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Research Article

GPS Satellites Orbits: Resonance

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The effects of perturbations due to resonant geopotential harmonics on the semimajor axis of GPS satellites are analyzed. For some GPS satellites, secular perturbations of about 4 m/day can be obtained by numerical integration of the Lagrange planetary equations considering in the disturbing potential the main secular resonant coefficients. Amplitudes for long-period terms due to resonant coefficients are also exhibited for some hypothetical satellites orbiting in the neighborhood of the GPS satellites orbits. The results are important to perform orbital maneuvers of GPS satellites such that they stay in their nominal orbits. Also, for the GPS satellites that are not active, the long-period effects due to the resonance must be taken into account in the surveillance of the orbital evolutions of such debris.

1. Introduction

The period of the orbits of the GPS satellites is about 12 hours, and the main perturbations acting on their orbits are caused by the nonuniform distribution of the Earth's mass, by the lunar and solar gravitational attractions and by the solar radiation pressure. In this paper, it is analyzed just some perturbations due to resonant terms of the geopotential coefficients. The resonance considered here is the 2 : 1 commensurability between the orbital period of the GPS satellites and the period of the Earth's rotation. As it is pointed out by Hugentobler [1] the resonance leads a daily drift rate in semimajor axis of up to 7 m/day.

Lagrange planetary equations, describing the temporal variation of the orbital elements, are used here to analyze the orbital perturbations of the GPS satellites under the influence of resonant coefficients.

2. Disturbing Potential

The geopotential acting on Earth's artificial satellite can be expressed as [2]

$$R = \frac{GM}{a}\sum_{n=2}^{\infty}\sum_{m=0}^{n}\left(\frac{a_e}{a}\right)^n (C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)P_{nm}(\sin\varphi). \tag{2.1}$$

Here GM is the geogravitational constant, r is geocentric distance of the satellite, a_e is the semimajor axis of the adopted Earth's ellipsoid, a is the orbital semimajor axis, C_{nm} and S_{nm} are the spherical harmonic coefficients. P_{nm} denotes the associated Legendre functions, φ, λ are the satellite's geocentric latitude and the longitude, n and m are, respectively, the degree and order of the harmonic coefficients.

The geocentric distance, the latitude, and the longitude of the satellite can be expressed in terms of the orbital elements and (2.1) becomes [2, 3]

$$R = \sum R_{nmpq} = \frac{GM}{a}\sum_{n=2}^{\infty}\sum_{m=0}^{n}\left(\frac{a_e}{a}\right)^n \sum_{p=0}^{n} F_{nmpq}(i)\sum_{q=-\infty}^{\infty} G_{npq}(e)S_{nmpq}(\omega,\Omega,M,\theta_T), \tag{2.2}$$

where $F_{nmpq}(i)$ and $G_{npq}(e)$ are, respectively, functions of the satellite's orbital inclination i and eccentricity e, Ω represents the right ascension of the orbital ascending node, ω is the argument of perigee, M is the mean anomaly, and θ_T is the Greenwich sidereal time. The function $S_{nmpq}(\omega,\Omega,M,\theta_T)$ can be expressed as

$$S_{nmpq}(\omega,\Omega,M,\theta_T) = \left[\begin{pmatrix} C_{nm} \\ S_{nmn} \end{pmatrix}\cos A + \begin{pmatrix} C_{nm} \\ S_{nmn} \end{pmatrix}\sin A\right]_{n-m,\text{odd}}^{n-m,\text{even}}, \tag{2.3}$$

and the argument A is given by

$$A = (n-2p)\omega + (n-2p+q)M + m(\Omega-\theta_T). \tag{2.4}$$

The functions $F_{nmpq}(i)$ and $G_{npq}(e)$ are presented as tables [2] and can be adapted to be used in computers.

3. Resonance in GPS Satellites

Resonance is associated with small divisors. For artificial Earth satellites whose orbital periods are in commensurability with the period of the Earth's rotation resonance can occur when [1, 3, 4]

$$\dot{A} = (n-2p)\dot{\omega} + (n-2p+q)\dot{M} + m(\dot{\Omega}-\dot{\theta}_T) \cong 0, \tag{3.1}$$

where $\dot{\theta}_T$ is the Earth's sidereal rotation.

For the GPS satellites where the commensurability is 2 : 1, there are bounds among the parameters n, m, p, q.

Table 1: Resonant parameters.

Degree n	Order m	p	q
3	2	1	0
4	4	1	0
2	2	1	1
4	2	2	1

Taking into account that $\dot{\omega}, \dot{\Omega} \ll \dot{\theta}_T$, (3.1) can be put as

$$(n - 2p + q)\dot{M} - m\dot{\theta}_T = 0, \tag{3.2}$$

since $\overline{n} = \dot{M} \cong 2\dot{\theta}_T$, we get

$$(n - 2p + q)2 \cong m. \tag{3.3}$$

Considering harmonics of order and degree up to order 4, Table 1, presents some values for the parameters satisfying the resonance condition for the GPS satellites.

Taking into account in the summations the conditions $(n - m)$ even and $(n - m)$ odd and putting [5]

$$\begin{aligned} C_{nm} &= K_{nm} \cos m\lambda_{nm}, \\ S_{nm} &= K_{nm} \sin m\lambda_{nm}, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} K_{nm} &= \sqrt{C_{nm}^2 + S_{nm}^2}, \\ \lambda_{nm} &= \frac{1}{m}\tan^{-1}\left(\frac{S_{nm}}{C_{nm}}\right), \end{aligned} \tag{3.5}$$

equation (2.2) can be written in general form as [1]

$$R_{nmpq} = \frac{GM}{a}\left(\frac{a_e}{a}\right)^n F_{nmp}(i) G_{npq}(e) \begin{pmatrix} \cos \mathrm{A} \\ \sin \mathrm{A} \end{pmatrix}_{n-m,\mathrm{odd}}^{n-m,\mathrm{even}} K_{nm}, \tag{3.6}$$

where

$$A = (n - 2p)\omega + (n - 2p + q)M + m(\Omega - \theta_T - \lambda_{nm}). \tag{3.7}$$

In order to analyze the orbital perturbations of GPS satellites due to the resonant coefficients presented in (Table 1), the Lagrange planetary equations will be used. However,

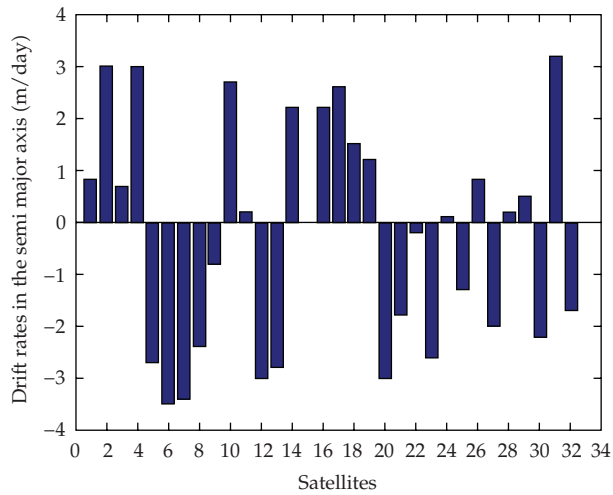

Figure 1: Drift rates in semimajor axis due to the resonant geopotential coefficient 32.

Table 2: Maximum values.

Resonants coefficients	Maximum drift rates m/day
C_{32}, S_{32}	−4.0
C_{44}, S_{44}	1.5
C_{22}, S_{22}	1.5
C_{42}, S_{42}	0.002

here this analysis will be concentrated initially in the secular effects of such perturbation on the orbital semimajor axis. Therefore, we have [2]

$$\frac{da}{dt} = \frac{2}{\overline{n}a}\frac{\partial R_{nmpq}}{\partial M}, \tag{3.8}$$

with $\overline{n}$ being the mean motion of the satellite.

Special care must be taken to compute $\partial R_{nmpq}/\partial M$. In fact, the parameters n, m, p, q given by Table 1 must be considered and each set of them gives a unique solution for (3.8).

A numerical integration of (3.8) was performed for a period of one day using IGS/POE (The International GNSS Service Precise Orbital Ephemeredes) given on June, 07, 2007. Transformations were performed to get the corresponding orbital elements.

Figures 1, 2, 3, and 4 show the daily variations of the semimajor axis a for all GPS satellites.

Figure 1 gives the variation due to the coefficients C_{32}, S_{32}, which are responsible by the greatest variation in the semimajor axis that, in this case, corresponds to the satellite PRN 06. Figures 2, 3, and 4 present, respectively, the daily variations as function of the coefficients $C_{44,}S_{44}, C_{22}, S_{22}, C_{42}, S_{42}$. Figure 5 represents the total daily variation due to the above considered coefficients.

Table 2 shows the maxima and minima values for the daily variation of the semimajor axis according to the resonant coefficients.

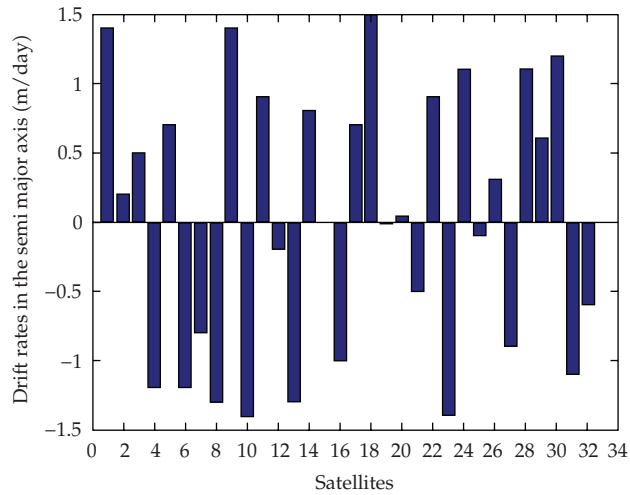

Figure 2: Drift rates in semimajor axis due to the resonant geopotential coefficient 44.

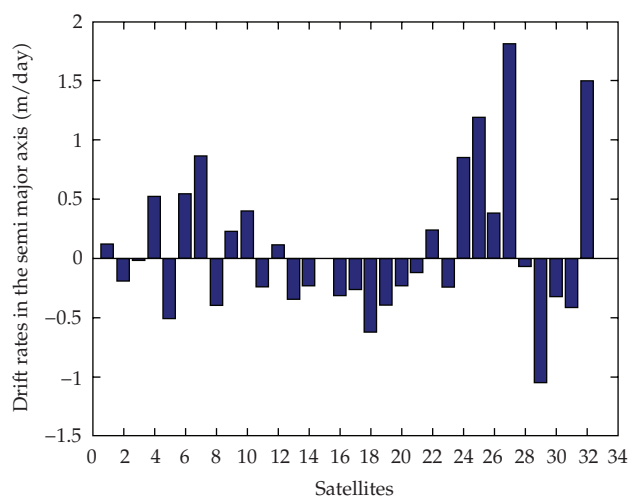

Figure 3: Drift rates in semimajor axis due to the resonant geopotential coefficient 22.

Figures 6 and 7 show the semimajor axis variation for the satellites PRN 02 and PRN 06 for a time interval of 200 days and taking into account the harmonic coefficients C_{32}, S_{32}. During this period and for the considered initial condition, it can be observed variations of about 600 m and 680 m, respectively.

4. Long-Period Perturbations

The effects of the resonance are enhanced when long periods are considered. Table 3 and Table 4, Figures 8, 9, 10, 11, 12, and 13 show some simulations using hypothetical satellites and a particular method to study effects of resonance on the orbits of artificial satellites [6–9].

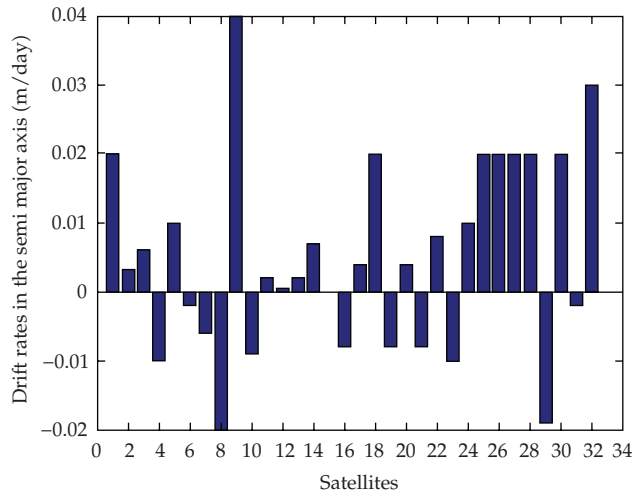

Figure 4: Drift rates in semimajor axis due to the resonant geopotential coefficient 42.

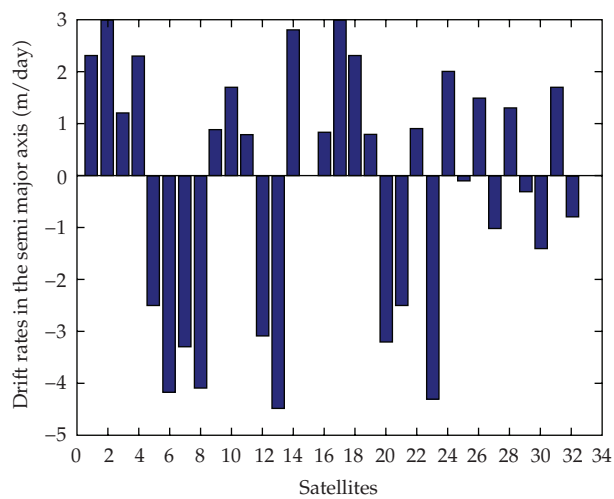

Figure 5: Drift rates in semimajor axis due to all resonant geopotential coefficients, Table 1.

Table 3 contains the amplitude and period of the variations of orbital elements for hypothetical satellites considering low eccentricity, small and high inclination, and the influence of the harmonics J_{20} and J_{22}

Figures 8 and 9 represent, respectively, the temporary variation of the semimajor axis and of the eccentricity of artificial satellites of the GPS type in the neighborhood of the 2 : 1 resonance region when the harmonics J_{20} and J_{22} are considered. By Figure 10, assuming several values for the semimajor axes in the neighborhood of the 2 : 1 resonance, it can be observed distinct behavior for their temporary variations.

Figure 11 represents the temporary variation for the semimajor axis in the neighborhood of the resonance 2 : 1 considering the influence of the harmonics J_{20} and J_{32} for a satellite of the GPS type with inclination of about 55°.

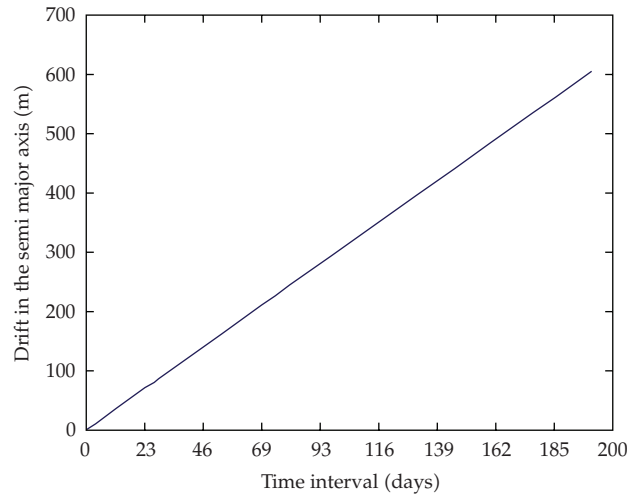

Figure 6: Semimajor axis variation (PRN 02) due to the resonant geopotential coefficient 32, for a time interval of 200 days.

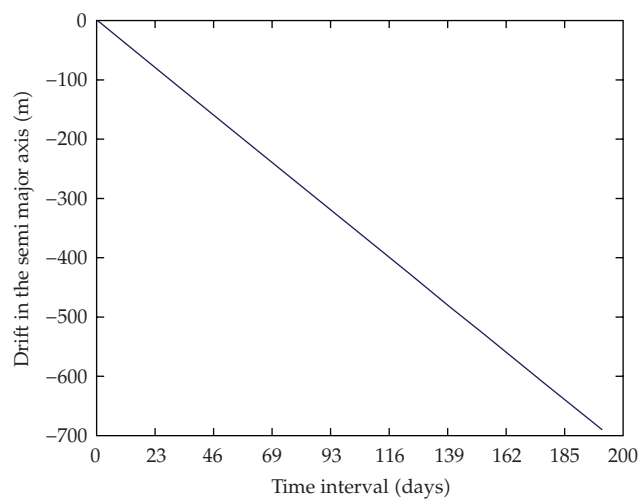

Figure 7: Semimajor axis variation (PRN 06) due to the resonant geopotential coefficient 32, for a time interval of 200 days.

Table 4 gives the influence of the resonance due to the harmonics J_{20} and J_{32} considering different inclinations, including that of the GPS type satellite. It can be observed that the amplitudes of the variations are smaller when compared with those when the harmonics J_{20} and J_{22} are taken into account. The more the satellites approach the region that was defined as a resonant region, the more the variations increase.

Figure 12 represents the temporary variation of the semimajor axis of a satellite of the GPS type considering resonance due to the harmonics J_{20}, J_{22}, and J_{32}.

Table 3: Amplitude and period of perturbations due to the 2 : 1 resonance: $J_2 + J_{22}$.

Orbital elements			Amplitude			Period
a_o = 26561.770 km	e	i	$\Delta a_{\max}$	$\Delta e_{\max}$	$\Delta i_{\max}(°)$	t (days)
$a_o + 0.430$	0.01	4°	12 km	0.014	8.7×10^{-4}	2500
$a_o + 0.718$	0.01	4°	12.5 km	0.0145	8.6×10^{-4}	7000
$a_o + 0.290$	0.01	4°	4.5 km	0.0065	3.6×10^{-4}	2000
$a_o + 0.500$	0.01	4°	3 km	0.0045	2.8×10^{-4}	1000
$a_o - 2.5$	0.01	4°	6.5 km	0.0085	4×10^{-4}	1700
$a_o - 5$	0.01	4°	1.4 km	0.0023	1.3×10^{-6}	500
a_o	0.05	55°	1.8 km	0.005	2×10^{-3}	3200
$a_o + 0.718$	0.005	55°	2.75 km	0.006	2.8×10^{-3}	3900
$a_o - 1.770$	0.005	55°	1.2 km	0.0045	1.4×10^{-3}	2000
$a_o + 5.129$	0.005	55°	900 m	0.0033	10^{-3}	1500
$a_o + 3.729$	0.005	55°	1.8 km	0.0055	1.9×10^{-3}	2500
a_o	0.05	63.4°	7.5 km	0.003	10.8×10^{-3}	1000
$a_o - 2$	0.05	63.4°	4 km	0.0015	5.7×10^{-3}	1000
a_o	0.05	87°	7 km	0.0037	13.9×10^{-3}	2000
$a_o - 2$	0.05	87°	3.5 km	0.002	10.3×10^{-3}	1900

Table 4: Amplitude and period of perturbations due to the 2 : 1 resonance: $J_2 + J_{32}$.

Orbital elements			Amplitude			Period
a_o = 26561.770 km	e	i	$\Delta a_{\text{máx}}$	$\Delta e_{\text{máx}}$	$\Delta i_{\text{máx}}(°)$	t (days)
$a_o - 3.77$	0.01	4°	120 m	0.0004	1.78×10^{-3}	500
$a_o + 0.718$	0.01	4°	245 m	0.0009	3.9×10^{-3}	1000
$a_o + 0.929$	0.01	4°	110 m	0.00045	1.78×10^{-3}	500
$a_o + 5$	0.01	4°	100 m	0.0003	1.15×10^{-3}	250
$a_o - 2.5$	0.005	4°	100 m	0.0009	4.6×10^{-5}	500
$a_o + 0.429$	0.005	4°	700 m	0.006	2.4×10^{-4}	1500
$a_o + 0.929$	0.005	55°	1.9 km	0.0085	6.8×10^{-4}	16000
$a_o + 1.73$	0.005	55°	250 m	0.0017	1.13×10^{-3}	16000
$a_o + 5$	0.005	55°	30 m	0.00055	2.3×10^{-4}	1500
$a_o - 3.77$	0.005	55°	47 m	0.00035	3.43×10^{-5}	500
$a_o - 0.429$	0.005	87°	110 m	0.00085	2.29×10^{-4}	1800
$a_o - 2.23$	0.005	87°	45 m	0.00035	8.59×10^{-5}	800
$a_o - 3.77$	0.005	87°	20 m	0.00014	4×10^{-5}	500

Figure 13 represents the time variation of the orbital semimajor axis of a satellite with eccentricity about 0.01 and inclination about 4° orbiting in a region near the 2 : 1 resonance. It is remarkable the oscillation of the semimajor axis in the region between by a = 26560.0 km and a = 26562.48 km. For instance, a variation of about 10 m in the initial semimajor axis (a = 26562.48 km) brings up a variation of about more than 10 km in its amplitude in a period of about 2000 days.

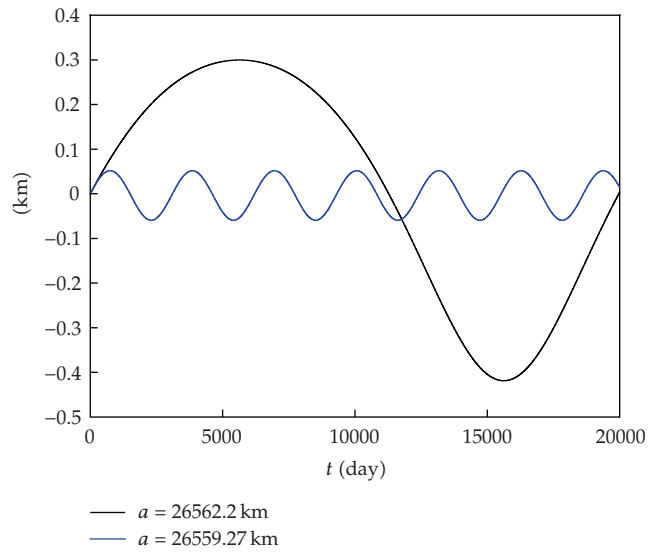

Figure 8: Δa versus time, $e = 0.005$, $i = 55°$.

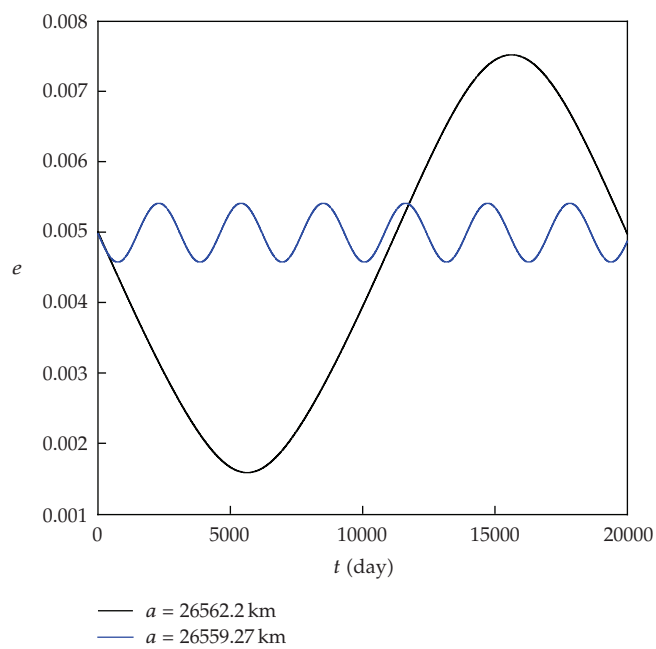

Figure 9: Eccentricity versus time, $e = 0.005$.

5. Conclusions

From the obtained results it can be seen that for the GPS satellites no negligible perturbations are provoked by resonant tesseral coefficients. The daily variation of the semimajor axis due to the C_{32}, S_{32} is less than 4 m/day and those values found for a time interval of 200 days

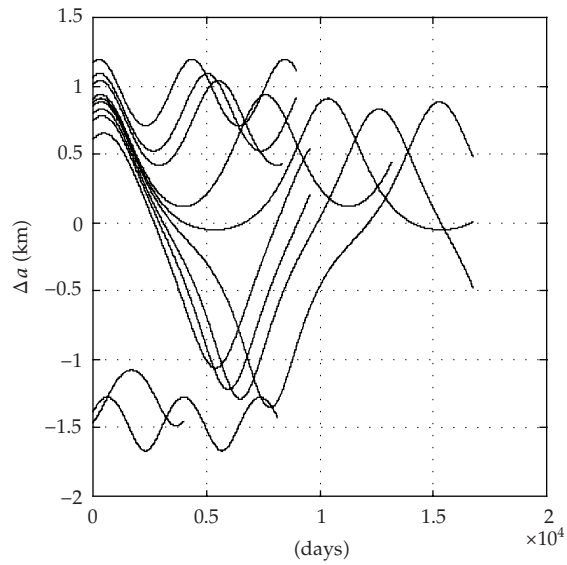

Figure 10: Δa versus time, $e = 0.005$, $i = 55°$, $a_0 = 26561.18$ km.

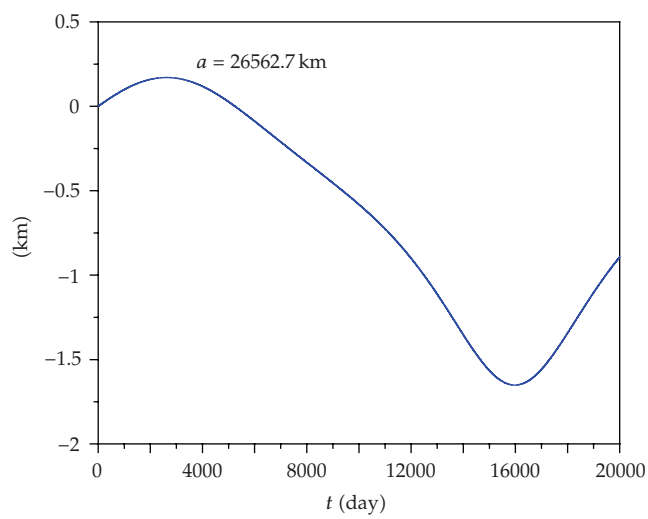

Figure 11: Δa versus time, $e = 0.005$, $i = 55°$, considering the harmonics J_{20} and J_{32}.

(~680 m) are compatible with the results presented by specialized literature (~7 m/day and ~1200 m) about this subject [1].

Taking into account the total effects of daily resonant perturbations, it can be observed that for some GPS satellites these effects are enhanced and for another satellites are attenuated but all of these effects are smaller than the amplitudes mentioned above. It was shown also that the effects of the resonance are very important for the analysis of long-period behavior of the GPS satellites orbits.

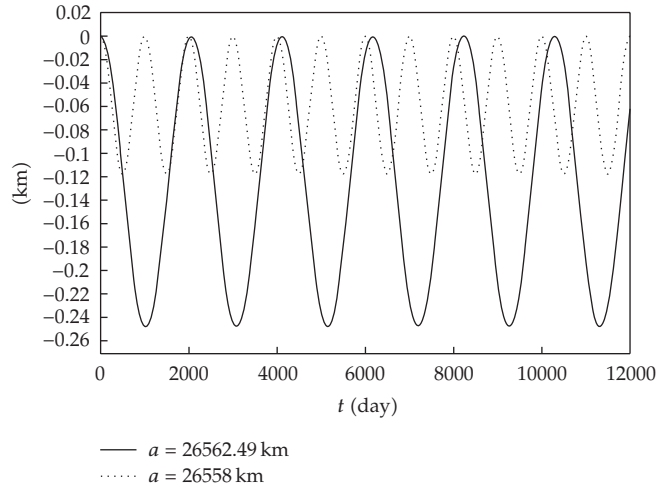

Figure 12: Time variation of semimajor axis considering the harmonics J_{20}, J_{22}, and $J_{3,2}$.

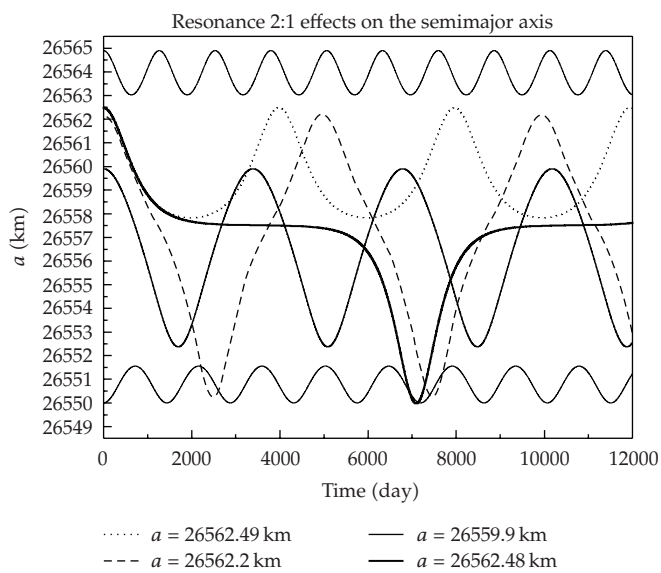

Figure 13: Time variation of the semimajor axis considering the harmonics J_{20} and J_{22} for $e = 0.005$, $i = 4°$.

An important aspect to be considered is the necessity to perform orbital maneuvers of GPS satellites in such way that they stay in their nominal orbits. Also, for the GPS satellites that are not active, the long-term effects due to the resonance must be taken into account in the surveillance of the orbital evolutions of such debris.

Acknowledgment

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