Research Article

Maximal Regularity for Flexible Structural Systems in Lebesgue Spaces

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We study abstract equations of the form $\lambda u'''(t) + u''(t) = c^2 Au(t) + c^2 \mu Au'(t) + f(t)$, $0 < \lambda < \mu$ which is motivated by the study of vibrations of flexible structures possessing internal material damping. We introduce the notion of $(\alpha; \beta; \gamma)$ -regularized families, which is a particular case of $(a; k)$ -regularized families, and characterize maximal regularity in L^p -spaces based on the technique of Fourier multipliers. Finally, an application with the Dirichlet-Laplacian in a bounded smooth domain is given.

1. Introduction

During the last few decades, the use of flexible structural systems had steadily increased importance. The study of a flexible aerospace structure involves problems of dynamical system theory governed by partial differential equations.

We consider here the problem of characterizing L^p -maximal regularity (or well-posedness) for a mathematical model of a flexible space structure like a thin uniform rectangular panel, for example, a solar cell array or a spacecraft with flexible attachments. This problem is motivated by both engineering and mathematical considerations.

The study of *vibrations* of flexible structures possessing internal material damping was first derived by Bose and Gorain [1]. The consideration of external forces leads to more general equations of the form

$$\alpha u'''(t) + u''(t) = \beta Au(t) + \gamma Au'(t) + f(t), \quad \alpha, \beta, \gamma \in \mathbb{R}, \quad (1.1)$$

where A is a closed linear operator acting in a Banach space X and f is an X -valued function. We emphasize that the abstract Cauchy problem associated with (1.1) is in general ill posed; see, for example, [2]. Also it is well known that in order to analyze well-posedness, a direct approach leads to better results than those obtained by a reduction to a first-order equation.

Maximal regularity in Hölder spaces for (1.1) has been recently characterized in [3]. In case $\alpha = 0$, there are more literatures. For example, stability of the solution was studied by Gorain in [4]. In [5], Gorain and Bose studied exact controllability and boundary stabilization. More recently, Batkai and Piazzera [6, page 188] have obtained the exact decay rate. We note that well-posedness in Lebesgue spaces in the case of a damped wave equation has been only recently considered by Chill and Srivastava in [7], and in Hölder spaces by Poblete [8]. We note that the class studied in [8] includes equations with delay. In particular, well-posedness of the homogeneous abstract Cauchy problem has been observed in [9] for $\alpha = 0$ under certain assumptions on A .

This paper is organized as follows. Section 2, collects results essentially contained in [10] and standard literature on R -boundedness and maximal regularity (see [11] and [12]). In Section 3 we study, by an operator theoretical method, sufficient conditions for existence of solutions for (1.1). We obtain two results: a description of the solution by means of certain regularized families (Proposition 3.1) and the existence of such families in the particular case of positive self-adjoint operators (Theorem 3.2). In Section 4, we succeed in characterizing well-posedness of (1.1) in terms of R -boundedness of a resolvent set which involves A (Theorem 4.2). This will be achieved in the Lebesgue spaces $L^p(\mathbb{R}, X)$, where X is a UMD space (see below the definition). The methods to obtain this goal are those incorporated in [13] where a similar problem in case of the first-order abstract Cauchy problem has been studied. Our main result (Theorem 4.2) is a combination of the well-known (and deep) result due to Weis [14] stated in Theorem 2.8 and a direct calculation involving the parameters α , β , and γ .

2. Preliminaries

Let $\alpha, \beta, \gamma > 0$ be given. In what follows we denote

$$\begin{aligned} k(t) &= \frac{1}{\alpha} \int_0^t (t-s)e^{-s/\alpha} ds = -\alpha + t + \alpha e^{-t/\alpha}, \quad t \in \mathbb{R}_+, \\ a(t) &= \beta k(t) + \frac{\gamma}{\alpha} \int_0^t e^{-s/\alpha} ds = -(\alpha\beta - \gamma) + \beta t + (\alpha\beta - \gamma)e^{-t/\alpha}, \quad t \in \mathbb{R}_+. \end{aligned} \tag{2.1}$$

In order to give an operator theoretical approach to (1.1) we introduce the following definition.

Definition 2.1. Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X . One calls A the generator of an (α, β, γ) -regularized family $\{R(t)\}_{t \geq 0} \subset \mathcal{B}(X)$ if the following conditions are satisfied.

- (R1) $R(t)$ is strongly continuous on $\mathbb{R}_+$ and $R(0) = 0$.
- (R2) $R(t)D(A) \subset D(A)$ and $AR(t)x = R(t)Ax$ for all $x \in D(A), t \geq 0$.

(R3) The following equation holds:

$$R(t)x = k(t)x + \int_0^t a(t-s)R(s)Ax \, ds \quad (2.2)$$

for all $x \in D(A)$, $t \geq 0$. In this case, $R(t)$ is called the (α, β, γ) -regularized family generated by A .

Remark 2.2. It is proved in [10], in the more general context of (a, k) -regularized families, that an operator A is the generator of an (α, β, γ) -regularized family if and only if there exists $\omega \geq 0$ and a strongly continuous function $R : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ such that $\{(\lambda^2 + \alpha\lambda^3)/(\beta + \gamma\lambda) : \operatorname{Re} \lambda > \omega\} \subset \rho(A)$ and

$$H(\lambda)x := \frac{1}{\beta + \gamma\lambda} \left(\frac{\lambda^2 + \alpha\lambda^3}{\beta + \gamma\lambda} - A \right)^{-1} x = \int_0^\infty e^{-\lambda t} R(t)x \, dt, \quad \operatorname{Re} \lambda > \omega, \quad x \in X. \quad (2.3)$$

Because of the uniqueness of the Laplace transform, we note that an (α, β, γ) -regularized family corresponds to an (a, k) -regularized family studied in [10]. In fact, we have

$$\hat{a}(\lambda) = \frac{\beta + \gamma\lambda}{\lambda^2 + \alpha\lambda^3}, \quad \hat{k}(\lambda) = \frac{1}{\lambda^2 + \alpha\lambda^3}, \quad \forall \operatorname{Re} \lambda > \omega. \quad (2.4)$$

As in the situation of C_0 -semigroups, we have diverse relations of an (α, β, γ) -regularized family and its generator. The following result is a direct consequence of [10, Proposition 3.1 and Lemma 2.2].

Proposition 2.3. *Let $R(t)$ be an (α, β, γ) -regularized family on X with generator A . Then the following hold.*

- (a) *For all $x \in D(A)$ one has $R(\cdot)x \in C^3(\mathbb{R}_+; X)$.*
- (b) *Let $x \in X$ and $t \geq 0$. Then $\int_0^t a(t-s)R(s)x \, ds \in D(A)$ and*

$$R(t)x = k(t)x + A \int_0^t a(t-s)R(s)x \, ds. \quad (2.5)$$

Results on perturbation, approximation, asymptotic behavior, representation, as well as ergodic-type theorems for (α, β, γ) -regularized families can be also deduced from the more general context of (a, k) -regularized families (see [10, 15–18]).

We will need the following results on Laplace transform (see [19, Theorem 2.5.1 and Corollary 2.5.2] for a detailed proof).

Lemma 2.4. *Suppose that $q : \mathbb{C}_+ \rightarrow \mathbb{C}$ is holomorphic and satisfies $\sup_{\operatorname{Re} \lambda > 0} |\lambda q(\lambda)| < \infty$ and let $b > 0$. Then there exists $f \in C(\mathbb{R}_+)$ with $\sup_{t > 0} |e^{-\omega t} t^{-b} f(t)| < \infty$ such that $q(\lambda) = \lambda^b \int_0^\infty e^{-\lambda t} f(t) dt$ for all $\operatorname{Re} \lambda > 0$.*

Lemma 2.5. Suppose that $q : \mathbb{C}_+ \rightarrow \mathbb{C}$ is holomorphic and satisfies $|\lambda q(\lambda)| + |\lambda^2 q'(\lambda)| \leq M$ for all $\operatorname{Re} \lambda > 0$. Then there exists a bounded function $f \in C(\mathbb{R}_+)$ such that $q(\lambda) = \int_0^\infty e^{-\lambda t} f(t) dt$ for all $\operatorname{Re} \lambda > 0$.

We introduce the means

$$\|(x_1, \dots, x_n)\|_R := \frac{1}{2^n} \sum_{\epsilon_j \in \{-1, 1\}^n} \left\| \sum_{j=1}^n \epsilon_j x_j \right\| \quad (2.6)$$

for $x_1, \dots, x_n \in X$.

Definition 2.6. Let X, Y be Banach spaces. A subset $\mathcal{T}$ of $\mathcal{B}(X, Y)$ is called R -bounded if there exists a constant $c \geq 0$ such that

$$\|(T_1 x_1, \dots, T_n x_n)\|_R \leq c \|(x_1, \dots, x_n)\|_R \quad (2.7)$$

for all $T_1, \dots, T_n \in \mathcal{T}, x_1, \dots, x_n \in X, n \in \mathbb{N}$. The least c such that (2.7) is satisfied is called the R -bound of $\mathcal{T}$ and is denoted as $R(\mathcal{T})$.

The notion of R -boundedness was implicitly introduced and used by Bourgain [20] and later on also by Zimmermann [21]. Explicitly it is due to Berkson and Gillespie [22] and to Clément et al. [23].

R -boundedness clearly implies boundedness. If $X = Y$, the notion of R -boundedness is strictly stronger than boundedness unless the underlying space is isomorphic to a Hilbert space [24, Proposition 1.17]. Some useful criteria for R -boundedness are provided in [11, 24].

Remark 2.7. (a) Let $\mathcal{S}, \mathcal{T} \subset \mathcal{B}(X, Y)$ be R -bounded sets, then $\mathcal{S} + \mathcal{T} := \{S + T : S \in \mathcal{S}, T \in \mathcal{T}\}$ is R -bounded.

(b) Let $\mathcal{T} \subset \mathcal{B}(X, Y)$ and $\mathcal{S} \subset \mathcal{B}(Y, Z)$ be R -bounded sets, then $\mathcal{S} \cdot \mathcal{T} := \{S \cdot T : S \in \mathcal{S}, T \in \mathcal{T}\} \subset \mathcal{B}(X, Z)$ is R -bounded and

$$R(\mathcal{S} \cdot \mathcal{T}) \leq R(\mathcal{S}) \cdot R(\mathcal{T}). \quad (2.8)$$

(c) Also, each subset $M \subset \mathcal{B}(X)$ of the form $M = \{\lambda I : \lambda \in \Omega\}$ is R -bounded whenever $\Omega \subset \mathbb{C}$ is bounded.

We recall that those Banach spaces X for which the Hilbert transform is bounded on $L^p(\mathbb{R}, X)$, for some $p \in (1, \infty)$, are called UMD spaces. For more information and details on the Hilbert transform and the UMD Banach spaces we refer to [12]. Examples of UMD spaces include Hilbert spaces, Sobolev spaces $W_p^s(\Omega)$, $1 < p < \infty$ (see [25]), Lebesgue spaces $L^p(\Omega, \mu)$, $1 < p < \infty$, $L^p(\Omega, \mu; X)$, $1 < p < \infty$, when X is a UMD space, and the Schatten-von Neumann classes $C_p(H)$, $1 < p < \infty$ of operators on Hilbert spaces.

After these preliminaries, we state the following operator-valued Fourier multiplier theorem. It is fundamental in our treatment. A proof can be founded in [11].

Theorem 2.8. Suppose that X is a UMD space and let $1 < p < \infty$. Let $M \in C^1(\mathbb{R} \setminus \{0\}; \mathcal{B}(X))$ be such that the following conditions are satisfied.

- (i) The set $\{M(\rho)\}_{\rho \in \mathbb{R} \setminus \{0\}}$ is R -bounded.
- (ii) The set $\{\rho M'(\rho)\}_{\rho \in \mathbb{R} \setminus \{0\}}$ is R -bounded.

Then the operator T defined by

$$Tf = \left(M(\cdot) \left[\widehat{f}(\cdot) \right] \right)^\vee \quad \text{where } f \in \mathcal{S}(X) \quad (2.9)$$

extends to a bounded operator from $L^p(\mathbb{R}, X)$ to $L^p(\mathbb{R}, X)$.

3. Existence of Solutions

Let $\alpha, \beta, \gamma \in (0, \infty)$. Consider the equation

$$u''(t) + \alpha u'''(t) = \beta Au(t) + \gamma Au'(t) + f(t), \quad (3.1)$$

with initial conditions $u(0) = u'(0) = u''(0) = 0$, where A is the generator of an (α, β, γ) -regularized family $R(t)$. By a solution of (3.1) we understand a function $u \in C(\mathbb{R}_+; D(A)) \cap C^3(\mathbb{R}_+; X)$ such that $u' \in C(\mathbb{R}_+; D(A))$ and verify (3.1).

Proposition 3.1. Let $R(t)$ be an (α, β, γ) -regularized family on X with generator A . If $f \in L_{\text{loc}}^1(\mathbb{R}_+, D(A^2))$, then $u(t)$ given by

$$u(t) = \int_0^t R(t-s)f(s)ds, \quad t \geq 0 \quad (3.2)$$

is a solution of (3.1).

Proof. Given that $x \in D(A)$, we obtain from Proposition 2.3 that $R(\cdot)x$, and hence u , is of class $C^3(\mathbb{R}_+, X)$. For all $x \in D(A)$, we have

$$R'(t)x = \left(1 - e^{-t/\alpha}\right)x + \int_0^t \left[\beta + (\gamma/\alpha - \beta)e^{-(t-s)/\alpha}\right]R(s)Ax \, ds. \quad (3.3)$$

If $x \in D(A^2)$, then $R'(t)x \in D(A)$. Moreover,

$$\begin{aligned} R''(t)x &= \frac{1}{\alpha}e^{-t/\alpha}x + \frac{\gamma}{\alpha}R(t)Ax + \int_0^t \left(\frac{\beta}{\alpha} - \frac{\gamma}{\alpha^2}\right)e^{-(t-s)/\alpha}AR(s)x \, ds, \\ R'''(t)x &= -\frac{1}{\alpha^2}e^{-t/\alpha}x + \frac{\gamma}{\alpha}R'(t)Ax + \frac{\beta}{\alpha}R(t)Ax - \frac{\gamma}{\alpha^2}AR(t)x \\ &\quad + \int_0^t \left(\frac{\gamma}{\alpha^3} - \frac{\beta}{\alpha^2}\right)e^{-(t-s)/\alpha}AR(s)x \, ds. \end{aligned} \quad (3.4)$$

Since $f \in L^1_{\text{loc}}(\mathbb{R}_+, D(A^2))$, from (3.2), we have that $u(t), u'(t) \in D(A)$ and

$$\begin{aligned} u'(t) &= \int_0^t R'(t-s)f(s)ds, \\ u''(t) &= \int_0^t R''(t-s)f(s)ds, \\ u'''(t) &= R''(0)f(t) + \int_0^t R'''(t-s)f(s)ds. \end{aligned} \quad (3.5)$$

Hence,

$$\begin{aligned} &u''(t) + \alpha u'''(t) - \beta Au(t) - \gamma Au'(t) \\ &= \int_0^t R''(t-s)f(s)ds + f(t) + \alpha \int_0^t R'''(t-s)f(s)ds \\ &\quad - \beta A \int_0^t R(t-s)f(s)ds - \gamma A \int_0^t R'(t-s)f(s)ds. \end{aligned} \quad (3.6)$$

By the other side, for all $x \in D(A^2)$, we obtain

$$\begin{aligned} &R''(t)x + \alpha R'''(t)x - \beta AR(t)x - \gamma AR'(t)x \\ &= \frac{1}{\alpha}e^{-t/\alpha}x + \frac{\gamma}{\alpha}AR(t)x + \int_0^t \left(\frac{\beta}{\alpha} - \frac{\gamma}{\alpha^2} \right) e^{-(t-s)/\alpha} AR(s)x ds - \frac{1}{\alpha}e^{-t/\alpha}x + \gamma AR'(t)x \\ &\quad + \beta AR(t)x - \frac{\gamma}{\alpha}AR(t)x + \int_0^t \left(\frac{\gamma}{\alpha^2} - \frac{\beta}{\alpha} \right) e^{-(t-s)/\alpha} AR(s)x ds - \beta AR(t)x - \gamma AR'(t)x \\ &= 0. \end{aligned} \quad (3.7)$$

Since $f(t) \in D(A^2)$ and A is closed, from (3.6) we conclude that $u(t)$ verify (3.1). $\square$

The following remarkable result provides a wide class of generators of (α, β, γ) -regularized families. In what follows we denote

$$\varphi(\lambda) := \frac{1}{\hat{a}(\lambda)} = \frac{\lambda^2(1 + \alpha\lambda)}{\beta + \gamma\lambda}. \quad (3.8)$$

Theorem 3.2. *Let $-B$ be a positive self-adjoint operator on a Hilbert space H such that*

$$\alpha\beta \leq \gamma. \quad (3.9)$$

Then B is the generator of a bounded (α, β, γ) -regularized family on H .

Proof. Since $-B$ is a positive self-adjoint operator in H , the spectrum $\sigma(B)$ is a subset of the negative real axis and the resolvent operator $(\mu - B)^{-1}$ is defined at least for all negative non real μ . Let $\lambda \in \mathbb{C}$ such that $\operatorname{Re} \lambda > 0$. If $\operatorname{Im} \varphi(\lambda) \neq 0$, then clearly $\varphi(\lambda) \in \rho(B)$. If $\operatorname{Im} \varphi(\lambda) = 0$, then we claim that $\operatorname{Re} \varphi(\lambda) > 0$. In fact, for $\lambda = a + bi \in \mathbb{C}$, $a > 0$, with a direct computation we obtain

$$\begin{aligned} \operatorname{Re} \varphi(\lambda) &= \frac{(a^2 - b^2)(1 + \alpha a)(\beta + \gamma a) - 2ab^2\alpha(\beta + \gamma a) + \alpha\gamma b^2(a^2 - b^2) + 2ab^2(1 + \alpha a)}{(\beta + \gamma a)^2 + \gamma^2 b^2}, \\ \operatorname{Im} \varphi(\lambda) &= \frac{ab(a^2 - b^2)(\beta + \gamma a) + 2ab(1 + \alpha a)(\beta + \gamma a) - \gamma b(a^2 - b^2)(1 + \alpha a) + 2ab^3\alpha\gamma}{(\beta + \gamma a)^2 + \gamma^2 b^2}. \end{aligned} \quad (3.10)$$

Note that $\operatorname{Im} \varphi(\lambda) = 0$ if and only if $b = 0$ or $\alpha(a^2 - b^2)(\beta + \gamma a) + 2a(1 + \alpha a)(\beta + \gamma a) - \gamma(a^2 - b^2)(1 + \alpha a) + 2ab^2\alpha\gamma = 0$.

Since $\alpha\beta \leq \gamma$, we have that

$$\begin{aligned} &\alpha(a^2 - b^2)(\beta + \gamma a) + 2a(1 + \alpha a)(\beta + \gamma a) - \gamma(a^2 - b^2)(1 + \alpha a) + 2ab^2\alpha\gamma \\ &= 2\alpha\gamma ab^2 + b^2(\gamma - \alpha\beta) + \gamma a^2 + 3\alpha\beta a^2 + 2\beta a + 2\alpha\gamma a^2 \\ &> 0. \end{aligned} \quad (3.11)$$

Hence, $\operatorname{Im} \varphi(\lambda) = 0$ if and only if $b = 0$. Since $a > 0$, a direct calculation gives

$$\operatorname{Re} \varphi(\lambda) = \frac{a^2(1 + \alpha a)}{\beta + \gamma a} > 0, \quad (3.12)$$

proving the claim. We conclude that $\varphi(\lambda) \in \rho(B)$ for all $\operatorname{Re} \lambda > 0$. Hence (see Kato [26, Section V.3.5]),

$$\|(\varphi(\lambda) - B)^{-1}\| = \frac{1}{\operatorname{dist}(\varphi(\lambda), \sigma(B))} \quad \forall \operatorname{Re} \lambda > 0. \quad (3.13)$$

Note that

$$\sup_{\operatorname{Re} \lambda > 0} \left(\frac{|\lambda|^2 + 1}{\operatorname{dist}(\varphi(\lambda), \sigma(B))} \right) < M, \quad (3.14)$$

since $\operatorname{dist}(\varphi(\lambda), \sigma(B))$ has order $|\lambda|^2$. Define $Q(\lambda) = (1/(\beta + \gamma\lambda)) (\varphi(\lambda) - B)^{-1}$. We have by (3.14) and (3.13) that for all $\operatorname{Re} \lambda > 0$

$$\|\lambda Q(\lambda)\| = \left\| \frac{\lambda}{(\beta + \gamma\lambda)} (\varphi(\lambda) - B)^{-1} \right\| \leq \frac{|\lambda|}{|\beta + \gamma\lambda|} \frac{1}{\operatorname{dist}(\varphi(\lambda), \sigma(B))} < M. \quad (3.15)$$

On the other hand,

$$\lambda^2 Q'(\lambda) = \frac{-\gamma\lambda}{\beta + \gamma\lambda} [\lambda Q(\lambda)] + [\lambda Q(\lambda)] \left[\lambda^2 (\varphi(\lambda) - B)^{-1} \right] \left[\lambda \frac{\hat{a}(\lambda)'}{\hat{a}(\lambda)} \right] \frac{1}{\lambda^2 \hat{a}(\lambda)}, \quad (3.16)$$

where

$$\frac{1}{\lambda^2 \hat{a}(\lambda)} = \frac{1 + \alpha\lambda}{\beta + \gamma\lambda}, \quad \lambda \frac{\hat{a}(\lambda)'}{\hat{a}(\lambda)} = -\frac{2\alpha\gamma\lambda^2 + (\gamma + 3\alpha\beta)\lambda + 2\beta}{(1 + \alpha\lambda)(\beta + \gamma\lambda)} \quad (3.17)$$

and, by (3.14),

$$\left\| \lambda^2 (\varphi(\lambda) - B)^{-1} \right\| \leq \frac{|\lambda^2|}{\text{dist}(\varphi(\lambda), \sigma(B))} < M \quad (3.18)$$

for all $\text{Re } \lambda > 0$. We conclude that $\sup_{\text{Re } \lambda > 0} \|\lambda^2 Q'(\lambda)\| < \infty$.

By Lemma 2.5 there exists a strongly continuous family $R(t)$ such that $\|R(t)\| \leq K$ and $Q(\lambda) = \hat{R}(\lambda)$ for $\text{Re } \lambda > 0$. In consequence, for all $\text{Re } \lambda > 0$ we have

$$\hat{R}(\lambda) = \frac{\varphi(\lambda)}{\lambda^2(1 + \alpha\lambda)} (\varphi(\lambda) - B)^{-1} = \frac{1}{\beta + \gamma\lambda} \left(\frac{\lambda^2 + \alpha\lambda^3}{\beta + \gamma\lambda} - B \right)^{-1}, \quad (3.19)$$

and, by Remark 2.2, it shows that $R(t)$ is a bounded (α, β, γ) -regularized family generated by B . $\square$

Since it is a known fact that the Dirichlet-Laplacian operator is a self-adjoint operator on $L^2(\Omega)$ and $\sigma(\Delta) \subset (-\infty, 0)$, we obtain the following corollary.

Corollary 3.3. *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and assume that $\alpha\beta \leq \gamma$. Then the Dirichlet-Laplacian operator Δ with domain $H^2(\Omega) \cap H_0^1(\Omega)$ is the generator of an (α, β, γ) -regularized family on $X = L^2(\Omega)$.*

Remark 3.4. In Theorem 3.2 the condition $\alpha\beta \leq \gamma$ is fundamental to have $\varphi(\lambda) \in \rho(B)$ for all $\lambda \in \mathbb{C}$ with $\text{Re } \lambda > 0$, which is the key in the proof. Figure 1 is the typical situation, where we have mapped by φ the lines $\text{Re}(\lambda) = 1, 2$, and 3 with $\alpha = 3, \beta = 1$, and $\gamma = 4$.

Note that in case $\alpha\beta > \gamma$ it can happen that $\varphi(\lambda) \in \sigma(B)$. For example, taking $\alpha = 1, \beta = 5$, and $\gamma = 1$, we obtain Figure 2 of $\varphi(\lambda)$ for $\text{Im}(\lambda) \in \mathbb{R}$ and $\text{Re}(\lambda) = 1$

4. L^p -Well-Posedness

Having presented preliminary material on R -boundedness and Fourier multipliers, we will now show how these tools can be used to handle (3.1). Our main result give concrete conditions on the operator A under which (3.1) has L^p -maximal regularity.

The definition of L^p -maximal regularity which we investigate in this section is given as follows.

Figure 1

Figure 2

Definition 4.1. One says that (3.1) has L^p -maximal regularity (or is L^p -well posed) on $\mathbb{R}_+$ if for each $f \in L^p(\mathbb{R}_+, X)$ there is a unique function $u \in W^{3,p}(\mathbb{R}_+, X) \cap W^{1,p}(\mathbb{R}_+, [D(A)]) \cap W^p(\mathbb{R}_+, [D(A)])$ such that (3.1) holds a.e.

The following is the main abstract result of this section. It completely characterizes the maximal regularity of solutions for (3.1) in Lebesgue spaces.

Theorem 4.2. Let X be a UMD space, $1 < p < \infty$, and let A be the generator of a bounded (α, β, γ) -regularized family $R(t)$. The following statements are equivalent.

- (i) Equation (3.1) has L^p -maximal regularity on $\mathbb{R}_+$.
- (ii) $b(\rho) := -\rho^2((1 + i\alpha\rho)/(\beta + i\gamma\rho)) \in \rho(A)$ for all $\rho \in \mathbb{R} \setminus \{0\}$ and the set

$$\left\{ \frac{\rho^3}{\beta + i\gamma\rho} R(b(\rho), A) \right\}_{\rho \in \mathbb{R} \setminus \{0\}} \text{ is } R\text{-bounded.} \quad (4.1)$$

Proof. (i) $\Rightarrow$ (ii). By (3.1) and Definition 4.1 together with Proposition 3.1, the convolution operator with kernel

$$K_4(t) := R'''(t)\chi_{(0,\infty)}(t), \quad t \in \mathbb{R}, \quad (4.2)$$

is a bounded operator from $L^p(\mathbb{R}, X)$ to $L^p(\mathbb{R}, X)$. Note that the Fourier transform $\tilde{R}(\rho)$ exists for $\rho \neq 0$ because $R(t)$ is bounded and $\hat{R}(\lambda)(\operatorname{Re} \lambda > 0)$ can be analytically extended from $\operatorname{Re} \lambda > 0$ to the imaginary axis. Then the symbol of this convolution operator is given by

$$M(\rho) = \frac{\rho^3}{\beta + i\gamma\rho} R(b(\rho), A), \quad \rho \in \mathbb{R} \setminus \{0\}, \quad (4.3)$$

and the conclusion follows from [11, Proposition 3.17].

(ii) $\Rightarrow$ (i). Define $N(\rho) := (1/(\beta + i\gamma\rho))R(b(\rho), A)$ and

$$N_1(\rho) := AN(\rho). \quad (4.4)$$

We check that the set $\{N_1(\rho)\}_{\rho \in \mathbb{R} \setminus \{0\}}$ is R -bounded.

Since $(b(\rho) - A)R(b(\rho), A) = I$, we have that $AR(b(\rho), A) = b(\rho)R(b(\rho), A) - I$. Replacing in (4.4)

$$N_1(\rho) = \frac{b(\rho)}{\beta + i\gamma\rho} R(b(\rho), A) - \frac{1}{\beta + i\gamma\rho} I = -\frac{1 + i\alpha\rho}{\beta + i\gamma\rho} \rho^2 N(\rho) - \frac{1}{\beta + i\gamma\rho} I. \quad (4.5)$$

Note that

$$\begin{aligned} \left| \frac{1 + i\alpha\rho}{\beta + i\gamma\rho} \right|^2 &= \frac{1 + \alpha^2\rho^2}{\beta^2 + \gamma^2\rho^2} < \frac{1}{\beta^2} + \frac{\alpha^2}{\gamma^2}, \\ \left| \frac{1}{\beta + i\gamma\rho} \right|^2 &= \frac{1}{\beta^2 + \gamma^2\rho^2} < \frac{1}{\beta^2}. \end{aligned} \quad (4.6)$$

Since the sum of R -bounded sets is R -bounded, see [11], we obtain that $\{N_1(\rho)\}$ is R -bounded.

We now check that the set $\{\rho N'_1(\rho)\}_{\rho \in \mathbb{R} \setminus \{0\}}$ is R -bounded. With a direct computation, we obtain

$$\begin{aligned} N'_1(\rho) &= b'(\rho)N(\rho) + b(\rho)N'(\rho) + \frac{i\gamma}{(\beta + i\gamma\rho)^2} I \\ &= \frac{2\alpha\gamma}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) - \frac{\gamma + 3\alpha\beta}{(\beta + i\gamma\rho)^2} i\rho^2 N(\rho) - \frac{2\beta}{(\beta + i\gamma\rho)^2} \rho N(\rho) \\ &\quad - \frac{i\gamma}{\beta + i\gamma\rho} b(\rho)N(\rho) - \frac{2\alpha\gamma\rho^2 - (\gamma + 3\alpha\beta)i\rho - 2\beta}{\beta + i\gamma\rho} \rho b(\rho)N(\rho)N(\rho) + \frac{i\gamma}{(\beta + i\gamma\rho)^2} I. \end{aligned} \quad (4.7)$$

Hence

$$\begin{aligned}
\rho N'_1(\rho) &= \frac{2\alpha\gamma\rho}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) - \frac{\gamma + 3\alpha\beta}{(\beta + i\gamma\rho)^2} i \rho^3 N(\rho) - \frac{2\beta}{(\beta + i\gamma\rho)^2} \rho^2 N(\rho) + \frac{i\gamma - \alpha\gamma\rho}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) \\
&\quad + \left(2\alpha\gamma\rho^2 - (\gamma + 3\alpha\beta)i\rho - 2\beta\right) \frac{1 + i\alpha\rho}{(\beta + i\gamma\rho)^2} \rho^4 N(\rho) N(\rho) + \frac{i\gamma\rho}{(\beta + i\gamma\rho)^2} I \\
&= \frac{2\alpha\gamma\rho}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) - \frac{\gamma + 3\alpha\beta}{(\beta + i\gamma\rho)^2} i \rho^3 N(\rho) - \frac{2\beta}{(\beta + i\gamma\rho)^2} \rho^2 N(\rho) + \frac{i\gamma - \alpha\gamma\rho}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) \\
&\quad + 2\alpha\gamma \frac{1 + i\alpha\rho}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) \rho^3 N(\rho) - (\gamma + 3\alpha\beta) \frac{i - \alpha\rho}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) \rho^2 N(\rho) \\
&\quad - 2\beta \frac{1 + i\alpha\rho}{\beta + i\gamma\rho} \rho^3 N(\rho) \rho N(\rho) + \frac{i\gamma\rho}{(\beta + i\gamma\rho)^2} I \\
&= \frac{\alpha\gamma\rho - 3\alpha\beta i}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) - \frac{2\beta}{(\beta + i\gamma\rho)^2} \rho^2 N(\rho) + 2\alpha\gamma \frac{1 + i\alpha\rho}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) \rho^3 N(\rho) \\
&\quad - (\gamma + 3\alpha\beta) \frac{i - \alpha\rho}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) \rho^2 N(\rho) - 2\beta \frac{1 + i\alpha\rho}{\beta + i\gamma\rho} \rho^3 N(\rho) \rho N(\rho) + \frac{i\gamma\rho}{(\beta + i\gamma\rho)^2} I.
\end{aligned} \tag{4.8}$$

Since the set $\{\rho^3 N(\rho)\}$ is R -bounded and the complex functions appearing in the above equality are bounded, we obtain the claim from the fact that the sum of R -bounded sets is again R -bounded. We employ now Theorem 2.8 to conclude that the operator T_1 defined by

$$T_1 f = \left(N_1(\cdot) \left[\widehat{f}(\cdot) \right] \right)^\vee \quad \text{where } f \in \mathcal{S}(X) \tag{4.9}$$

extends to a bounded operator from $L^p(\mathbb{R}, X)$ to $L^p(\mathbb{R}, X)$.

Define

$$N_2(\rho) := \frac{\rho}{\beta + i\gamma\rho} AR(b(\rho), A). \tag{4.10}$$

We will prove that the sets $\{N_2(\rho)\}_{\rho \in \mathbb{R} \setminus \{0\}}$ and $\{\rho N'_2(\rho)\}_{\rho \in \mathbb{R} \setminus \{0\}}$ are R -bounded.

In fact, note that $N_2(\rho) = \rho N_1(\rho) = -((1 + i\alpha\rho)/(\beta + i\gamma\rho))\rho^3 N(\rho) - (\rho/(\beta + i\gamma\rho))I$. Hence the set $\{N_2(\rho)\}$ is R -bounded. Moreover, we have

$$\begin{aligned} \rho N'_2(\rho) &= \rho^2 N'_1(\rho) + \rho N_1(\rho) \\ &= \frac{\alpha\gamma\rho^2 - 3\alpha\beta\rho i}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) - \frac{2\beta}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) + 2\alpha\gamma \frac{\rho + i\alpha\rho^2}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) \rho^3 N(\rho) \\ &\quad - (\gamma + 3\alpha\beta) \frac{i - \alpha\rho}{(\beta + i\gamma\rho)^2} \rho^3 N(\rho) \rho^3 N(\rho) - 2\beta \frac{1 + i\alpha\rho}{\beta + i\gamma\rho} \rho^3 N(\rho) \rho^2 N(\rho) \\ &\quad + \frac{i\gamma\rho^2}{(\beta + i\gamma\rho)^2} I + N_2(\rho), \end{aligned} \tag{4.11}$$

obtaining that the set $\{\rho N'_2(\rho)\}_{\rho \in \mathbb{R} \setminus \{0\}}$ is R -bounded. By Theorem 2.8 we conclude that the operator T_2 defined by

$$T_2 f = \left(N_2(\cdot) \left[\widehat{f}(\cdot) \right] \right)^\vee \quad \text{where } f \in \mathcal{S}(X) \tag{4.12}$$

extends to a bounded operator from $L^p(\mathbb{R}, X)$ to $L^p(\mathbb{R}, X)$.

Finally, define

$$N_3(\rho) := \frac{\rho^2}{\beta + i\gamma\rho} R(b(\rho), A) = \rho^2 N(\rho). \tag{4.13}$$

The set $\{N_3(\rho)\}_{\rho \in \mathbb{R} \setminus \{0\}}$ is R -bounded from hypothesis and also note that the set $\{\rho N'_3(\rho)\}_{\rho \in \mathbb{R} \setminus \{0\}}$ is R -bounded, since

$$\begin{aligned} \rho N'_3(\rho) &= 2\rho^2 N(\rho) - \frac{i\gamma}{\beta + i\gamma\rho} \rho^3 N(\rho) - \frac{2\alpha\gamma}{\beta + i\gamma\rho} \rho^3 N(\rho) \rho^3 N(\rho) \\ &\quad + \frac{\gamma + 3\alpha\beta}{\beta + i\gamma\rho} i \rho^3 N(\rho) \rho^2 N(\rho) + \frac{2\beta}{\beta + i\gamma\rho} \rho N(\rho) \rho^3 N(\rho). \end{aligned} \tag{4.14}$$

Again by Theorem 2.8 we conclude that the operator T_3 defined by

$$T_3 f = \left(N_3(\cdot) \left[\widehat{f}(\cdot) \right] \right)^\vee \quad \text{where } f \in \mathcal{S}(X) \tag{4.15}$$

extends to a bounded operator from $L^p(\mathbb{R}, X)$ to $L^p(\mathbb{R}, X)$. From (4.9), (4.12), and (4.15) and since it is clear that (3.1) has L^p -maximal regularity if the convolution operator with each one of the kernels

$$K_1(t) := AR(t)\chi_{(0,\infty)}(t), \quad K_2(t) := AR'(t)\chi_{(0,\infty)}(t), \quad K_3(t) := R''(t)\chi_{(0,\infty)}(t), \quad t \in \mathbb{R}, \quad (4.16)$$

is a bounded operator from $L^p(\mathbb{R}, X)$ to $L^p(\mathbb{R}, X)$ (see [11]), we conclude (i) and the proof is complete. $\square$

Of course, R -boundedness in (4.1) can be replaced by boundedness in case $X = H$ is a Hilbert space.

Corollary 4.3. *The solution u of (3.1), under the conditions given by Theorem 4.2, satisfies the following maximal regularity property: $u, u' \in L^p(\mathbb{R}_+; [D(A)])$ and $Au, Au', u'', u''' \in L^p(\mathbb{R}_+; X)$. Moreover, there exists a constant $C > 0$ independent of $f \in L^p(\mathbb{R}_+; X)$ such that*

$$\|u\|_p + \|u'\|_p + \|u''\|_p + \|u'''\|_p + \|Au\|_p + \|Au'\|_p \leq C\|f\|_p. \quad (4.17)$$

The proof follows by the closed-graph theorem.

As an example, we consider for $A = \Delta$ the vibration equation subject to the action of an external force. Explicitly, we consider

$$\begin{aligned} v_{tt}(t, x) + \lambda v_{ttt}(t, x) &= c^2(\Delta v(t, x) + \mu \Delta v_t(t, x)) + f(t, x) \text{ in }]0, T] \times \Omega, \\ v(t, x) &= 0 \text{ on }]0, T] \times \Omega, \\ v(0, x) &= 0 \text{ in } \Omega, \\ v_t(0, x) &= 0 \text{ in } \Omega, \\ v_{tt}(0, x) &= 0 \text{ in } \Omega \end{aligned} \quad (4.18)$$

in a smooth bounded region $\Omega \subset \mathbb{R}^n$. Also, we assume that $f \in L^2(\mathbb{R}; L^2(\mathbb{R}^n))$. We have the following application in the Hilbert space setting.

Theorem 4.4. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$. Suppose that $0 < \lambda < \mu$. Then the initial value problem (4.18) defined on $L^2(\Omega)$ with Dirichlet boundary conditions has L^2 -maximal regularity on $\mathbb{R}_+$.*

Proof. Let $\alpha = \lambda$, $\beta = c^2$, and $\gamma = c^2\mu$ and note that $\alpha\beta < \gamma$ if and only if $\lambda < \mu$. By Corollary 3.3 we conclude that Δ generates a bounded (α, β, γ) -regularized family on $L^2(\Omega)$.

Figure 3

Note that we have $b(\rho) = -\rho^2((1 + i\alpha\rho)/(\beta + i\gamma\rho)) \in \rho(\Delta)$ and there exists a constant $C > 0$ such that

$$\begin{aligned} \left\| \frac{\rho^3}{\beta + i\gamma\rho} R(b(\rho), \Delta) \right\| &= \left\| \rho \frac{b(\rho)}{1 + i\alpha\rho} (b(\rho) - \Delta)^{-1} \right\| \\ &= \frac{|\rho|}{|1 + i\alpha\rho|} \frac{|b(\rho)|}{\text{dist}(b(\rho), \lambda_1(\Omega))} \leq C, \end{aligned} \quad (4.19)$$

for all $\rho \in \mathbb{R}$. Here $\lambda_1(\Omega)$ is the first eigenvalue of the Dirichlet-Laplacian. Hence, by Theorem 4.2 the assertion follows. $\square$

Remark 4.5. In Figure 3, we show $b(\rho)$ in case $\lambda = 3$, $\mu = 4$, and $c^2 = 1$.

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