Research Article

Controllability of Second-Order Equations in $L^2(\Omega)$

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Received 24 August 2010; Accepted 13 November 2010

Academic Editor: Christos H. Skiadas

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We present a simple proof of the interior approximate controllability for the following broad class of second-order equations in the Hilbert space $L^2(\Omega)$: $\ddot{y} + Ay = 1_\omega u(t)$, $t \in (0, \tau]$, $y(0) = y_0$, $\dot{y}(0) = y_1$, where $\Omega$ is a domain in $\mathbb{R}^N$ ($N \geq 1$), $y_0, y_1 \in L^2(\Omega)$, $\omega$ is an open nonempty subset of $\Omega$, $1_\omega$ denotes the characteristic function of the set $\omega$, the distributed control $u$ belongs to $L^2(0, \tau; L^2(\Omega))$, and $A : D(A) \subset L^2(\Omega) \rightarrow L^2(\Omega)$ is an unbounded linear operator with the following spectral decomposition: $A = \sum_{j=1}^\infty \lambda_j \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}$, with the eigenvalues $\lambda_j$ given by the following formula: $\lambda_j = \frac{m^2}{m^2 + m^2}$, $j = 1, 2, 3, \ldots$ and $m \geq 1$ is a fixed integer number, multiplicity $\gamma_j$ is equal to the dimension of the corresponding eigenspace, and $\{\phi_{j,k}\}$ is a complete orthonormal set of eigenvectors (eigenfunctions) of $A$. Specifically, we prove the following statement: if for an open nonempty set $\omega \subset \Omega$ the restrictions $\phi_{j,k}^\omega = \phi_{j,k}|_{\omega}$ of $\phi_{j,k}$ to $\omega$ are linearly independent functions on $\omega$, then for all $\tau \geq 2/\pi^{m-1}$ the system is approximately controllable on $[0, \tau]$. As an application, we prove the controllability of the 1D wave equation.

1. Introduction

This paper has been motivated by the work in [1] and the articles [2, 3], where a new technique is used to prove the interior approximate controllability of some diffusion process. Particularly in [3], where the authors prove the interior approximate controllability of the following broad class of reaction diffusion equations in the Hilbert space $Z = L^2(\Omega)$ given by

$$z' = -Az + 1_\omega u(t), \quad t \in [0, \tau],$$

where $\Omega$ is a domain in $\mathbb{R}^n$, $\omega$ is an open nonempty subset of $\Omega$, $1_\omega$ denotes the characteristic function of the set $\omega$, the distributed control $u \in L^2(0, \tau; L^2(\Omega))$ and $A : D(A) \subset Z \rightarrow Z$ is an
unbounded linear operator with the spectral decomposition

\[ Az = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}. \]  

(1.2)

The eigenvalues \(0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty\) of \(A\) have finite multiplicity \(\gamma_j\) equal to the dimension of the corresponding eigenspace, and \(\{\phi_{j,k}\}\) is a complete orthonormal set of eigenvectors of \(A\). The operator \(-A\) generates a strongly continuous semigroup \(\{T_A(t)\}_{t \geq 0}\) given by

\[ T_A(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}. \]  

(1.3)

As a consequence of this result, the controllability of the following heat equation follows trivially by putting \(A = -\Delta\)

\[ z_t = \Delta z + 1_\omega u(t, x), \quad \text{in } (0, \tau) \times \Omega, \]
\[ z = 0, \quad \text{on } (0, \tau) \times \partial \Omega, \]
\[ z(0, x) = z_0(x), \quad \text{in } \Omega. \]  

(1.4)

Following [1–3], in this paper, we study the interior approximate controllability of the following broad class of second-order equations in the Hilbert space \(L^2(\Omega)\):

\[ \ddot{y} + Ay = 1_\omega u(t), \quad t \in (0, \tau], \]
\[ y(0) = y_0, \quad \dot{y}(0) = y_1, \]  

(1.5)

where the eigenvalues \(\lambda_j\) of the operator of \(A\) are given by the following formula:

\[ \lambda_j = j^{2m} \pi^{2m}, \quad j = 1, 2, 3, \ldots, \quad m \geq 1 \text{ is a fixed integer number.} \]  

(1.6)

Specifically, we prove the following statement: if for an open nonempty set \(\omega \subset \Omega\) the restrictions \(\phi_{j,k}^\omega = \phi_{j,k}|_{\omega}\) of \(\phi_{j,k}\) to \(\omega\) are linearly independent functions on \(\omega\), then for all \(\tau \geq 2/\pi^{m-1}\) the system (1.5) is approximately controllable on \([0, \tau]\). Moreover, we can exhibit a sequence of controls steering the system from an initial state to a final state in a prefixed time (see Theorem 2.8).

This result implies the interior controllability of the following well-known examples of partial differential equations.
Example 1.1. The 1D Wave Equation

\[ \begin{align*}
    y_{tt} - \Delta y &= 1_\omega u(t, x), & \text{in } (0, \tau] \times \Omega, \\
    y &= 0, & \text{on } (0, \tau) \times \partial\Omega, \\
    y(0, x) &= y_0(x), & y_t(0, x) = y_1(x), & x \in \Omega,
\end{align*} \]

where \( \omega \) is an open nonempty subset of \( \Omega = [0, 1] \), \( 1_\omega \) denotes the characteristic function of the set \( \omega \), and the distributed control \( u \in L^2(0, \tau; L^2(\Omega)) \).

Example 1.2. The Model of Vibrating String Equation

\[ \begin{align*}
    w_{tt} + \Delta^2 w &= 1_\omega u(t, x), & \text{in } (0, \tau] \times \Omega, \\
    w &= \Delta w = 0, & \text{on } (0, \tau) \times \partial\Omega, \\
    w(0, x) &= \phi_0(x), & w_t(0, x) = \psi_0(x), & x \in \Omega,
\end{align*} \]

where \( \Omega = [0, 1] \), \( \omega \) is an open nonempty subset of \( \Omega \), \( u \in L^2([0, \tau]; L^2(\Omega)) \), \( \phi_0, \psi_0 \in L^2(\Omega) \).

2. Main Results

In this section, we will prove the main result of this work; to this end, we consider by \( X = U = L^2(\Omega) \) and the linear unbounded operator \( A : D(A) \subset X \to X \) can be written as follows.

(a) For all \( x \in D(A) \), we have

\[ Ax = \sum_{j=1}^{\infty} \lambda_j E_j x, \]

where

\[ E_j x = \sum_{k=1}^{\gamma_j} \langle \xi_j, \phi_{j,k} \rangle \phi_{j,k}. \]

So, \( \{ E_j \} \) is a family of complete orthogonal projections in \( X \) and \( x = \sum_{j=1}^{\infty} E_j x, \ x \in X \).

(b) The semigroup \( \{ T_A(t) \} \) generated by \(-A\) can be written as follows:

\[ T_A(t) x = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j x. \]
(c) The fractional powered spaces $X^r$ are given by

$$X^r = D(A^r) = \left\{ x \in X : \sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n x\|^2 < \infty \right\}, \quad r \geq 0,$$

(2.4)

with the norm

$$\|x\|_r = \|A^r x\| = \left\{ \sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n x\|^2 \right\}^{1/2}, \quad x \in X^r,$$

(2.5)

and

$$A^r x = \sum_{n=1}^{\infty} \lambda_n E_n x.$$

Also, for $r \geq 0$, we define $Z_r = X^r \times X$, which is a Hilbert space endowed with the norm given by

$$\left\| \begin{bmatrix} y \\ v \end{bmatrix} \right\|_{Z_r}^2 = \|y\|^2_r + \|v\|^2.$$

(2.6)

**Proposition 2.1.** The operator $P_j : Z_r \to Z_r$, $j \geq 0$, defined by

$$p_j = \begin{bmatrix} E_j & 0 \\ 0 & E_j \end{bmatrix}, \quad j \geq 1,$$

(2.7)

is a continuous (bounded) orthogonal projections in the Hilbert space $Z_r$.

**Proof.** First we will show that $P_j(Z_r) \subset Z_r$, which is equivalent to show that $E_j(X') \subset X^r$. In fact, let $x$ be in $X'$ and consider $E_j x$. Then,

$$\sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n x\|^2 = \lambda_j^{2r} \|E_j x\|^2 < \infty.$$

(2.8)

Therefore, $E_j x \in X^r$, for all $x \in X'$.

Now, we will prove that this projection is bounded. In fact, from the continuous inclusion $X^r \subset X$, there exists a constant $k > 0$ such that

$$\|x\| \leq k \|x\|_r, \quad \forall x \in X^r.$$

(2.9)

Then, for all $x \in X'$, we have the following estimate

$$\|E_j x\|_r^2 = \sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n x\|^2 = \lambda_j^{2r} \|E_j x\|^2 \leq \lambda_j^{2r} \|x\|^2 \leq \lambda_j^{2r} k^2 \|x\|_r^2.$$

(2.10)
Hence, $\|E_j x\| \leq \lambda_j \|x\|$, which implies the continuity of $E_j : X' \to X'$. So, $P_j$ is a continuous projection on $Z_r$.

Hence, with the change of variable $y' = v$, the system (1.5) can be written as a first-order system of ordinary differential equations in the Hilbert space $Z_{1/2} = X^{1/2} \times X$ as follows:

$$z' = Az + B_\omega u, \quad z(0) = z_0, \quad z \in Z_{1/2}, \quad t \in (0, \tau],$$

(2.11)

where

$$z = \begin{bmatrix} y \\ v \end{bmatrix}, \quad B_\omega = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I_x \\ -A & 0 \end{bmatrix}$$

(2.12)

is an unbounded linear operator with domain $D(A) = D(A) \times D(A^{1/2})$.

The proof of the following theorem follows in the same way as [4, Theorem 3.1], by putting $c = 0$ and $d = 1$ or directly from [5, Lemma 2.1] or [6, Lemma 3.1].

**Theorem 2.2.** The operator $A$ given by (2.12) is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t \in \mathbb{R}}$ given by

$$T(t)z = \sum_{j=1}^\infty e^{A_j t} P_j z, \quad z \in Z, \quad t \geq 0,$$

(2.13)

where $\{P_j\}_{j \geq 1}$ is a complete family of orthogonal projections in the Hilbert space $Z_{1/2}$ given by

$$P_j = \text{diag}[E_j, E_j], \quad j \geq 1,$$

$$A_j = R_j P_j, \quad R_j = \begin{bmatrix} 0 & 1 \\ -\lambda_j & 0 \end{bmatrix}.$$

(2.14)

Also,

$$A_j^* = R_j^* P_j, \quad R_j^* = \begin{bmatrix} 0 & -1 \\ \lambda_j & 0 \end{bmatrix}.$$

(2.15)

Moreover, $e^{A_j t} = e^{R_j t} P_j$ and the eigenvalues of $R_j$ are $\sqrt{\lambda_j} i$ and $-\sqrt{\lambda_j} i$.

Now, before proving the main theorem, we will give the definition of approximate controllability for this system. To this end, for all $z_0 \in Z_{1/2}$ and $u \in L^2(0, \tau; U)$, the initial value problem

$$z' = Az + B_\omega u(t), \quad z \in Z, \quad t \in (0, \tau],$$

$$z(0) = z_0,$$

(2.16)
where the control function \( u \) belongs to \( \mathcal{L}^2(0, \tau; U) \), admits only one mild solution given by

\[
z(t) = T(t)z_0 + \int_0^t T(t - s)B_\omega u(s)ds, \quad t \in [0, \tau]. \tag{2.17}
\]

**Definition 2.3 (Approximate Controllability).** The system (2.16) is said to be \textit{approximately controllable} on \([0, \tau]\) if for every \( z_0, z_1 \in \mathcal{Z}_{1/2}, \varepsilon > 0 \) there exists \( u \in \mathcal{L}^2(0, \tau; U) \) such that the solution \( z(t) \) of (2.17) corresponding to \( u \) verifies

\[
z(0) = z_0, \quad \|z(\tau) - z_1\| < \varepsilon. \tag{2.18}
\]

Consider the following bounded linear operator:

\[
G : \mathcal{L}^2(0, \tau; \mathcal{Z}) \longrightarrow \mathcal{Z}, \quad Gu = \int_0^\tau T(\tau - s)B_\omega u(s)ds,
\tag{2.19}
\]

whose adjoint operator \( G^* : \mathcal{Z} \rightarrow \mathcal{L}^2(0, \tau; \mathcal{Z}) \) is given by

\[
(G^*z)(s) = B_\omega^* T^*(\tau - s)z, \quad \forall s \in [0, \tau], \quad \forall z \in \mathcal{Z}.
\tag{2.20}
\]

The following lemma is trivial

**Lemma 2.4.** The equation (2.16) is \textit{approximately controllable} on \([0, \tau]\) if, and only if, \( \overline{\text{Rang}(G)} = \mathcal{Z} \).

The following result is well known from linear operator theory.

**Lemma 2.5.** Let \( W \) and \( Z \) be Hilbert spaces and \( G^* \in \mathcal{L}(Z, W) \) the adjoint operator of the linear operator \( G \in \mathcal{L}(W, Z) \). Then,

\[
\overline{\text{Rang}(G)} = \mathcal{Z} \iff \text{Ker}(G^*) = \{0\}. \tag{2.21}
\]

As a consequence of the foregoing Lemma, one can prove the following result.

**Lemma 2.6.** Let \( W \) and \( Z \) be Hilbert spaces and \( G^* \in \mathcal{L}(Z, W) \) the adjoint operator of the linear operator \( G \in \mathcal{L}(W, Z) \). Then, \( \overline{\text{Rang}(G)} = \mathcal{Z} \) if, and only if, one of the following statements holds:

\begin{enumerate}
\item \( \text{Ker}(G^*) = \{0\} \),
\item \( (GG^*z, z) > 0, \ z \neq 0 \in Z \),
\item \( \lim_{\alpha \to 0} \alpha (\alpha I + GG^*)^{-1}z = 0 \),
\item \( \sup_{\alpha > 0} \|\alpha (\alpha I + GG^*)^{-1}\| \leq 1 \).
\end{enumerate}

The following theorem follows directly from (2.20) and Lemmas 2.4 and 2.6.

**Theorem 2.7.** The equation (2.16) is \textit{approximately controllable} on \([0, \tau]\) if

\[
B_\omega^* T^*(t)z = 0, \quad \forall t \in [0, \tau], \quad \implies z = 0. \tag{2.22}
\]
Now, we are ready to formulate and prove the main theorem of this work.

**Theorem 2.8 (Main Theorem).** If for an open nonempty set \( \omega \subset \Omega \) the restrictions \( \tilde{\phi}_{j,k} = \tilde{\phi}_{j,k}|_{\omega} \) of \( \phi_{j,k} \) to \( \omega \) are linearly independent functions on \( \omega \), then for all \( \tau \geq 2/\pi m - 1 \) the system (2.16) is approximately controllable on \([0, \tau]\). Moreover, a sequence of controls steering the system (2.16) from initial state \( z_0 \) to an \( \epsilon \) neighborhood of the final state \( z_1 \) at time \( \tau > 0 \) is given by

\[
u_\alpha(t) = B_{\omega}^* T^*(\tau - t)(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0),
\]

and the error of this approximation \( E_\alpha \) is given by

\[
E_\alpha = \alpha(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0).
\]

**Proof.** We will apply Theorem 2.7 to prove the controllability of system (2.16). To this end, we observe that the adjoint of operator \( B_{\omega} \) is by

\[
B_{\omega}^* = \begin{bmatrix} 0 & 1 \omega \end{bmatrix},
\]

\[
T^*(t)z = \sum_{j=1}^{\infty} e^{A_j t} P_j z, \quad z \in Z, \quad t \geq 0.
\]

Therefore,

\[
B_{\omega}^* T^*(t)z = \sum_{j=1}^{\infty} B_{\omega} e^{A_j t} P_j z.
\]

On the other hand, we have that

\[
e^{K_j t} = \begin{bmatrix} \cos \sqrt{\lambda_j} t & -\frac{1}{\sqrt{\lambda_j}} \sin \sqrt{\lambda_j} t \\ \sqrt{\lambda_j} \sin \sqrt{\lambda_j} t & \cos \sqrt{\lambda_j} t \end{bmatrix} = \begin{bmatrix} \cos j^m \pi m t & -\frac{1}{j^m \pi m} \sin j^m \pi m t \\ j^m \pi m \sin j^m \pi m t & \cos j^m \pi m t \end{bmatrix}. \tag{2.27}
\]

Suppose for all \( z \in Z_{1/2} \) that

\[
B_{\omega}^* T^*(t)z = \sum_{j=1}^{\infty} \{ j^m \pi m \sin(j^m \pi m t)(1_\omega E_j z_1) + \cos(j^m \pi m t)(1_\omega E_j z_2) \} = 0, \quad \forall t \in [0, \tau]. \tag{2.28}
\]

Then, if we make the change of variable \( s = \pi m t \), we obtain that

\[
\sum_{j=1}^{\infty} \{ j^m \pi m \sin(j^m s)(1_\omega E_j z_1) + \cos(j^m s)(1_\omega E_j z_2) \} = 0, \quad \forall s \in [0, \pi k t]. \tag{2.29}
\]
Since $\tau \geq 2/\pi^{m-1}$ we get that

$$\sum_{j=1}^{\infty} \left\{ j^m \pi^m \sin(j^m s) \left( 1_{\omega} E_j z_1 \right) + \cos(j^m s) \left( 1_{\omega} E_j z_2 \right) \right\} = 0, \quad \forall s \in [0,2\pi]. \quad (2.30)$$

On the other hand, it is well known that \{1, \cos(ns), \sin(ns) : n = 1, 2, 3, \ldots\} is an orthogonal base of $L^2[0,2\pi]$, which implies that \{\cos(j^m s), \sin(j^m s) : j = 1, 2, 3, \ldots\} is an orthogonal set in $L^2[0,2\pi]$, and therefore

$$(1_{\omega} E_j z_1)(x) = 0, \quad (1_{\omega} E_j z_2)(x) = 0, \quad \forall x \in \Omega, \ j = 1, 2, 3, \ldots, \quad (2.31)$$

that is,

$$(E_j z_1)(x) = 0, \quad (E_j z_2)(x) = 0, \quad \forall x \in \omega, \ j = 1, 2, 3, \ldots, \quad (2.32)$$

that is,

$$\sum_{k=1}^{\gamma_j} \langle z_1, \phi_{j,k} \rangle \phi_{j,k}(x) = 0, \quad \sum_{k=1}^{\gamma_j} \langle z_2, \phi_{j,k} \rangle \phi_{j,k}(x) = 0, \quad \forall x \in \omega, \ j = 1, 2, 3, \ldots. \quad (2.33)$$

Since the restrictions $\phi_{j,k}^\omega = \phi_{j,k}|_\omega$ of $\phi_{j,k}$ to $\omega$ are linearly independent functions on $\omega$, we get that

$$\langle z_1, \phi_{j,k} \rangle = 0, \quad \langle z_2, \phi_{j,k} \rangle = 0, \quad j = 1, 2, 3, \ldots; \ k = 1, 2, \ldots, \gamma_j. \quad (2.34)$$

Therefore,

$$P_j(z) = \begin{bmatrix} E_j(z_1) \\ E_j(z_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (2.35)$$

Hence, $z = \sum_{j=1}^{\gamma_j} P_j z = 0$, and the proof of the approximate controllability of the system (2.16) is completed.

Now, given the initial and the final states $z_0$ and $z_1$, we consider the sequence of controls

$$u_\alpha(\cdot) = B_{\alpha}^* T^*(\tau - \cdot)(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0)$$

$$= G^*(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0), \quad \alpha > 0. \quad (2.36)$$
Then,

\[ G_{\alpha} = GG^*(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0) \]
\[ = (\alpha I + GG^* - \alpha I)(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0) \] (2.37)
\[ = z_1 - T(\tau)z_0 - \alpha(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0). \]

From part (c) of Lemma 2.6, we know that

\[ \lim_{\alpha \to 0} \alpha(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0) = 0. \] (2.38)

Therefore,

\[ \lim_{\alpha \to 0} G_{\alpha} = z_1 - T(\tau)z_0, \] (2.39)

that is,

\[ \lim_{\alpha \to 0} \left\{ T(\tau)z_0 + \int_0^\tau T(\tau - s)B_{\omega}u_{\alpha}(s)\,ds \right\} = z_1. \] (2.40)

This completes the proof of the theorem. \( \square \)

The following basic theorem will be used to prove an important consequence of the foregoing theorem.

**Theorem 2.9** (see [7, Theorem 1.23, page 20]). Suppose \( \Omega \subset \mathbb{R}^n \) is open, nonempty, and connected set, and \( f \) is real analytic function in \( \Omega \) with \( f = 0 \) on a nonempty open subset \( \omega \) of \( \Omega \). Then, \( f \equiv 0 \) in \( \Omega \).

**Corollary 2.10.** If \( \phi_{j,k} \) are analytic functions on \( \Omega \), then for all open nonempty set \( \omega \subset \Omega \) and all \( \tau \geq 2/\pi(m - 1) \) the system (2.16) is approximately controllable on \( [0, \tau] \).

**Proof.** It is enough to prove that, for all open nonempty set \( \omega \subset \Omega \) the restrictions \( \phi_{j,k}^{\omega} = \phi_{j,k}|_{\omega} \) of \( \phi_{j,k} \) to \( \omega \) are linearly independent functions on \( \omega \), which follows directly from **Theorem 2.9**. \( \square \)

### 3. Applications

For the applications, we will use Corollary 2.10 and the following fact.

**Theorem 3.1** (see [3]). The eigenfunctions of the operator \(-\Delta\) with Dirichlet boundary conditions on \( \Omega \) are real analytic functions in \( \Omega \).
In this section, we will prove the approximate controllability of (1.7) and (1.8). Specifically, we will prove the following theorem.

**Theorem 3.2.** For all open nonempty set \( \omega \subset \Omega = [0,1] \), we have the following statements.

(a) For all \( \tau \geq 2 \) the system (1.7) is approximately controllable on \([0, \tau]\).

(b) For all \( \tau \geq 2/\pi \) the system (1.8) is approximately controllable on \([0, \tau]\).

**Proof.** Let \( X = L^2(\Omega) \) and consider the linear unbounded operator \(-\Delta : D(-\Delta) \subset X \to X\) with \(D(-\Delta) = H_0^1(\Omega) \cap H^2(\Omega)\). In this case, the eigenvalues and the eigenfunctions of \( A = -\Delta \) are given, respectively, by

\[
\lambda_j = j^2 \pi^2, \quad \phi_j(x) = \sqrt{2} \sin(j \pi x), \quad j = 1, 2, 3, \ldots,
\]

\[
Ax = -\Delta x = \sum_{j=1}^{\infty} \lambda_j \langle x, \phi_j \rangle \phi_j = \sum_{j=1}^{\infty} j^2 \pi^2 \langle x, \phi_j \rangle \phi_j. \tag{3.1}
\]

Then, \( m = 1 \) and \( \tau \geq 2 \). So, (a) follows from Corollary 2.10.

To prove (b), we consider the operator

\[
Ax = (-\Delta)^2 x = \sum_{j=1}^{\infty} \lambda_j^2 \langle x, \phi_j \rangle \phi_j = \sum_{j=1}^{\infty} j^4 \pi^4 \langle x, \phi_j \rangle \phi_j. \tag{3.2}
\]

Then, \( m = 2 \) and \( \tau \geq 2/\pi \). So, (b) follows from Corollary 2.10. \( \square \)

### 4. Final Remark

The result presented in this paper can be formulated in a more general setting. Indeed, we can consider the following second-order evolution equation in a general Hilbert space \( X \):

\[
\dot{y} + Ay = Cu(t), \quad t \in (0, \tau],
\]

\[
y(0) = y_0, \quad \dot{y}(0) = y_1, \tag{4.1}
\]

where, \( A : D(A) \subset X \to X \) is an unbounded linear operator in \( X \) with the spectral decomposition given by (1.2), the control \( u \in L^2(0, \tau; X) \) and \( C : X \to X \) is a linear and bounded operator (linear and continuous). In this case, the characteristic function set is a particular operator \( C \), and the following theorem is a generalization of Theorem 2.8.

**Theorem 4.1.** If the vectors \( C^* \phi_{j,k} \) are linearly independent in \( X \), then for all \( \tau \geq 2/\pi^{m-1} \) the system (4.1) is approximately controllable on \([0, \tau]\). Moreover, a sequence of controls steering the system (2.16) from initial state \( z_0 \) to an \( \varepsilon \) neighborhood of the final state \( z_1 \) at time \( \tau > 0 \) is given by

\[
u_a(t) = B^* T^*(\tau - t)(\alpha I + GG^*)^{-1}(z_1 - T(\tau)z_0), \tag{4.2}
\]
and the error of this approximation $E_\alpha$ is given by

$$E_\alpha = \alpha(aI + G G^*)^{-1}(z_1 - T(\tau)z_0),$$

where the operator $B$ is given by $B = \begin{bmatrix} 0 \\ C \end{bmatrix}$.

The novelty of this result is based on the fact that, it is general, rigorous, applicable, and easily comprehensible by those young mathematician who are located in places away from majors research center.

**Acknowledgment**

This work was supported by the CDCHT-ULA-project: C-1667-09-05-AA and BCV.

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