Research Article

Periodic and Solitary Wave Solutions to the Fornberg-Whitham Equation

Jiangbo Zhou and Lixin Tian
Nonlinear Scientific Research Center, Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu 212013, China
Correspondence should be addressed to Jiangbo Zhou, zhoujiangbo@yahoo.cn
Received 27 November 2008; Revised 4 February 2009; Accepted 6 March 2009
Recommended by Katica R. Stevanovic Hedrih

New travelling wave solutions to the Fornberg-Whitham equation
\[ u_t - u_{xxt} + u_x + uu_x = uu_{xxx} + 3u_xu_{xx} \]
are investigated. They are characterized by two parameters. The expressions for the periodic and solitary wave solutions are obtained.

Copyright © 2009 J. Zhou and L. Tian. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

Recently, Ivanov [1] investigated the integrability of a class of nonlinear dispersive wave equations:

\[ u_t - u_{xxt} + \partial_x\left( \kappa u + \alpha u^2 + \beta u^3 \right) = \nu u_x u_{xx} + \gamma uu_{xxx}, \quad (1.1) \]

where and \( \alpha, \beta, \gamma, \kappa, \nu \) are real constants.

The important cases of (1.1) are as follows. The hyperelastic-rod wave equation

\[ u_t - u_{xxt} + 3uu_x = \gamma (2u_x u_{xx} + uu_{xxx}) \quad (1.2) \]

has been recently studied as a model, describing nonlinear dispersive waves in cylindrical compressible hyperelastic rods [2–7]. The physical parameters of various compressible materials put \( \gamma \) in the range from \(-29.4760\) to \(3.4174\) [2, 4].

The Camassa-Holm equation

\[ u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \quad (1.3) \]
A short conclusion is given in Section 4. The Degasperis-Procesi equation

\[ u_t - u_{xxt} + 4uu_x = 3u_u u_{xx} + uu_{xxx}, \]  

(1.4)

models nonlinear shallow water dynamics. It is completely integrable [1] and has a variety of travelling wave solutions including solitary wave solutions, peakon solutions and shock waves solutions [19–26].

The Fornberg-Whitham equation

\[ u_t - u_{xxt} + uu_x = uu_{xxx} + 3u_u u_{xx} \]  

(1.5)

appeared in the study qualitative behaviors of wave-breaking [27]. It admits a wave of greatest height, as a peaked limiting form of the travelling wave solution [28], \( u(x, t) = A \exp(-1/2|x - 4/3t|) \), where \( A \) is an arbitrary constant. It is not completely integrable [1].

The regularized long-wave or BBM equation

\[ u_t - u_{xxt} + uu_x + uu_{xx} = 0 \]  

(1.6)

and the modified BBM equation

\[ u_t - u_{xxt} + uu_x + 3u^2u_x = 0 \]  

(1.7)

have also been investigated by many authors [29–37].

Many efforts have been devoted to study (1.2)–(1.4), (1.6), and (1.7), however, little attention was paid to study (1.5). In [38], we constructed two types of bounded travelling wave solutions \( u(\xi) (\xi = x - ct) \) to (1.5), which are defined on semifinal bounded domains and called kink-like and antikink-like wave solutions. In this paper, we continue to study the travelling wave solutions to (1.5). Following Vakhnenko and Parkes’s strategy in [39], we obtain some periodic and solitary wave solutions \( u(\xi) \) to (1.5) which are defined on \( (-\infty, +\infty) \). The travelling wave solutions obtained in this paper are obviously different from those obtained in our previous work [38]. To the best of our knowledge, these solutions are new for (1.5). Our work may help people to know deeply the described physical process and possible applications of the Fornberg-Whitham equation.

The remainder of the paper is organized as follows. In Section 2, for completeness and readability, we repeat Appendix A in [39], which discusses the solutions to a first-order ordinary differential equation. In Section 3, we show that, for travelling wave solutions, (1.5) may be reduced to a first-order ordinary differential equation involving two arbitrary integration constants \( a \) and \( b \). We show that there are four distinct periodic solutions corresponding to four different ranges of values of \( a \) and restricted ranges of values of \( b \). A short conclusion is given in Section 4.
2. Solutions to a First-Order Ordinary Differential Equation

This section is due to Vakhnenko and Parkes (see Appendix A in [39]). For completeness and readability, we repeat it in the following.

Consider solutions to the following ordinary differential equation

$$(\varphi \varphi_{\xi})^2 = \varepsilon^2 f(\varphi),$$

(2.1)

where

$$f(\varphi) = (\varphi - \varphi_1)(\varphi - \varphi_2)(\varphi - \varphi_3)(\varphi - \varphi_4),$$

(2.2)

and $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ are chosen to be real constants with $\varphi_1 \leq \varphi_2 \leq \varphi \leq \varphi_3 \leq \varphi_4$.

Following [40] we introduce $\zeta$ defined by

$$\frac{d\xi}{d\zeta} = \frac{\varphi}{\varepsilon},$$

(2.3)

so that (2.1) becomes

$$(\varphi_{\xi})^2 = f(\varphi).$$

(2.4)

Equation (2.4) has two possible forms of solution. The first form is found using result 254.00 in [41]. Its parametric form is

$$\varphi = \frac{\varphi_2 - \varphi_1 \text{sn}^2(w \mid m)}{1 - \text{sn}^2(w \mid m)},$$

$$\xi = \frac{1}{\varepsilon p} (\varepsilon \varphi_1 + (\varphi_2 - \varphi_1) \text{Pi}(n, w \mid m)), \quad (2.5)$$

with $w$ as the parameter, where

$$m = \frac{(\varphi_3 - \varphi_2)(\varphi_4 - \varphi_1)}{(\varphi_4 - \varphi_2)(\varphi_3 - \varphi_1)}, \quad p = \frac{1}{2} \sqrt{(\varphi_4 - \varphi_2)(\varphi_3 - \varphi_1)}, \quad w = p \zeta, \quad (2.6)$$

$$n = \frac{\varphi_3 - \varphi_2}{\varphi_4 - \varphi_1}. \quad (2.7)$$

In (2.5) $\text{sn}(w \mid m)$ is a Jacobian elliptic function, where the notation is as used in [42, Chapter 16], and the notation is as used in [42, Section 17.2.15].
The solution to (2.1) is given in parametric form by (2.5) with \( w \) as the parameter. With respect to \( w \), \( \varphi \) in (2.5) is periodic with period \( 2K(m) \), where \( K(m) \) is the complete elliptic integral of the first kind. It follows from (2.5) that the wavelength \( \lambda \) of the solution to (2.1) is

\[
\lambda = \frac{2}{\varepsilon p} \left| \varphi_1 K(m) + (\varphi_2 - \varphi_1) \Pi(n \mid m) \right|, \tag{2.8}
\]

where \( \Pi(n \mid m) \) is the complete elliptic integral of the third kind.

When \( \varphi_3 = \varphi_4, m = 1, \) (2.5) becomes

\[
\varphi = \frac{\varphi_2 - \varphi_1 n \tanh^2 w}{1 - n \tanh^2 w},
\]
\[
\xi = \frac{1}{\varepsilon} \left( \frac{w\varphi_1}{p} - 2 \tanh^{-1} (\sqrt{n} \tanh w) \right). \tag{2.9}
\]

The second form of solution of (2.5) is found using result 255.00 in [41]. Its parametric form is

\[
\varphi = \frac{\varphi_3 - \varphi_4 n \text{sn}^2 (w \mid m)}{1 - n \text{sn}^2 (w \mid m)},
\]
\[
\xi = \frac{1}{\varepsilon p} (w\varphi_4 - (\varphi_4 - \varphi_3) \Pi(n; w \mid m)), \tag{2.10}
\]

where \( m, p, w \) are as in (2.6), and

\[
n = \frac{\varphi_3 - \varphi_2}{\varphi_4 - \varphi_2}. \tag{2.11}
\]

The solution to (2.1) is given in parametric form by (2.10) with \( w \) as the parameter. The wavelength \( \lambda \) of the solution to (2.1) is

\[
\lambda = \frac{2}{\varepsilon p} \left| \varphi_1 K(m) - (\varphi_4 - \varphi_3) \Pi(n \mid m) \right|. \tag{2.12}
\]

When \( \varphi_1 = \varphi_2, m = 1, \) (2.10) becomes

\[
\varphi = \frac{\varphi_3 - \varphi_4 n \tanh^2 w}{1 - n \tanh^2 w},
\]
\[
\xi = \frac{1}{\varepsilon} \left( \frac{w\varphi_2}{p} + 2 \tanh^{-1} (\sqrt{n} \tanh w) \right). \tag{2.13}
\]
3. Periodic and Solitary Wave Solutions to Equation (1.5)

Equation (1.5) can also be written in the form

\[(u_t + uu_x)_{xx} = u_t + uu_x + u_x.\]  

(3.1)

Let \( u = \varphi(\xi) + c \) with \( \xi = x - ct \) be a travelling wave solution to (3.1), then it follows that

\[ (\varphi \varphi_\xi)_\xi = \varphi \varphi_\xi + \varphi_\xi. \]  

(3.2)

Integrating (3.2) twice with respect to \( \xi \), we have

\[ (\varphi \varphi_\xi)^2 = \frac{1}{4} \left( \varphi^4 + \frac{8}{3} \varphi^3 + a \varphi^2 + b \right), \]  

(3.3)

where \( a \) and \( b \) are two arbitrary integration constants.

Equation (3.3) is in the form of (2.1) with \( \varepsilon = 1/2 \) and \( f(\varphi) = (\varphi^4 + 8/3 \varphi^3 + a \varphi^2 + b) \).

For convenience we define \( g(\varphi) \) and \( h(\varphi) \) by

\[ f(\varphi) = \varphi^2 g(\varphi) + b, \quad \text{where} \quad g(\varphi) = \varphi^2 + \frac{8}{3} \varphi + a, \]  

(3.4)

and define \( \varphi_L, \varphi_R, b_L, \) and \( b_R \) by

\[ \varphi_L = -\frac{1}{2} \left( 2 + \sqrt{4 - 2a} \right), \quad \varphi_R = -\frac{1}{2} \left( 2 - \sqrt{4 - 2a} \right), \]  

\[ b_L = -\varphi_L^2 g(\varphi_L) = \frac{a^2}{4} - 2a + \frac{8}{3} (2 - a) \sqrt{4 - 2a}, \]  

(3.5)

\[ b_R = -\varphi_R^2 g(\varphi_R) = \frac{a^2}{4} - 2a + \frac{8}{3} (2 - a) \sqrt{4 - 2a}. \]

Obviously, \( \varphi_L, \varphi_R \) are the roots of \( h(\varphi) = 0 \).

In the following, suppose that \( a < 2 \) and \( a \neq 0 \) such that \( f(\varphi) \) has three distinct stationary points: \( \varphi_L, \varphi_R, 0 \) and comprise two minimums separated by a maximum. Under this assumption, (3.3) has periodic and solitary wave solutions that have different analytical forms depending on the values of \( a \) and \( b \) as follows.

1. \( a < 0 \)

In this case \( \varphi_L < 0 < \varphi_R \) and \( f(\varphi_L) < f(\varphi_R) \). For each value \( a < 0 \) and \( 0 < b < b_R \) (a corresponding curve of \( f(\varphi) \) is shown in Figure 1(a)), there are periodic inverted loop-like solutions to (3.3) given by (2.5) so that \( 0 < m < 1 \), and with wavelength given by (2.8); see Figure 2(a), for an example.
The case \( a < 0 \) and \( b = b_R \) (a corresponding curve of \( f(\varphi) \) is shown in Figure 1(b)) corresponds to the limit \( \varphi_3 = \varphi_4 = \varphi_R \) so that \( m = 1 \), and then the solution is an inverted loop-like solitary wave given by (2.9) with \( \varphi_2 \leq \varphi < \varphi_R \) and

\[
\varphi_1 = -\frac{1}{6} \left( 2 + 3\sqrt{4 - 2a + 2\sqrt{4 + 6\sqrt{4 - 2a}}} \right),
\]

\[
\varphi_2 = -\frac{1}{6} \left( 2 + 3\sqrt{4 - 2a - 2\sqrt{4 + 6\sqrt{4 - 2a}}} \right);
\]

see Figure 3(a), for an example.

(2) \( 0 < a < 16/9 \)

In this case \( \varphi_L < \varphi_R < 0 \) and \( f(\varphi_L) < f(0) \). For each value \( 0 < a < 16/9 \) and \( b_R < b < 0 \) (a corresponding curve of \( f(\varphi) \) is shown in Figure 1(c)), there are periodic hump-like solutions to (3.3) given by (2.5) so that \( 0 < m < 1 \), and with wavelength given by (2.8); see Figure 2(b), for an example.

The case \( 0 < a < 16/9 \) and \( b = 0 \) (a corresponding curve of \( f(\varphi) \) is shown in Figure 1(d)) corresponds to the limit \( \varphi_3 = \varphi_4 = 0 \) so that \( m = 1 \), and then the solution can be given by (2.9) with \( \varphi_1 \) and \( \varphi_2 \) given by the roots of \( g(\varphi) = 0 \), namely

\[
\varphi_1 = -\frac{4}{3} - \frac{1}{3}\sqrt{16 - 9a}, \quad \varphi_2 = -\frac{4}{3} + \frac{1}{3}\sqrt{16 - 9a}.
\]

In this case we obtain a weak solution, namely, the periodic upward-cusp wave

\[
\varphi = \varphi(\xi - 2j \xi_m), \quad (2j - 1)\xi_m < \xi < (2j + 1)\xi_m, \quad j = 0, \pm 1, \pm 2, \ldots,
\]
see Figure 2

Figure 2: Periodic solutions to (3.3) with $0 < m < 1$. (a) $a = -50, b = 233$ so $m = 0.7885$; (b) $a = 1.5, b = -0.05$ so $m = 0.6893$; (c) $a = 16/9, b = -0.1$ so $m = 0.8254$; (d) $a = 1.9, b = -0.24$ so $m = 0.6121$.

where

\[
\varphi(\xi) = \left(\varphi_2 - \varphi_1 \tanh^2 \left(\frac{\xi}{4}\right)\right) \cosh^2 \left(\frac{\xi}{4}\right),
\]

(3.9)

\[
\xi_m = 4 \tanh^{-1} \sqrt{\frac{\varphi_2}{\varphi_1}},
\]

(3.10)

see Figure 3(b), for an example.

(3) $a = 16/9$

In this case $\varphi_L < \varphi_R < 0$ and $f(\varphi_L) = f(0)$. For $a = 16/9$ and each value $b_R < b < 0$ (a corresponding curve of $f(\varphi)$ is shown in Figure 1(e)), there are periodic hump-like solutions to (3.3) given by (2.10) so that $0 < m < 1$, and with wavelength given by (2.12); see Figure 2(c), for an example.

The case $a = 16/9$ and $b = 0$ (a corresponding curve of $f(\varphi)$ is shown in Figure 1(f)) corresponds to the limit $\varphi_1 = \varphi_2 = \varphi_L = -4/3$ and $\varphi_3 = \varphi_4 = 0$ so that $m = 1$. In this case neither (2.9) nor (2.13) is appropriate. Instead we consider (3.3) with $f(\varphi) = 1/4\varphi^2(\varphi + 4/3)^2$ and note that the bound solution has $-4/3 < \varphi \leq 0$. On integrating (3.3) and setting $\varphi = 0$ at $\xi = 0$ we obtain a weak solution

\[
\varphi = \frac{4}{3} \exp \left(-\frac{1}{2} \xi \right) - \frac{4}{3} \xi,
\]

(3.11)

that is, a single peakon solution with amplitude $4/3$, see Figure 3(c).

(4) $16/9 < a < 2$

In this case $\varphi_L < \varphi_R < 0$ and $f(\varphi_L) > f(0)$. For each value $16/9 < a < 2$ and $b_R < b < b_L$ (a corresponding curve of $f(\varphi)$ is shown in Figure 1(g)), there are periodic hump-like
solutions to (3.3) given by (2.10) so that $0 < m < 1$, and with wavelength given by (2.12); see Figure 2(d), for an example.

The case $16/9 < a < 2$ and $b = b_L$ (a corresponding curve of $f(\varphi)$ is shown in Figure 1(h)) corresponds to the limit $\varphi_1 = \varphi_2 = \varphi_L$ so that $m = 1$, and then the solution is a hump-like solitary wave given by (2.13) with $\varphi_L < \varphi \leq \varphi_3$ and

\[
\varphi_3 = \frac{1}{6} \left(-2 + 3\sqrt{4 - 2a} - 2\sqrt{4 - 6\sqrt{4 - 2a}}\right),
\]

\[
\varphi_4 = \frac{1}{6} \left(-2 + 3\sqrt{4 - 2a} + 2\sqrt{4 - 6\sqrt{4 - 2a}}\right),
\]

see Figure 3(d), for an example.

On the above, we have obtained expressions of parametric form for periodic and solitary wave solutions $\varphi(\xi)$ to (3.3). So in terms of $u = \varphi(\xi) + c$, we can get expressions for the periodic and solitary wave solutions $u(\xi)$ to (1.5).

4. Conclusion

In this paper, we have found expressions for new travelling wave solutions to the Fornberg-Whitham equation. These solutions depend, in effect, on two parameters $a$ and $m$. For $m = 1$, there are inverted loop-like ($a < 0$), single peaked ($a = 16/9$), and hump-like ($16/9 < a < 2$)
solitary wave solutions. For $m = 1, 0 < a < 16/9$ or $0 < m < 1, a < 2$, and $a \neq 0$, there are periodic hump-like wave solutions.

Acknowledgments

The authors are deeply grateful to an anonymous referee for the important comments and suggestions. Zhou acknowledges funding from Startup Fund for Advanced Talents of Jiangsu University (No. 09JDG013). Tian’s work was partially supported by NSF of China (No. 90610031).

References
