$$\begin{aligned}\mu\nabla^2 u_k(\xi) + (\lambda+\mu)\theta_{,k}(\xi) &= a^{-1}\int_0^\infty \iint_{-\infty}^{\infty} F(x)G_{,k}(x,\xi)dx_1dx_2dx_3 \\ &\quad - \iint_{-\infty}^{\infty} T(y)\frac{\partial}{\partial n_{y_1}}G_{,k}(y,\xi)\ dy_2dy_3 = T_{,k}(\xi).\end{aligned} \tag{2.38}$$

So, the displacements, described by Poisson's type integral formula (2.34), satisfy (2.4). Finally, we have to show that the displacements in (2.34) satisfy the mechanical boundary conditions in (2.5). Of course, these conditions are satisfied, because, as we have shown above, the kernels of Poisson's type integral formula (2.34) satisfied the same mechanical boundary conditions in (2.24) that are analogical to (2.5). So, now we are sure that all items of the theorem are proved. It means that the theorem is proved.

3. An Example of Application of the Obtained Poisson's Type Integral Formula

Let us solve (2.4)–(2.6) and to determine thermoelastic displacements $u_k(\xi)$ in half-space if on the segment $[y_1 = 0, -b \le y_2 \le b; y_3 = 0]$ of its boundary plane $\Gamma(y_1 = 0;\ -\infty < y_2, y_3 < \infty)$, the constant temperature

$$T(y) = \begin{cases} T_0 = \text{const}, & y \in [y_1 = 0, -b \le y_2 \le b; y_3 = 0]; \\ 0, & y \notin [y_1 = 0, -b \le y_2 \le b; y_3 = 0] \end{cases} \tag{3.1}$$

is given. To solve this BVP we use the integral formula in (2.8) and (2.9) for $F(x) = 0$ and $T(y) = T_0 = \text{const}$:

$$\mathbf{u}(\xi) = -T_0\int_{-b}^{b} \mathbf{Q}(\mathrm{y};\xi)dy_2, \quad y \equiv (0, y_2, 0),\ -b \le y_2 \le b, \tag{3.2}$$

where $\mathbf{Q}(\mathrm{y};\xi)$ is determined by the matrix (2.9). The formulas in (3.1) and (2.9) can be presented in terms of components of thermoelastic displacements u_k; $k = 1,2,3$ as follows (see (2.9) and (2.34)):

$$u_k(\xi) = -\frac{\gamma T_0}{4\pi(\lambda+2\mu)}\int_{-b}^{b}\frac{\partial^2}{\partial\xi_1\partial\xi_k}R(\mathrm{y}_2;\xi)dy_2, \quad R(\mathrm{y}_2;\xi) = \sqrt{\xi_1^2 + (y_2-\xi_2)^2 + \xi_3{}^2}, \quad \xi \equiv (\xi_1,\xi_2,\xi_3). \tag{3.3}$$

So, taking the respective integrals over the boundary segment $[y_1 = 0, -b \le y_2 \le b; y_3 = 0]$, we obtain the following thermoelastic displacements at an arbitrary inner point $\xi \equiv (\xi_1, \xi_2, \xi_3)$:

$$\begin{aligned} u_k(\xi) &= -\frac{\gamma T_0}{4\pi(\lambda+2\mu)}\int_{-b}^{b}\frac{\partial^2}{\partial\xi_k\partial\xi_1}R(\mathrm{y}_2;\xi)\,dy_2 = -\frac{\gamma T_0}{4\pi(\lambda+2\mu)}\frac{\partial}{\partial\xi_k}\int_{-b}^{b}\frac{\xi_1}{R(\mathrm{y}_2;\xi)}dy_2 \\ &= -\frac{\gamma T_0}{4\pi(\lambda+2\mu)}\frac{\partial}{\partial\xi_k}\left[\xi_1\ln\left|y_2-\xi_2+R(\mathrm{y}_2;\xi)\right|_{-b}^{b}\right], \\ u_k(\xi) &= -\frac{\gamma T_0}{4\pi(\lambda+2\mu)}\frac{\partial}{\partial\xi_k}[\xi_1 L(\xi)], \end{aligned} \tag{3.4}$$

where

$$L(\xi)=\ln\left|\frac{b_2-\xi_2+R(b;\xi)}{-(b+\xi_2)+R(-b;\xi)}\right|, \quad R(b;\xi)=\sqrt{\xi_1^2+(b-\xi_2)^2+\xi_3{}^2},\ R(-b;\xi)=\sqrt{\xi_1^2+(b+\xi_2)^2+\xi_3{}^2}. \tag{3.5}$$

Check whether thermoelastic displacements $u_k(\xi)$ in (3.4) and (3.5) satisfy the stated in this section BVP, described by (2.4)–(2.6) and (3.1). For this, first calculate

$$\mu\nabla^2 u_k(\xi) = -\mu\frac{\gamma T_0}{4\pi(\lambda+2\mu)}\nabla^2\frac{\partial}{\partial\xi_k}[\xi_1 L(\xi)] = -\frac{\gamma\mu T_0}{2\pi(\lambda+2\mu)}\frac{\partial^2}{\partial\xi_k\partial\xi_1}L(\xi), \tag{3.6}$$

$$\begin{aligned} (\lambda+\mu)\frac{\partial}{\partial\xi_k}\Theta^{(k)}(\xi) &= (\lambda+\mu)\frac{\partial^2}{\partial\xi_k\partial\xi_j}u_j(\xi) = -\frac{(\lambda+\mu)\gamma T_0}{4\pi(\lambda+2\mu)}\frac{\partial^3}{\partial\xi_k\partial\xi_j^2}[\xi_1 L(\xi)] \\ &= -\frac{(\lambda+\mu)\gamma T_0}{4\pi(\lambda+2\mu)}\frac{\partial}{\partial\xi_k}\nabla^2[\xi_1 L(\xi)] = -\frac{(\lambda+\mu)\gamma T_0}{2\pi(\lambda+2\mu)}\frac{\partial^2}{\partial\xi_k\partial\xi_1}L(\xi). \end{aligned} \tag{3.7}$$

In (3.6) and (3.7) the Euler's formula was used:

$$\nabla^2(f\cdot\varphi) = \varphi\nabla^2 + f\nabla^2\varphi + 2f_{,j}\varphi_{,j}. \tag{3.8}$$

Also it was taken into account that the function $L(\xi)$, described by (3.5), is a harmonic function, so that $\nabla^2 L(\xi) = 0$. Next calculate the temperature $T(\xi)$ using the well-known Poisson's integral formula in (2.37) rewritten in the case when on the segment $[y_1 = 0, -b \le y_2 \le b; y_3 = 0]$ of its boundary plane $\Gamma(y_1 = 0;\ -\infty < y_2, y_3 < \infty)$ the constant temperature $T = T_0$ is given:

$$T(\xi) = -T_0\int_{-b}^{b}\frac{\partial}{\partial n_{y_1}}G(y_2,\xi)\,dy_2 = -T_0\frac{\partial}{\partial\xi_1}\int_{-b}^{b}\frac{dy_2}{2\pi R(\mathrm{y}_2;\xi)}, \qquad T(\xi) = -\frac{T_0}{2\pi}\frac{\partial}{\partial\xi_1}L(\xi). \tag{3.9}$$

It is easy to see that temperature in (3.9) satisfies the BVP in heat conduction, described by Poisson equation (2.6) [for $F(x) = 0$] and boundary condition in (3.1). So, substituting the expressions in (3.6), (3.7), and (3.9) into (2.4) we can see that this equation is satisfied. It

remains to show that the thermoelastic displacements $u_k(\xi)$ in (3.4) and (3.5) satisfy the boundary conditions in (2.5), and also that they vanish at infinity. From (3.4) it is easy to see that for $k = 2,3$ the thermoelastic displacements $u_2(\xi)$ and $u_3(\xi)$ for $\xi_1 = 0$ satisfy the second and third boundary conditions in (2.5). To check the first boundary condition in (2.5) calculate the normal stresses $\sigma_{11}(\xi)$ using the Duhamel-Neumann law, the displacements in (3.4) and the temperature in (3.9):

$$\begin{aligned}
&\sigma_{11} = 2\mu U_{1,1} + \lambda U_{j,j} - \gamma T, \qquad i,j = 1,2,3, \qquad \sigma_{11} = (\lambda + 2\mu)U_{1,1} - \lambda(U_{2,2} + U_{3,3}) - \gamma T,\\
&\sigma_{11} = -(\lambda + 2\mu)\frac{\gamma T_0}{4\pi(\lambda + 2\mu)}\frac{\partial^2}{\partial \xi_1^2}[\xi_1 L(\xi)] - \lambda(U_{2,2} + U_{3,3}) - \gamma T,\\
&\sigma_{11} = -\frac{\gamma T_0}{4\pi}\left(2 + \xi_1\frac{\partial}{\partial \xi_1}\right)\frac{\partial}{\partial \xi_1}L(\xi) - \lambda(U_{2,2} + U_{3,3}) + \frac{T_0}{2\pi}\frac{\partial}{\partial \xi_1}L(\xi).
\end{aligned} \tag{3.10}$$

As on the boundary plane (for $\xi_1 = 0$) $u_2 = u_3 = 0$ (which means also $u_{2,2} = u_{3,3} = 0$), then from the last equation it follows $\sigma_{11} = 0$ [the first boundary condition in (2.5) is satisfied].

Finally, calculating the derivatives of expressions in (3.4) and (3.5), we obtain the thermoelastic displacements of the stated in this section BVP in the explicit form

$$\begin{aligned}
u_1(\xi) &= -\frac{\gamma T_0}{4\pi(\lambda + 2\mu)}\left\{\ln\left|\frac{b_2 - \xi_2 + R(b;\xi)}{-(b + \xi_2) + R(-b;\xi)}\right|\right.\\
&\qquad \left. + \xi_1^2\left[R^{-1}(b;\xi)(b_2 - \xi_2 + R(b;\xi))^{-1} - R^{-1}(-b;\xi)(-(b_2 + \xi_2) + R(-b;\xi))^{-1}\right]\right\};\\
u_2(\xi) &= \frac{\gamma T_0}{4\pi(\lambda + 2\mu)}\xi_1\left[R^{-1}(b;\xi) - R^{-1}(-b;\xi)\right];\\
u_3(\xi) &= -\frac{\gamma T_0}{4\pi(\lambda + 2\mu)}\xi_1\xi_3\left[R^{-1}(b;\xi)(b_2 - \xi_2 + R(b;\xi))^{-1} - R^{-1}(-b;\xi)(-(b_2 + \xi_2) + R(-b;\xi))^{-1}\right],
\end{aligned} \tag{3.11}$$

which vanish at infinity.

4. Conclusions

(1) The Poisson's type integral formula, obtained in this paper, is new, useful, and completely ready to be efficiently applied for computing of the thermoelastic displacements $u_k(\xi)$ in half-space (see example of its application given in Section 3). The main advantage of the obtained integral formula is that the searched half-space thermoelastic displacements are expressed directly via given inside heat source, boundary temperature, and known kernels. So it is not necessary to determine intermediately the inner temperature field or to solve, as in traditional methods, additional BVP.

(2) The most difficult problems in the proposed here method are the problems of deriving the Green's functions $G(x,\xi)$ in heat conduction and the functions of influence for volume dilatation $\Theta^{(k)}(x,\xi)$ in the elasticity theory. Note that for

canonical Cartesian domains, these problems were solved successfully in the handbook [11], where about 190 Green's functions for Laplace's equation and 250 influence functions for volume dilatation $\Theta^{(k)}(x,\xi)$ are presented. At last, computing the integral over volume in (1.4) of the product of functions $G(x,\xi)$ and $\Theta^{(k)}(x,\xi)$ is also solved successfully [see the recommendations of the appendix]. So, for canonical Cartesian domains, the proposed in thermoelasticity method will work successfully. This means that the presented paper opened great possibilities for researchers to derive many new Poisson's type integral formulas, not only for half-space, but also for many other canonical Cartesian domains. These new Poisson's type integral formulas are very useful to solve effectively not only deterministic BVPs of thermoelasticity, but also the stochastic ones [20].

(3) The approach presented in this paper in thermoelasticity for Cartesian canonical domains can be extended onto spherical [21, 22], polar [23], cylindrical, and other canonical domains of any orthogonal systems of coordinates. This extension will be done when the lists of the respective functions $G(x,\xi)$ and $\Theta^{(k)}(x,\xi)$ are completed.

(4) The approach presented in this paper is valid also for other physical phenomena as electroelasticity, magnetoelasticity, and poroelasticity, described by the same BVP as in thermoelasticity.

Appendix

The improper integral in (2.23) was taken using:

(a) the following equalities on the boundary plane of the half-space:

$$[R(x,z)-R_1(x,z)]|_{z=y\equiv(0,y_2,y_3)}=0, \qquad \left[R^{-1}(z,\xi)-R_1^{-1}(z,\xi)\right]\Big|_{z=y\equiv(0,y_2,y_3)}=0, \tag{A.1}$$

(b) the relations

$$\nabla_z^2 R(x,z)=2R^{-1}(x,z), \qquad \nabla_z^2 R_1(x,z)=2R_1^{-1}(x,z), \tag{A.2}$$

(c) the following property of Dirac's function:

$$\int_V f(x)\delta(x,\xi)dV(x)=f(\xi), \tag{A.3}$$

(d) the Green's formula inside the half-space

$$\int_V \left(\varphi\nabla^2 f-f\nabla^2\varphi\right)dV=\int_\Gamma\left[\varphi\left(\frac{\partial f}{\partial n}\right)-f\left(\frac{\partial\varphi}{\partial n}\right)\right]d\Gamma, \tag{A.4}$$

where

$$f=[R(x,z)-R_1(x,z)], \qquad \varphi=(4\pi)^{-1}\left[R^{-1}(z,\xi)-R_1^{-1}(z,\xi)\right]. \tag{A.5}$$

So, using the recommended equations (A.1)–(A.5), we obtain

$$\begin{aligned}
U_k(x,\xi) &= \gamma\int_0^\infty\iint_{-\infty}^\infty G(x,z)\Theta^{(k)}(z,\xi)dz_1dz_2dz_3\\
&= -\gamma\int_0^\infty\iint_{-\infty}^\infty \frac{1}{4\pi}\left(R^{-1}-R_1^{-1}\right)\frac{1}{4\pi(\lambda+2\mu)}\frac{\partial}{\partial\xi_k}\left(R^{-1}-R_1^{-1}\right)dz_1dz_2dz_3\\
&= -\frac{\gamma}{8\pi(\lambda+2\mu)}\frac{\partial}{\partial\xi_k}\int_0^\infty\iint_{-\infty}^\infty \nabla_z^2[R(x,z)-R_1(x,z)]\frac{1}{4\pi}\left[R^{-1}(z,\xi)-R_1^{-1}(z,\xi)\right]dz_1dz_2dz_3\\
&= -\frac{\gamma}{8\pi(\lambda+2\mu)}\frac{\partial}{\partial\xi_k}\int_0^\infty\iint_{-\infty}^\infty [R(x,z)-R_1(x,z)]\frac{1}{4\pi}\nabla_z^2\left[R^{-1}(z,\xi)-R_1^{-1}(z,\xi)\right]dz_1dz_2dz_3\\
&= \frac{\gamma}{8\pi(\lambda+2\mu)}\frac{\partial}{\partial\xi_k}\int_0^\infty\iint_{-\infty}^\infty [R(x,z)-R_1(x,z)]\delta(z-\xi)dz_1dz_2dz_3\\
&= \frac{\gamma}{8\pi(\lambda+2\mu)}\frac{\partial}{\partial\xi_k}[R(x,\xi)-R_1(x,\xi)],
\end{aligned} \tag{A.6}$$

which coincides with the final expression in (2.23).

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References

[1] B. A. Boley and J. H. Weiner, *Theory of Thermal Stresses*, John Wiley & Sons, New York, NY, USA, 1960.
[2] A. D. Kovalenko, *The Basis of Thermoelasticity*, Naukova Dumka, Kiev, Ukraine, 1970.
[3] E. Melan and H. Parkus, *Thermoelastic Stresses Caused by the Stationary Heat Fields*, Fizmatgiz, Moscow, Russia, 1958.
[4] V. M. Mayzel, *The Temperature Problem of the Theory of Elasticity*, AN SSSR, Kiev, Ukraine, 1951.
[5] W. Nowacki, *Thermoelasticity*, Pergamon Press, Oxford, UK; PWN, Warszawa, Poland, 1962.
[6] W. Nowacki, *TheTheory of Elasticity*, Mir, Moscow, Russia, 1975.
[7] V. D. Sheremet, *The Integral Equations and Green's Matrices of the Method of Elements of Influence in the Mechanics of Solids*, Thesis of the doctor of physical, Mathematical Sciences, Technical University of Moldova, Chisinau, Moldova, 1995.
[8] V. D. Şeremet, "The modification of Maizeli's formula in the stationary thermoelasticity," *Bulletin of Academy of Science of Republic of Moldova, Mathematics*, no. 3, pp. 19–22, 1997.
[9] V. D. Şeremet, "Some new influence functions and integral solutions in theory of thermal stresses," in *Proceedings of the IVth International Congress on Thermal Stresses*, pp. 423–427, Osaka, Japan, June 2001.
[10] V. D. Şeremet, "New results in 3-D thermoelasticity," in *Proceedings of the 14th U.S. National Congress of Theoretical and Applied Mechanics, Virginia Tech*, Blacksburg, Va, USA, June 2002.
[11] V. D. Şeremet, *Handbook of Green's Functions and Matrices*, WIT Press, Southampton, UK, 2003.
[12] V. Sheremet, "New formulae for dynamical thermal stresses," *Journal of Thermal Stresses*, vol. 25, no. 2, pp. 123–153, 2002.

[13] V. Sheremet, "Generalization of Green's formulae in thermoelasticity," *Collection: Multiscale Green's Functions for Nanostructures, National Science Digital Library of USA*, NSF, p. 4, 2003.
[14] V. Şeremet, *Influence Elements Method*, Edit. Center, Agrar. Univ. of Moldova, Chisinau, Moldova, 2003.
[15] V. D. Şeremet, I. Vlad, and A. Şeremet, "New integral formulae in thermoelasticity," in *Proceedings of the 16th ASCE Engineering Mechanics Conference (EM '03)*, pp. 1–10, Washington University, Seattle, Wash, USA, July 2003.
[16] V. D. Sheremet, D. Precupan, I. Vlad, and A. V. Sheremet, "The constructing of Green's matrices in cylindrical coordinates," in *Proceedings of the 17th Engineering Mechanics Conference of the American Society of Civil Engineers*, pp. 1–9, University of Delaware Newark, Newark, Del, USA, June 2004.
[17] V. Şeremet, G. Bonnet, and T. Speianu, "New results in construction of the Green's matrices in spherical coordinates," in *Proceedings of the Inaugural International Conference of the Engineering Mechanics Institute (EM '08)*, pp. 1–7, University of Minnesota, USA, May 2008.
[18] V. M. Budac, A. A. Samarskii, and A. N. Tihonov, *Set of Problems on Mathematical Physics*, Gostehnizdat, Moscow, Russia, 1980.
[19] A. D. Kartashov, *Analytic Methods in the Theory of Heat Conductibility of Rigid Bodies*, High School, Moscow, Russia, 1980.
[20] K. Sabelfeld, "Expansion of random boundary excitations for elliptic PDEs," *Monte Carlo Methods and Applications*, vol. 13, no. 5-6, pp. 405–453, 2007.
[21] V. D. Şeremet, I. Vlad, and A. Şeremet, "New influence functions for thermoelastic spherical shells," in *Proceedings of the 5th International Congress on Thermal Stresses (ICTS '03), Virginia Tech.*, pp. 1–4, Blacksburg, Va, USA, June 2003.
[22] V. Şeremet, G. Bonnet, and T. Speianu, "Influence functions and integral formulae for spherical thermoelastic bodies," in *Proceedings of the 22nd International Congress of Theoretical and Applied Mechanics (ICTAM '08)*, pp. 1–2, Adelaide University, Adelaide, Australia, August 2008.
[23] V. Şeremet and G. Bonnet, *Encyclopedia of Domain Green's Functions (Thermo-Magnetoelectrostatics of Solids in Rectangular and Polar Coordinates)*, Edit. Center, Agrar. Univ. of Moldova, Chisinau, Moldova, 2008.