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Research Article

Application of Variational Iteration Method to Fractional Hyperbolic Partial Differential Equations

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The solution of the fractional hyperbolic partial differential equation is obtained by means of the variational iteration method. Our numerical results are compared with those obtained by the modified Gauss elimination method. Our results reveal that the technique introduced here is very effective, convenient, and quite accurate to one-dimensional fractional hyperbolic partial differential equations. Application of variational iteration technique to this problem has shown the rapid convergence of the sequence constructed by this method to the exact solution.

1. Introduction

It is known that various problems in fluid mechanics (dynamics, elasticity) and other areas of physics lead to fractional partial differential equations. Methods of solutions of problems for fractional differential equations have been studied extensively by many researchers (see [1–11]).

The variational iteration method (VIM), which was proposed by He (see, e.g., [12–21]), was successfully applied to autonomous ordinary and partial differential equations and other fields. He [15] was the first research who applied the VIM to fractional differential equations. Odibat and Momani [22] extended the application of this method to provide approximate solutions for initial value problems of nonlinear partial differential equations of fractional order. VIM [23–25] is relatively a new approach to provide an analytical approximation to linear and nonlinear problems which is particularly valuable tools for scientists and applied mathematicians. Yulita et al. [26] used the VIM to obtain analytical solutions of fractional heat- and wave-like equations with variable coefficients. In the Ashyralyev et al. [27], the mixed boundary value problem for the multidimensional fractional hyperbolic equation is considered. The first order of accuracy in t and the second order of accuracy in space variables for the approximate solution of problem were presented. The

stability estimates for the solution of this difference scheme and its first- and second-order difference derivatives were established. A procedure of modified Gauss elimination method [28] was used for solving this difference scheme in the case of one-dimensional fractional hyperbolic partial differential equations.

In this paper, we apply variational iteration method to fractional hyperbolic partial differential equations and then we compare the results with those obtained using modified Gauss elimination method [27].

2. Definitions

Definition 2.1. A reel function $f(x)$, $x > 0$, is said to be in the space C_M, $M \in R$, if there exists a real number $p(> M)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0,\infty)$, and it is said to be in the space C_M^m if $f^m \in C_M$, $m \in N$.

Definition 2.2. If $f(x) \in C[a,b]$ and $a < x < b$, then

$$I_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \tag{2.1}$$

where $-\infty < \alpha < \infty$ is called the Riemann-Liouville fractional integral operator of order α.

Definition 2.3. For $0 < \alpha < 1$, we let

$$D_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^{\alpha}} dt, \tag{2.2}$$

which is called the Riemann-Liouville fractional derivative operator of order α.

3. Variational teration Method

In the present paper, the mixed boundary value problem for the multidimensional fractional hyperbolic equation

$$\begin{gathered} \frac{\partial^2 u(x,t)}{\partial t^2} - \sum_{r=1}^{m} (a_r(x) u_{x_r})_{x_r} + D_t^{1/2} u(x,t) = f(x,t), \\ x = (x_1, \ldots, x_m) \in \Omega, \quad 0 < t < 1, \\ u(x,0) = 0, \quad u_t(x,0) = 0, \quad x \in \overline{\Omega}, \\ u(x,t) = 0, \quad x \in S \end{gathered} \tag{3.1}$$

is considered. Here Ω is the unit open cube in the m-dimensional Euclidean space $\mathbb{R}^m : \Omega = \{x = (x_1, \ldots, x_m) : 0 < x_j < 1,\ 1 \le j \le m\}$ with boundary S, $\overline{\Omega} = \Omega \cup S$, $a_r(x)$, $(x \in \Omega)$, and $f(x,t)$ $(t \in (0,1),\ x \in \Omega)$ are given smooth functions and $a_r(x) \ge a > 0$.

The correction functional for (3.1) can be approximately expressed as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left[\frac{\partial^2 u(x,s)}{\partial s^2} - \sum_{r=1}^{m} (a_r(x)\tilde{u}_{x_r})_{x_r} + D_s^{1/2}\tilde{u}(x,s) - f(x,s) \right] ds, \tag{3.2}$$

where λ is a general Lagrangian multiplier [29] and $\tilde{u}$ is considered as a restricted variation as a restricted variation [21], that is, $\delta\tilde{u} = 0$, and $u_0(x,t)$ is its initial approximation. Using the above correction functional stationary and noticing that $\delta\tilde{u} = 0$, we obtain

$$\begin{aligned} \delta u_{n+1}(x,t) &= \delta u_n(x,t) + \int_0^t \delta\lambda \left[\frac{\partial u_n^2(x,s)}{\partial s^2} \right] ds, \\ \delta u_{n+1}(x,t) &= \delta u_n(x,t) - \frac{\partial \lambda}{\partial s} \delta u_n(x,s)\Big|_{s=t} + \lambda \frac{\partial}{\partial s} \delta u_n(x,s)\Big|_{s=t} \\ &\quad + \int_0^t \frac{\partial^2 \lambda(t,s)}{\partial s^2} \delta u_n(x,s) ds = 0. \end{aligned} \tag{3.3}$$

From the above relation for any δu_n, we get the Euler-Lagrange equation:

$$\frac{\partial \lambda^2(t,s)}{\partial s^2} = 0 \tag{3.4}$$

with the following natural boundary conditions:

$$\begin{aligned} 1 - \frac{\partial \lambda(t,s)}{\partial s}\Big|_{s=t} &= 0, \\ \lambda(t,s)|_{s=t} &= 0. \end{aligned} \tag{3.5}$$

Therefore, the Lagrange multiplier can be identified as follows:

$$\lambda(t,s) = s - t. \tag{3.6}$$

Substituting the identified Lagrange multiplier into (3.2), following variational iteration formula can be obtained:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (s-t) \left[\frac{\partial^2 u(x,s)}{\partial s^2} - \sum_{r=1}^{m} (a_r(x) u_{x_r})_{x_r} + D_s^{1/2} u(x,s) - f(x,s) \right] ds. \tag{3.7}$$

In this case, let an initial approximation be $u_0(x,t) = u(x,0) + t u_t(x,0)$. Then approximate solution takes the form $u(x,t) = \lim_{n\to\infty} u_n(x,t)$.

3.1. The Difference Scheme

The discretization of problem (3.1) is carried out in two steps. In the first step, let us define the grid space

$$\widetilde{\Omega}_h = \{x = x_r = (h_1 r_1, \ldots, h_m r_m),\ r = (r_1, \ldots, r_m),\ 0 \le r_j \le N_j,\ h_j N_j = 1,\ j = 1, \ldots, m\},$$
$$\Omega_h = \widetilde{\Omega}_h \cap \Omega, \qquad S_h = \widetilde{\Omega}_h \cap S. \tag{3.8}$$

We introduce the Banach space $L_{2h} = L_2(\widetilde{\Omega}_h)$ of the grid functions $\varphi^h(x) = \{\varphi(h_1 r_1, \ldots, h_m r_m)\}$ defined on $\widetilde{\Omega}_h$, equipped with the norm

$$\left\|\varphi^h\right\|_{L_2(\widetilde{\Omega}_h)} = \left(\sum_{x \in \overline{\Omega}_h} \left|\varphi^h(x)\right|^2 h_1 \cdots h_m\right)^{1/2}. \tag{3.9}$$

To the differential operator A^x generated by problem (3.1), we assign the difference operator A_h^x by the formula

$$A_h^x u_x^h = -\sum_{r=1}^{m} \left(a_r(x) u_{\overline{x}_r}^h\right)_{x_r, j_r} \tag{3.10}$$

acting in the space of grid functions $u^h(x)$, satisfying the conditions $u^h(x) = 0$ for all $x \in S_h$. It is known that A_h^x is a self-adjoint positive definite operator in $L_2(\widetilde{\Omega}_h)$. With the help of A_h^x we arrive at the initial boundary value problem

$$\frac{d^2 v^h(x,t)}{dt^2} + A_h^x v^h(x,t) + D_t^{1/2} v^h(x,t) = f^h(x,t), \quad 0 \le t \le 1,\ x \in \Omega_h,$$
$$v^h(x,0) = 0, \quad \frac{dv^h(x,0)}{dt} = 0, \quad x \in \widetilde{\Omega} \tag{3.11}$$

for an finite system of ordinary fractional differential equations.

In the second step, applying the first order of approximation formula $(1/\sqrt{\pi}) \sum_{m=1}^{k} (\Gamma(k - m + 1/2)/(k - m)!)((u(t_k) - u(t_{k-1}))/\tau^{1/2})$ for $D_t^{1/2} u(t)$ and using

the first order of accuracy stable difference scheme for hyperbolic equations (see [30]), one can present the first order of accuracy difference scheme:

$$\frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+A_h^x u_{k+1}^h+\frac{1}{\sqrt{\pi}}\sum_{m=1}^{k}\frac{\Gamma(k-m+1/2)}{(k-m)!}\frac{\left(u_m^h-u_{m-1}^h\right)}{\tau^{1/2}}=f_k^h(x),\quad x\in\widetilde{\Omega}_h,$$
$$f_k^h(x)=f(x_n,t_k),\quad t_k=k\tau,\ 1\le k\le N-1,\ N\tau=1,$$
$$\frac{u_1^h(x)-u_0^h(x)}{\tau}=0,\quad u_0^h(x)=0,\ x\in\widetilde{\Omega}_h \tag{3.12}$$

for the approximate solution of problem (3.1). Here $\Gamma(k-m+1/2)=\int_0^\infty t^{k-m-1/2}e^{-t}dt$.

3.2. Example 1

For the numerical result, the mixed problem

$$D_t^2u(x,t)-D_t^{1/2}u(x,t)-u_{xx}(x,t)=f(x,t),$$
$$f(x,t)=\left(2-\frac{8t^{3/2}}{3\sqrt{\pi}}+(\pi t)^2\right)\sin\pi x,\quad 0<t,\ x<1,$$
$$u(x,0)=0,\quad u_t(x,0)=0,\quad 0\le x\le 1,$$
$$u(t,0)=u(t,1)=0,\quad 0\le t\le 1 \tag{3.13}$$

for solving the one-dimensional fractional hyperbolic partial differential equation is considered.

According to the formula (3.7), the iteration formula for (3.13) is given by

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t(s-t)\left[\frac{\partial u_n^2(x,s)}{\partial s^2}-D_s^{1/2}u_n(x,s)-\frac{\partial u_n^2(x,s)}{\partial x^2}\right.$$
$$\left.-\left(2-\frac{8s^{3/2}}{3\sqrt{\pi}}+(\pi s)^2\right)\sin(\pi x)\right]ds. \tag{3.14}$$

Now we start with an initial approximation:

$$u_0(x,t)=u(x,0)+tu_t(x,0). \tag{3.15}$$

Using the above iteration formula (3.14), we can obtain the other components as

$$
\begin{aligned}
u_0(x,t) &= 0,\\
u_1(x,t) &= \left(-\frac{128}{420\sqrt{\pi}}t^{7/2}+\frac{\pi^2t^4}{12}+t^2\right)\sin(\pi x),\\
u_2(x,t) &= \frac{1}{41580\sqrt{\pi}}\sin(\pi x)t^{5/2}\left[512\pi^2t^3+12672t-693\sqrt{\pi}t^{5/2}\right]\\
&\quad+\frac{128}{10395}\sin(\pi x)t^{11/2}\pi^{3/2}-\frac{1}{360}\pi^4\sin(\pi x)t^6-\frac{1}{12}\pi^2\sin(\pi x)t^4\\
&\quad+\frac{1}{420\sqrt{\pi}}\left[-128t^{7/2}+35\pi^{5/2}t^4+420t^2\sqrt{\pi}\right]\sin(\pi x),\\
&\vdots
\end{aligned}
\tag{3.16}
$$

and so on; in the same manner the rest of the components of the iteration formula (3.14) can be obtained using the Maple package.

3.3. Example 2

We consider one-dimensional fractional hyperbolic partial differential equation as follows:

$$
\begin{gathered}
D_t^2u(x,t)+D_t^{1/2}u(x,t)-u_{xx}(x,t)+u(x,t)=f(x,t),\\
f(x,t)=\left(2\left(1-x-\exp(-x)\right)\right)+\frac{8t^{3/2}}{3\sqrt{\pi}}\left(1-x-\exp(-x)\right)+(1-x)t^2x,\quad 0<t,\ x<1,\\
u(x,0)=0,\quad u_t(x,0)=0,\quad 0\le x\le 1,\\
u(t,0)=u(t,1)=0,\quad 0\le t\le 1.
\end{gathered}
\tag{3.17}
$$

The iteration formula for (3.17) is given by

$$
\begin{aligned}
u_{n+1}(x,t)=u_n(x,t)+\int_0^t(s-t)\Bigg[&\frac{\partial u_n^2(x,s)}{\partial s^2}+D_s^{1/2}u_n(x,s)-\frac{\partial u_n^2(x,s)}{\partial x^2}+u(x,s)\\
&-\Bigg[\left(2\left(1-x-\exp(-x)\right)\right)\\
&\quad+\frac{8t^{3/2}}{3\sqrt{\pi}}\left(1-x-\exp(-x)\right)+(1-x)t^2\Bigg]\Bigg]ds.
\end{aligned}
\tag{3.18}
$$

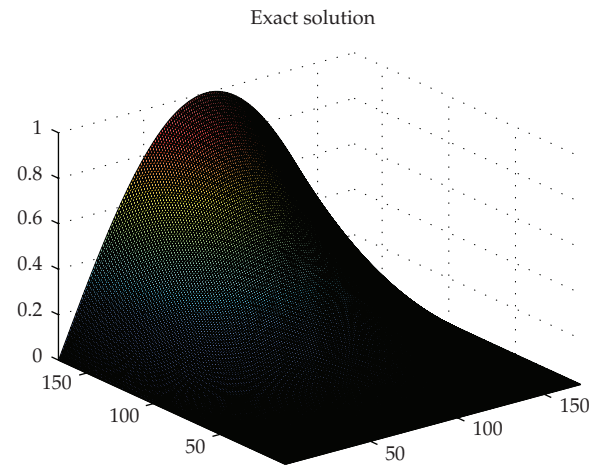

Figure 1: The surface shows the exact solution $u(x,t)$ for (3.13).

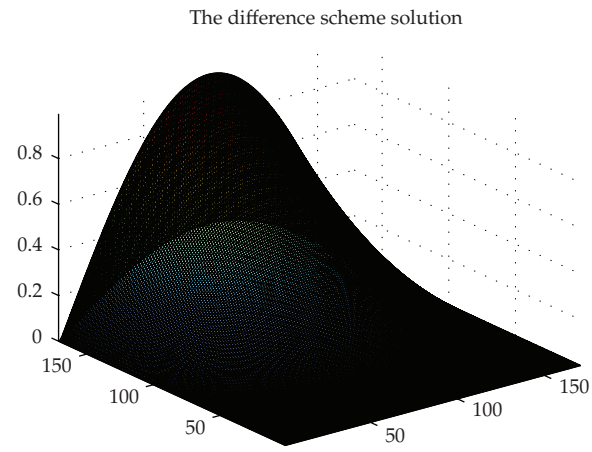

Figure 2: Difference scheme solution [27] for (3.13).

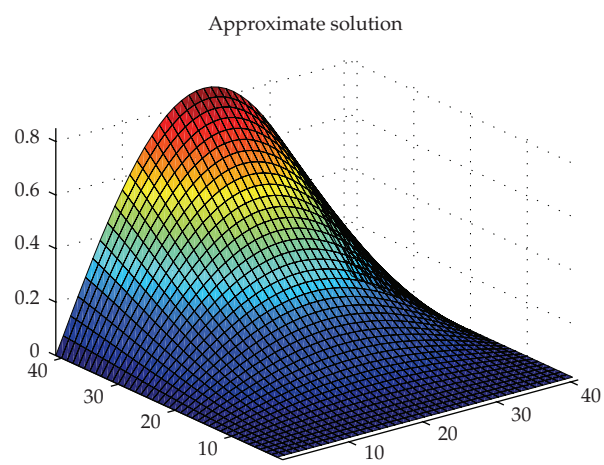

Figure 3: Variational iteration method for (3.13).

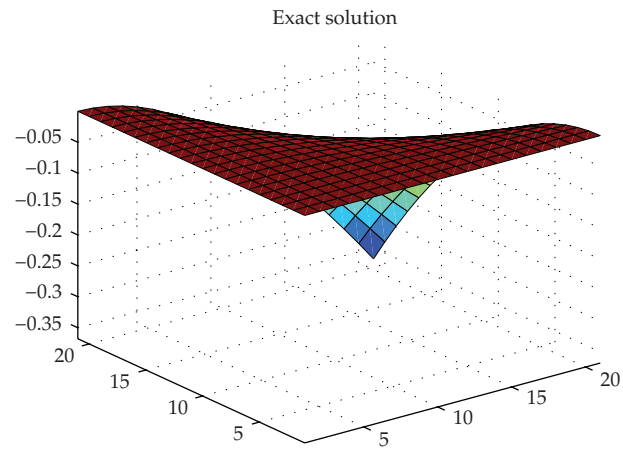

Figure 4: The surface shows the exact solution $u(x,t)$ for (3.17).

When an initial approximation is $u_0(x,t) = u(x,0) + tu_t(x,0)$, we have the other components as

$$u_o(x,t) = 0,$$

$$u_1(x,t) = \left(1 - x - \exp(-x)\right)\left(\frac{32}{105}t^{7/2} + t^2\right) + \frac{t^4}{12}(1-x),$$

$$\begin{aligned}
u_2(x,t) \;=\; & \left(1 - x - \exp(-x)\right)\left(\frac{32}{105}t^{7/2} + t^2\right) + \frac{t^4}{12}(1-x) \\
& + \frac{1}{420}\left[128t^{7/2}\left(-1 + x + \exp(-x)\right) + 35t^4(x-1) + 420t^2\left(-1 + x + \exp(-x)\right)\right] \\
& + \frac{1}{41580\sqrt{\pi}}\left[12672t^{7/2}\left(-1 + x + \exp(-x)\right)\right. \\
& \qquad \left. + 512t^{11/2}(-1+x) + 693t^5\left(-1 + x + \exp(-x)\right)\right] \\
& - \frac{1}{41580}\left(512\exp(-x)t^{11/2} + 3465t^4\exp(-x)\right) \\
& + \frac{1}{83160}\left[1024t^{11/2}\left(-1 + x + \exp(-x)\right) + 231t^6(-1+x) + 6930t^4\left(-1 + x + \exp(-x)\right)\right] \\
& - \frac{1}{420\sqrt{\pi}}\left[128t^{7/2}\left(-1 + x + \exp(-x)\right) + 35t^4\sqrt{\pi}(-1+x)\right. \\
& \qquad \left. + 420t^2\sqrt{\pi}\left(-1 + x + \exp(-x)\right)\right] \\
& \qquad \vdots
\end{aligned} \tag{3.19}$$

and so on. For (3.18), the rest of the components of the iteration formula can be obtained using the Maple 10 package.

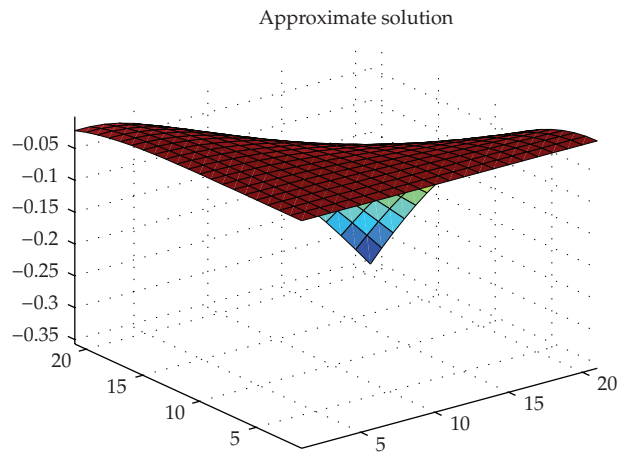

Figure 5: Variational iteration method for (3.17).

4. Conclusions

Variational iteration method is a powerful and efficient technique in finding exact and approximate solutions for one-dimensional fractional hyperbolic partial differential equations. The solution procedure is very simple by means of variational theory, and only a few steps lead to highly accurate solutions which are valid for the whole solution domain. The results of applying variational iteration method are exactly the same as those obtained by modified Gauss elimination method [27]. All Computations are performed by Maple 10 package program.

Figure 1 shows the exact solution of (3.13). Figure 2 shows difference scheme solution of (3.13). Figure 3 shows approximate solution by VIM for (3.13). Figure 4 shows the exact solution of (3.17). Figure 5 shows approximate solution by VIM for (3.17).

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