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*Research Article*

# A Note on Finite Quadrature Rules with a Kind of Freud Weight Function

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We introduce a finite class of weighted quadrature rules with the weight function $|x|^{-2a}\exp(-1/x^2)$ on $(-\infty,\infty)$ as $\int_{-\infty}^{\infty}|x|^{-2a}\exp(-1/x^2)f(x)dx = \sum_{i=1}^{n} w_i f(x_i) + R_n[f]$, where $x_i$ are the zeros of polynomials orthogonal with respect to the introduced weight function, $w_i$ are the corresponding coefficients, and $R_n[f]$ is the error value. We show that the above formula is valid only for the finite values of $n$. In other words, the condition $a \geq \{\max n\} + 1/2$ must always be satisfied in order that one can apply the above quadrature rule. In this sense, some numerical and analytic examples are also given and compared.



## 1. Introduction

Recently in [1] the differential equation

$$x^2\left(px^2+q\right)\Phi_n''(x) + x\left(rx^2+s\right)\Phi_n'(x) - \left(n\left(r+(n-1)p\right)x^2 + \frac{\left(1-(-1)^n\right)s}{2}\right)\Phi_n(x) = 0 \tag{1.1}$$

is introduced, and its explicit solution is shown by

$$\begin{aligned} &S_n\left(\begin{array}{cc} r & s \\ p & q \end{array}\middle| x\right) \\ &\quad = \sum_{k=0}^{[n/2]} \binom{\left[\frac{n}{2}\right]}{k} \left(\prod_{i=0}^{[n/2]-(k+1)} \frac{\left(2i+(-1)^{n+1}+2[n/2]\right)p+r}{\left(2i+(-1)^{n+1}+2\right)q+s}\right) x^{n-2k}. \end{aligned} \tag{1.2}$$

It is also called the generic equation of classical symmetric orthogonal polynomials [1, 2]. If this equation is written in a self-adjoint form then the first-order equation

$$x\frac{d}{dx}\left(\left(px^2+q\right)W(x)\right)=\left(rx^2+s\right)W(x) \tag{1.3}$$

is derived. The solution of (1.3) is known as an analogue of Pearson distributions family and can be indicated as

$$W\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right)=\exp\left(\int\frac{(r-2p)x^2+s}{x(px^2+q)}dx\right). \tag{1.4}$$

In general, there are four main subclasses of distributions family (1.4) (as subsolutions of (1.3)) whose explicit probability density functions are, respectively,

$$K_1W\left(\begin{matrix} -2a-2b-2, & 2a \\ -1, & 1 \end{matrix}\middle| x\right)=\frac{\Gamma(a+b+3/2)}{\Gamma(a+1/2)\Gamma(b+1)}x^{2a}\left(1-x^2\right)^b, \quad -1\le x\le 1,\quad a+\frac{1}{2}>0,\quad b+1>0, \tag{1.5}$$

$$K_2W\left(\begin{matrix} -2, & 2a \\ 0, & 1 \end{matrix}\middle| x\right)=\frac{1}{\Gamma(a+1/2)}x^{2a}\exp\left(-x^2\right),\quad -\infty<x<\infty,\ a+\frac{1}{2}>0, \tag{1.6}$$

$$K_3W\left(\begin{matrix} -2a-2b+2, & -2a \\ 1, & 1 \end{matrix}\middle| x\right)=\frac{\Gamma(b)}{\Gamma(b+a-1/2)\Gamma(-a+1/2)}\frac{x^{-2a}}{(1+x^2)^b}, \quad -\infty<x<\infty,\quad b>0,\quad a<\frac{1}{2},\quad b+a>\frac{1}{2}, \tag{1.7}$$

$$K_4W\left(\begin{matrix} -2a+2, & 2 \\ 1, & 0 \end{matrix}\middle| x\right)=\frac{1}{\Gamma(a-1/2)}x^{-2a}\exp\left(-\frac{1}{x^2}\right),\quad -\infty<x<\infty,\ a>\frac{1}{2}. \tag{1.8}$$

The values $K_i; i=1,2,3,4$ play the normalizing constant role in these distributions. Moreover, the value of distribution vanishes at $x=0$ in each four cases, that is, $W(0;p,q,r,s)=0$ for $s\neq 0$. Hence, (1.4) is called in [1] "The dual symmetric distributions family."

As a special case of $W(x;p,q,r,s)$, let us choose the values $p=1$, $q=0$, $r=-2a+2$, and $s=2$ corresponding to distribution (1.8) here and replace them in (1.1) to get

$$x^4\Phi_n''(x)+2x\left((1-a)x^2+1\right)\Phi_n'(x)-\left(n(n+1-2a)x^2+1-(-1)^n\right)\Phi_n(x)=0. \tag{1.9}$$

If (1.9) is solved, the polynomial solution of monic type

$$\overline{S}_n\left(\begin{matrix} -2a+2 & 2 \\ 1 & 0 \end{matrix}\middle| x\right) = \prod_{i=0}^{[n/2]-1} \frac{2}{2i+2[n/2]+(-1)^{n+1}+2-2a} \times \sum_{k=0}^{[n/2]} \binom{\left[\frac{n}{2}\right]}{k} \left(\prod_{i=0}^{[n/2]-(k+1)} \frac{2i+2[n/2]+(-1)^{n+1}+2-2a}{2}\right) x^{n-2k} \tag{1.10}$$

is obtained. According to [1], these polynomials are finitely orthogonal with respect to a special kind of Freud weight function, that is, $x^{-2a}\exp(-1/x^2)$, on the real line $(-\infty,\infty)$ if and only if $a \geq \{\max n\} + 1/2$; see also [3, 4]. In other words, we have

$$\begin{aligned} &\int_{-\infty}^{\infty} |x|^{-2a} \exp\left(-\frac{1}{x^2}\right) \overline{S}_n\left(\begin{matrix} -2a+2 & 2 \\ 1 & 0 \end{matrix}\middle| x\right) \overline{S}_m\left(\begin{matrix} -2a+2 & 2 \\ 1 & 0 \end{matrix}\middle| x\right) dx \\ &\quad = \left(\prod_{i=1}^{n} \frac{2(-1)^i(i-a)+2a}{(2i-2a+1)(2i-2a-1)}\right) \Gamma\left(a-\frac{1}{2}\right) \delta_{n,m}, \end{aligned} \tag{1.11}$$

if and only if $m,n = 0,1,2,\ldots$, $N = \max\{m,n\} \leq a - 1/2$, $(-1)^{2a} = 1$ and

$$\delta_{n,m} = \begin{cases} 0, & \text{if } n \neq m, \\ 1, & \text{if } n = m. \end{cases} \tag{1.12}$$

Furthermore, the polynomials (1.10) also satisfy a three-term recurrence relation as

$$\overline{S}_{n+1}(x) = x\,\overline{S}_n(x) - \frac{2(-1)^n(n-a)+2a}{(2n-2a+1)(2n-2a-1)}\,\overline{S}_{n-1}(x), \quad \overline{S}_0(x) = 1,\ \overline{S}_1(x) = x,\ n \in \mathbb{N}. \tag{1.13}$$

But the polynomials $\overline{S}_n(x;1,0,-2a+2,2)$ are suitable tool to finitely approximate arbitrary functions, which satisfy the Dirichlet conditions (see, e.g., [5]). For example, suppose that $N = \max\{m,n\} = 3$ and $a > 7/2$ in (1.10). Then, the function $f(x)$ can finitely be approximated as

$$\begin{aligned} f(x) \cong\ & C_0\overline{S}_0(x;1,0,-2a+2,2) + C_1\overline{S}_1(x;1,0,-2a+2,2) \\ & + C_2\overline{S}_2(x;1,0,-2a+2,2) + C_3\overline{S}_3(x;1,0,-2a+2,2), \end{aligned} \tag{1.14}$$

where

$$C_m = \int_{-\infty}^{\infty} \frac{|x|^{-2a}\exp(-1/x^2)\overline{S}_m\left(\begin{matrix} -2a+2 & 2 \\ 1 & 0 \end{matrix}\middle| x\right) f(x)dx}{\left(\prod_{i=1}^{m}\left(\left(2\ (-1)^i(i-a)+2a\right)/(2i-2a+1)(2i-2a-1)\right)\Gamma(a-1/2)\right)}, \tag{1.15}$$

for $m = 0,1,2,3$.

Clearly (1.14) is valid only when the general function $x^m|x|^{-2a}\exp(-1/x^2)\ f(x)$ in (1.15) is integrable for any $m = 0, 1, 2, 3$. This means that the finite set $\{\overline{S}_i(x; 1, 0, -2a + 2, 2)\}_{i=0}^{3}$ is a basis space for all polynomials of degree at most three. So if $f(x) = a_3x^3+a_2x^2+a_1x+a_0$, the approximation (1.14) is exact. By noting this, here is a good position to express an application of the mentioned polynomials in weighted quadrature rules [6, 7] by a straightforward example. Let us consider a two-point approximation as

$$\int_{-\infty}^{\infty} |x|^{-2a} \exp\left(-\frac{1}{x^2}\right) f(x)dx \cong w_1 f(x_1) + w_2 f(x_2), \tag{1.16}$$

provided that $a > 5/2$. According to the described themes, (1.16) must be exact for all elements of the basis $f(x) = \{x^3, x^2, x, 1\}$ if and only if $x_1, x_2$ are two roots of $\overline{S}_2(x; 1, 0, -2a + 2, 2)$. For instance, if $a = 3 > 5/2$ then (1.16) should be changed to

$$\int_{-\infty}^{\infty} x^{-6} \exp\left(-\frac{1}{x^2}\right) f(x)dx \cong w_1 f\left(\sqrt{\frac{2}{3}}\right) + w_2 f\left(-\sqrt{\frac{2}{3}}\right), \tag{1.17}$$

in which $\sqrt{2/3}$ and $-\sqrt{2/3}$ are zeros of $\overline{S}_2(x; 1, 0, -4, 2)$, and $w_1, w_2$ are computed by solving the linear system

$$w_1 + w_2 = \int_{-\infty}^{\infty} x^{-6} \exp\left(-\frac{1}{x^2}\right) dx = \frac{3}{4}\sqrt{\pi}, \qquad \sqrt{\frac{2}{3}}\ (w_1 - w_2) = \int_{-\infty}^{\infty} x^{-5} \exp\left(-\frac{1}{x^2}\right) dx = 0. \tag{1.18}$$

Hence, after solving (1.18) the final form of (1.16) is known as

$$\int_{-\infty}^{\infty} x^{-6} \exp\left(-\frac{1}{x^2}\right) f(x)dx \cong \frac{3}{8}\sqrt{\pi}\left(f\left(\sqrt{\frac{2}{3}}\right) + f\left(-\sqrt{\frac{2}{3}}\right)\right). \tag{1.19}$$

This approximation is exact for all arbitrary polynomials of degree at most 3.

## 2. Application of Polynomials (1.10) in Weighted Quadrature Rules: General Case

As we know, the general form of weighted quadrature rules is given by

$$\int_{\alpha}^{\beta} w(x) f(x)dx = \sum_{i=1}^{n} w_i f(x_i) + R_n[f], \tag{2.1}$$

in which the weights $\{w_i\}_{i=1}^n$ and the nodes $\{x_i\}_{i=1}^n$ are unknown values, $w(x)$ is a positive function, and $[\alpha,\beta]$ is an arbitrary interval; see, for example, [6, 7]. Moreover the residue $R_n[f]$ is determined (see, e.g., [7]) by

$$R_n[f] = \frac{f^{(2n)}(\xi)}{(2n)!}\int_\alpha^\beta w(x)\prod_{i=1}^n (x-x_i)^2 dx, \quad \alpha < \xi < \beta. \tag{2.2}$$

It can be proved in (2.1) that $R_n[f] = 0$ for any linear combination of the sequence $\{1, x, x^2, \ldots, x^{2n-1}\}$ if and only if $\{x_i\}_{i=1}^n$ are the roots of orthogonal polynomials of degree $n$ with respect to the weight function $w(x)$ on the interval $[\alpha,\beta]$. For more details, see [6]. Also, it is proved that to derive $\{w_i\}_{i=1}^n$ in (2.1), it is not required to solve the following linear system of order $n \times n$:

$$\sum_{i=1}^n w_i x_i^j = \int_\alpha^\beta w(x)x^j dx \quad \text{for } j = 0, 1, \ldots, 2n-1, \tag{2.3}$$

rather, one can directly use the relation

$$\frac{1}{w_i} = \widehat{P}_0^2(x_i) + \widehat{P}_1^2(x_i) + \cdots + \widehat{P}_{n-1}^2(x_i) \quad \text{for } i = 1, 2, \ldots, n, \tag{2.4}$$

where $\widehat{P}_i(x)$ are orthonormal polynomials of $P_i(x)$ defined as

$$\widehat{P}_i(x) = \left(\int_\alpha^\beta w(x)P_i^2(x)dx\right)^{-1/2} P_i(x). \tag{2.5}$$

In this way, as it is shown in [8, 9], $\widehat{P}_i(x)$ satisfies a particular type of three-term recurrence as

$$x\widehat{P}_{n-1}(x) = \alpha_n \widehat{P}_n(x) + \beta_n \widehat{P}_{n-1}(x) + \alpha_{n-1}\widehat{P}_{n-2}(x). \tag{2.6}$$

Now, by noting these comments and the fact that the symmetric polynomials $\overline{S}_n(x; 1, 0, -2a+2, 2)$ are finitely orthogonal with respect to the weight function $W(x,a) = |x|^{-2a}\exp(-1/x^2)$ on the real line, we can define a finite class of quadrature rules as

$$\int_{-\infty}^{\infty} |x|^{-2a}\exp\left(-\frac{1}{x^2}\right) f(x)dx = \sum_{j=1}^n w_j f(x_j) + R_n[f], \tag{2.7}$$

in which $x_j$ are the roots of $\overline{S}_n(x; 1, 0, -2a+2, 2)$ and $w_j$ are computed by

$$\frac{1}{w_j} = \sum_{i=0}^{n-1}\left(\overline{S}_i^*(1, 0, -2a+2, 2; x_j)\right)^2, \quad \text{for } j = 0, 1, 2, \ldots, n. \tag{2.8}$$

Moreover, for the residue value we have

$$R_n[f] = \frac{f^{(2n)}(\xi)}{(2n)!}\int_{-\infty}^{\infty}|x|^{-2a}\exp\left(-\frac{1}{x^2}\right)\prod_{j=1}^{n}(x-x_j)^2dx, \quad \xi\in\mathbb{R}. \tag{2.9}$$

### 2.1. An Important Remark

It is important to note that by applying the change of variable $1/x^2 = t$ in the left-hand side of (2.7) the orthogonality interval $(-\infty,\infty)$ changes to $[0,\infty)$ and subsequently

$$\int_{-\infty}^{\infty}|x|^{-2a}\exp\left(-\frac{1}{x^2}\right)f(x)dx = \int_0^{\infty}t^{a-3/2}e^{-t}f\left(\frac{1}{\sqrt{t}}\right)dt. \tag{2.10}$$

As it is observed, the right-hand integral of (2.10) contains the well-known Laguerre weight function $x^u e^{-x}$ for $u = a-3/2$. Hence, one can use Gauss-Laguerre quadrature rules [8, 9] with the special parameter $u = a-3/2$. This process changes (2.7) in the form

$$\int_{-\infty}^{\infty}|x|^{-2a}\exp\left(-\frac{1}{x^2}\right)f(x)dx = \sum_{j=1}^{n}w_j^{(a-3/2)}f\left(\frac{1}{\sqrt{x_j^{(a-3/2)}}}\right) + R_n\left[f\left(\frac{1}{\sqrt{x}}\right)\right], \tag{2.11}$$

in which $x_j^{(a-3/2)}$ are the zeros of Laguerre polynomials $L_n^{(a-3/2)}(x)$. But, there is a large disadvantage for formula (2.11). According to (2.2) or (2.9), the residue of integration rules generally depends on $f^{(2n)}(\xi)$; $\alpha<\xi<\beta$. Thus, by noting (2.11) we should have

$$\frac{d^{2n}f(1/\sqrt{x})}{dx^{2n}} = \sum_{i=0}^{2n}\phi_i(x)f^{(i)}\left(\frac{1}{\sqrt{x}}\right), \tag{2.12}$$

where $\phi_i(x)$ are real functions to be computed and $f^{(i)}$, $i = 0,1,2,\ldots,2n$, are the successive derivatives of function $f(x)$.

As we observe in (2.12), $f(x)$ cannot be in the form of an arbitrary polynomial function in order that the right-hand side of (2.12) is equal to zero. In other words, (2.11) is not exact for the basis space $f(x) = x^j$, $j = 0,1,2,\ldots,2n-1$. This is the main disadvantage of using (2.11), as the examples of next section support this claim.

## 3. Examples

*Example 3.1.* Since a 2-point formula was presented in (1.19), in this example we consider a 3-point integration formula. For this purpose, we should first note that according to (1.11) the condition $a > 7/2$ is necessary. Hence, let us, for instance, assume that $a = 4$. After some computations the related quadrature rule would take the form

$$\int_{-\infty}^{\infty}x^{-8}\exp\left(-\frac{1}{x^2}\right)f(x)dx = \frac{3}{16}\sqrt{\pi}\left(3f\left(\sqrt{\frac{2}{3}}\right)+4f(0)+3f\left(-\sqrt{\frac{2}{3}}\right)\right)+R_3[f], \tag{3.1}$$

where

$$R_3[f] = \frac{f^{(6)}(\xi)}{6!}\int_{-\infty}^{\infty} x^{-8}\exp\left(-\frac{1}{x^2}\right)\left(\overline{S}_3\left(\begin{array}{cc} -6 & 2 \\ 1 & 0 \end{array}\middle| x\right)\right)^2 dx$$
$$= \frac{\sqrt{\pi}}{1080} f^{(6)}(\xi), \quad \xi \in \mathbf{R}, \tag{3.2}$$

and $x_1 = \sqrt{2/3}$, $x_2 = 0$, and $x_3 = -\sqrt{2/3}$ are the roots of $\overline{S}_3(x;1,0,-6,2) = x^3 - (2/3)x$. Moreover, $w_1, w_2, w_3$ can be computed by

$$\frac{1}{w_j} = \sum_{i=0}^{2}\left(\overline{S}_i^*(x_j;1,0,-6,2)\right)^2, \quad j = 1,2,3, \tag{3.3}$$

in which

$$\overline{S}_i^*(x_j;1,0,-6,2) = \frac{\overline{S}_i(x_j;1,0,-6,2)}{\left\langle \overline{S}_i(x_j;1,0,-6,2), \overline{S}_i(x_j;1,0,-6,2)\right\rangle^{1/2}}. \tag{3.4}$$

*Example 3.2.* To have a 4-point formula, we should again note that $a > 9/2$ is a necessary condition. In this sense, if, for example, $a = 5$ then we eventually get

$$\int_{-\infty}^{\infty} x^{-10}\exp\left(-\frac{1}{x^2}\right) f(x)dx$$
$$= \frac{15}{64}\sqrt{\pi}\left(7 - 2\sqrt{10}\right)\left(f\left(\sqrt{\frac{10+2\sqrt{10}}{15}}\right) + f\left(-\sqrt{\frac{10+2\sqrt{10}}{15}}\right)\right)$$
$$+ \frac{15}{64}\sqrt{\pi}\left(7 + 2\sqrt{10}\right)\left(f\left(\sqrt{\frac{10-2\sqrt{10}}{15}}\right) + f\left(-\sqrt{\frac{10-2\sqrt{10}}{15}}\right)\right) + R_4[f], \tag{3.5}$$

where

$$R_4[f] = \frac{f^{(8)}(\xi)}{8!}\int_{-\infty}^{\infty} x^{-10}\exp\left(-\frac{1}{x^2}\right)\left(\overline{S}_4\left(\begin{array}{cc} -8 & 2 \\ 1 & 0 \end{array}\middle| x\right)\right)^2 dx = \frac{\sqrt{\pi}}{75600} f^{(8)}(\xi), \quad \xi \in \mathbf{R}. \tag{3.6}$$

Clearly this formula is exact for the basis elements $f(x) = x^j$, $j = 0,1,2,\ldots,7$, and the nodes of quadrature (3.5) are the roots of $\overline{S}_4(x;1,0,-8,2) = x^4 - (4/3)x^2 + 4/15$.

## 4. Numerical results

In this section, some numerical examples are given and compared. The numerical results related to the 2-point formula (1.19) are presented in Table 1, the results related to 3-point

**Table 1:** $\int_{-\infty}^{+\infty} x^{-6}\exp(-1/x^2)f(x)dx$.

| $f(x)$ | Approx. value (2-point) | Exact value | Error |
|---|---|---|---|
| $\cos x$ | 0.9103037512 | 0.9382539141 | 0.0279501629 |
| $\exp(-2/x^2)$ | 0.0661839608 | 0.0852772257 | 0.0190932649 |
| $\exp(-\cos x)$ | 0.6702559297 | 0.6812645398 | 0.0110086101 |

**Table 2:** $\int_{-\infty}^{+\infty} x^{-8}\exp(-1/x^2)f(x)dx$.

| $f(x)$ | Approx. value (3-point) | Exact value | Error |
|---|---|---|---|
| $\exp(-\cos x)$ | 1.494420894 | 1.492841821 | 0.001579073 |
| $\sqrt{1+\sin x^2}$ | 3.866024228 | 3.866700560 | 0.000676332 |
| $\sqrt{1+\cos x^2}$ | 4.544708979 | 4.561266761 | 0.016557782 |

**Table 3:** $\int_{-\infty}^{+\infty} x^{-10}\exp(-1/x^2)f(x)dx$.

| $f(x)$ | Approx. value (4-point) | Exact value | Error |
|---|---|---|---|
| $\sqrt{1+\cos x^2}$ | 16.21776936 | 16.21978539 | 0.002016030 |
| $(1+x^2)^{-1/2}$ | 10.30987753 | 10.31704740 | 0.007116987 |
| $\exp(-x^2-2)$ | 1.198219038 | 1.199125136 | 0.000906098 |

formula (3.1) are given in Table 2, and finally the results related to 4-point formula (3.5) are presented in Table 3.

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