Research Article

On Multiple Generalized $w$-Genocchi Polynomials and Their Applications

Lee-Chae Jang
Department of Mathematics and Computer Science, Konkuk University, Chungju, Chungcheongbuk-do 380-701, Republic of Korea

Correspondence should be addressed to Lee-Chae Jang, leechae.jang@kku.ac.kr

Received 26 September 2010; Accepted 3 December 2010

Academic Editor: Ben T. Nohara

Copyright © 2010 Lee-Chae Jang. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We define the multiple generalized $w$-Genocchi polynomials. By using fermionic $p$-adic invariant integrals, we derive some identities on these generalized $w$-Genocchi polynomials, for example, fermionic $p$-adic integral representation, Witt’s type formula, explicit formula, multiplication formula, and recurrence formula for these $w$-Genocchi polynomials.

1. Introduction

Let $p$ be a fixed odd prime number. Throughout this paper, $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ will, respectively, denote the ring of integers, the ring of $p$-adic integers, the field of $p$-adic rational numbers, the complex number field, and the $p$-adic completion of the algebraic closure of $\mathbb{Q}_p$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$.

The $q$-basic natural numbers are defined by

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}$$ (1.1)

for $n \in \mathbb{N}$, and the binomial coefficient is defined as

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1) \cdots (n-k+1)}{k!}.$$ (1.2)
The binomial formulas are well known that
\[(1 - b)^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i b^i, \quad \frac{1}{(1 - b)^n} = \sum_{i=0}^{n} \binom{n + i - 1}{i} b^i \tag{1.3}\]
(see, [1, 2]). When one talks of \(q\)-extension, \(q\) is variously considered as an indeterminate, a complex number \(q \in \mathbb{C}\), or a \(p\)-adic number \(q \in \mathbb{C}_p\). If \(q \in \mathbb{C}\), one normally assumes that \(|q| < 1\). If \(q \in \mathbb{C}_p\), one normally assumes that \(|q - 1|_p < 1\). We use the notation
\[\left[ x \right]_q = \frac{1 - q^x}{1 - q}, \quad \left[ x \right]_{-q} = \frac{1 - (-q)^x}{1 + q}, \tag{1.4}\]
see [1–13] for all \(x \in \mathbb{Z}_p\). Note that \(\lim_{q \to 1} [x]_q = x\) for \(x \in \mathbb{Z}_p\) in presented \(p\)-adic case.

Let \(UD(\mathbb{Z}_p)\) be denoted by the set of uniformly differentiable function on \(\mathbb{Z}_p\). For \(f \in UD(\mathbb{Z}_p)\), an invariant \(p\)-adic \(q\)-integral on \(\mathbb{Z}_p\) is defined as
\[I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{n \to \infty} \frac{1 + q}{1 + q^p} \sum_{x=0}^{p^n-1} f(x) (-q)^x. \tag{1.5}\]

Thus, we have the following integral relation:
\[\lim_{q \to 1} q I_{-q}(f_1) + I_{-q}(f) = (1 + q)f(0), \tag{1.6}\]
where \(f_1(x) = f(x + 1)\), and the fermionic \(p\)-adic invariant integral relation:
\[I_{-1}(f) = \lim_{q \to 1} I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x), \tag{1.7}\]
\[I_{-1}(f_1) + I(f) = 2f(0). \tag{1.8}\]

Now, we recall that the definitions of \(w\)-Euler polynomials and \(w\)-Genocchi polynomials are defined as
\[\frac{2e^{x t}}{we^t + 1} = \sum_{n=0}^{\infty} E_{n,w}(x) \frac{t^n}{n!},\]
\[\frac{2te^{x t}}{we^t + 1} = \sum_{n=0}^{\infty} G_{n,w}(x) \frac{t^n}{n!}, \quad t \in \mathbb{R}, \quad w \in \mathbb{C}, \tag{1.9}\]
with \(|1 - w|_p < 1\), respectively. In the special case \(x = 0\), \(E_{n,w}(0) = E_{n,w}\), and \(G_{n,w}(0) = G_{n,w}\) are called \(w\)-Euler numbers and \(w\)-Genocchi numbers (see [2, 9]).
In [13], Bayard and Simsek have studied multiple generalized Bernoulli polynomials as follows:

\[
\prod_{j=1}^{r} \left( \frac{a_j t + \log(w^{a_j})}{(we^t)^{a_j} - 1} \right) e^t = \sum_{n=0}^{\infty} B_{n,w}^{(r)}(x; a_1, \ldots, a_r) \frac{t^n}{n!}, \quad |t + \log(|w|)| < \min \left\{ \frac{\pi}{a_1}, \ldots, \frac{\pi}{a_p} \right\},
\]

where \(a_1, \ldots, a_r\) are strictly positive real numbers.

The purpose of this paper is to define another construction of multiple generalized \(w\)-Genocchi polynomials and numbers, which are different from multiple generalized Bernoulli polynomials and numbers in [13]. By using fermionic \(p\)-adic invariant integrals, we derive some identities on these generalized \(w\)-Genocchi polynomials, for example, fermionic \(p\)-adic integral representation, Witt’s type formula, explicit formulas, multiplication formula, and recurrence formula for these \(w\)-Genocchi polynomials.

### 2. Multiple Generalized \(w\)-Genocchi Polynomials and Numbers

Let \(r \in \mathbb{N}\) and \(a_1, \ldots, a_r\) be strictly positive real numbers. The multiple generalized \(w\)-Genocchi polynomials \(G_{n,w}^{(r)}(x; a_1, \ldots, a_r)\) are defined as

\[
\prod_{j=1}^{r} \left( \frac{2t^{a_j}}{(we^t)^{a_j} - 1} \right) e^{xt} = \sum_{n=0}^{\infty} G_{n,w}^{(r)}(x; a_1, \ldots, a_r) \frac{t^n}{n!}, \quad \text{for } t \in \mathbb{R}, w \in \mathbb{C},
\]

where \(|\log w + t| < \min_{1 \leq j \leq r} \{\pi/a_j\}\). The values of \(G_{n,w}^{(r)}(x; a_1, \ldots, a_r)\) at \(x = 0\) are called the multiple generalized \(w\)-Genocchi numbers: when \(r = 1, w = 1,\) and \(a_j = 0\) (\(j = 1, \ldots, r\)), the polynomials or numbers are called the ordinary Genocchi polynomials or numbers.

It is known that

\[
t \int_{\mathbb{Z}_p} w^z e^{(z+x)} d\mu_{-1}(z) = \frac{2t}{we^t + 1} = \sum_{n=0}^{\infty} G_{n,w}(x) \frac{t^n}{n!},
\]

and

\[
th \prod_{j=1}^{r} \int_{\mathbb{Z}_p} w^{z_1+z_2+\cdots+z_r} e^{(z_1+\cdots+z_r+x)} d\mu_{-1}(z_1) \cdots d\mu_{-1}(z_r) = \left( \frac{2t}{we^t + 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} G_{n,w}^{(r)}(x) \frac{t^n}{n!}.
\]

In fact, let us take \(t \in \mathbb{R}, \quad w \in \mathbb{C},\) and we apply the above difference integral formula (1.8) for \(f(z) = w^{az} e^{iaz}\), then we obtain

\[
\frac{2}{(we^t)^a + 1} e^{tx} = \int_{\mathbb{Z}_p} w^{az} e^{(iaz+x)} d\mu_{-1}(z).
\]
By (2.3), we easily see that

\[
\prod_{j=1}^{r} \frac{(2t)^r}{(w t^j)^n + 1} e^{\cdot t} = t^r \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{a_1z_1 + \cdots + a_rz_r} e^{l(a_1z_1 + \cdots + a_rz_r + x)} d\mu_{-1}(z_1) \cdots d\mu_{-1}(z_r) \\
= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{a_1z_1 + \cdots + a_rz_r} (a_1z_1 + \cdots + a_rz_r + x)^n \\
\times d\mu_{-1}(z_1) \cdots d\mu_{-1}(z_r) \frac{t^{n+r}}{n!},
\]

(2.4)

\[
G_{0,w}^{(r)}(x; a_1, \ldots, a_r) = \cdots = G_{r-1,w}^{(r)}(x; a_1, \ldots, a_r) = 0.
\]

(2.5)

By (2.4) and (2.5), we obtain the following fermionic p-adic integral representation formula for these numbers.

**Theorem 2.1** (p-adic integral representation). Let \( r \in \mathbb{N} \) and \( a_1, \ldots, a_r \) be strictly positive real numbers. Then one has a fermionic p-adic invariant integral representation for the multiple generalized w-Genocchi polynomials \( G_{n,w}^{(r)}(x; a_1, \ldots, a_r) \) as follows:

\[
G_{n,w}^{(r)}(x; a_1, \ldots, a_r) = \prod_{j=1}^{r} \frac{(2t)^r}{(w t^j)^n + 1} e^{\cdot t} = t^r \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{a_1z_1 + \cdots + a_rz_r} e^{l(a_1z_1 + \cdots + a_rz_r + x)} d\mu_{-1}(z_1) \cdots d\mu_{-1}(z_r) \\
= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{a_1z_1 + \cdots + a_rz_r} (a_1z_1 + \cdots + a_rz_r + x)^n \\
\times d\mu_{-1}(z_1) \cdots d\mu_{-1}(z_r) \frac{t^{n+r}}{n!},
\]

(2.6)

for \( n \geq r \) and

\[
G_{0,w}^{(r)}(x; a_1, \ldots, a_r) = \cdots = G_{r-1,w}^{(r)}(x; a_1, \ldots, a_r) = 0.
\]

(2.7)

We remark that if we set \( r = 1 \) and \( a_1 = 1 \), then we have the following equation:

\[
G_{n,w}^{(1)}(x; 1) = C_{n+1,w}^{(r)}(x) = E_{n,w}(x).
\]

(2.8)

The generalized w-Genocchi polynomials are given by

\[
\int_{\mathbb{Z}_p} w^{az} e^{l(az + x)} d\mu_{-1}(z) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} w^{az} (az + x)^n d\mu_{-1}(z)t^n.
\]

(2.9)
By comparing the coefficients on both sides in (2.9), we obtain the following identity on the generalized \(w\)-Genocchi polynomials

\[
\frac{G_{n,w}(x; a)}{n!} = \int_{\mathbb{Z}_p} w^{az}(a x + x)^n d\mu_{-1}(z).
\] (2.10)

Similarly, from (2.4), we can obtain the following Witt’s type formula for the multiple generalized \(w\)-Genocchi polynomials.

**Theorem 2.2** (Witt’s type formula). Let \( r \in \mathbb{N} \) and \( a_1, \ldots, a_r \) be strictly positive real numbers. Then one has

\[
\frac{G^{(r)}_{n,w}(x; a_1, \ldots, a_r)}{n!} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{a_1 z_1 + \cdots + a_r z_r} (a_1 z_1 + \cdots + a_r z_r + x)^n d\mu_{-1}(z_1) \cdots d\mu_{-1}(z_r).
\] (2.11)

From (2.4), we can directly calculate the following:

\[
G^{(r)}_{n,w}(x; a_1, \ldots, a_r)
\]

\[
= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{a_1 z_1 + \cdots + a_r z_r} (a_1 z_1 + \cdots + a_r z_r + x)^n \times d\mu_{-1}(z_1) \cdots d\mu_{-1}(z_r) n!
\]

\[
= \sum_{i=0}^{n} \binom{n}{i} x^{n-i} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} w^{a_1 z_1 + \cdots + a_r z_r} (a_1 z_1 + \cdots + a_r z_r)^i d\mu_{-1}(z_1) \cdots d\mu_{-1}(z_r) n!
\]

\[
= \sum_{i=0}^{n} \binom{n}{i}^2 (n-i)! x^{n-i} G^{(r)}_{i,w}(a_1, \ldots, a_r).
\] (2.12)

From (2.12), we get the following explicit formula.

**Theorem 2.3** (explicit formula). Let \( r \in \mathbb{N} \) and \( a_1, \ldots, a_r \) be strictly positive real numbers. Then one has

\[
G^{(r)}_{n,w}(x; a_1, \ldots, a_r) = \sum_{i=0}^{n} \binom{n}{i}^2 (n-i)! x^{n-i} G^{(r)}_{i,w}(a_1, \ldots, a_r).
\] (2.13)
Next we discuss the multiplication formula for the multiple generalized \( w \)-Genocchi polynomials as follows:

\[
G_{n,w}^{(r)}(x; a_1, \ldots, a_r)
= \int_{\mathbb{R}_+} \cdots \int_{\mathbb{R}_+} w^a(z_1 + \cdots + a_r z_r + x)^n \, d\mu_1(z_1) \cdots d\mu_r(z_r) n!
= \lim_{N \to \infty} \sum_{z_1, \ldots, z_r = 0}^{mN-1} w^{a_1 z_1 + \cdots + a_r z_r} (a_1 z_1 + \cdots + a_r z_r + x)^n (-1)^{z_1 + \cdots + z_r}
= m^n \sum_{t_1, \ldots, t_r = 0}^{m-1} w^{a_1 t_1 + \cdots + a_r t_r} (-1)^{t_1 + \cdots + t_r} \lim_{N \to \infty} \sum_{y_1, \ldots, y_r = 0}^{pN-1} (-1)^{m(y_1 + \cdots + y_r)}
\times (\frac{x + a_1 t_1 + \cdots + a_r t_r}{m} + a_1 y_1 + \cdots + a_r y_r)^n n!
= m^n \sum_{t_1, \ldots, t_r = 0}^{m-1} w^{a_1 t_1 + \cdots + a_r t_r} (-1)^{t_1 + \cdots + t_r} n! \int_{\mathbb{R}_+} \cdots \int_{\mathbb{R}_+} (w^m)^a y_1 + \cdots + a_r y_r
\times \left( \frac{x + a_1 t_1 + \cdots + a_r t_r}{m} + a_1 y_1 + \cdots + a_r y_r \right)^n \, d\mu_1(y_1) \cdots d\mu_r(y_r)
= m^n \sum_{t_1, \ldots, t_r = 0}^{m-1} w^{a_1 t_1 + \cdots + a_r t_r} (-1)^{t_1 + \cdots + t_r} \times G_{n,w}^{(r)} \left( \frac{x + a_1 t_1 + \cdots + a_r t_r}{m}; a_1, \ldots, a_r \right).\tag{2.14}
\]

Thus, we obtain the following multiplication formula for the multiple generalized \( w \)-Genocchi polynomials.

**Theorem 2.4 (multiplication formula).** Let \( r \in \mathbb{N} \) and \( a_1, \ldots, a_r \) be strictly positive real numbers. For any \( m \in \mathbb{N} \), one has

\[
G_{n,w}^{(r)}(mx; a_1, \ldots, a_r)
= m^n \sum_{t_1, \ldots, t_r = 0}^{m-1} w^{a_1 t_1 + \cdots + a_r t_r} (-1)^{t_1 + \cdots + t_r} \times G_{n,w}^{(r)} \left( \frac{x + a_1 t_1 + \cdots + a_r t_r}{m}; a_1, \ldots, a_r \right).\tag{2.15}
\]

**Corollary 2.5.** (1) If one sets \( w = a_1 = \cdots = a_r = 1 \) and \( r, n \in \mathbb{N} \), then one obtains Raabe type formula for multiple Genocchi polynomials \( G_{n}^{(r)}(x) \) as follows:

\[
G_{n}^{(r)}(mx) = m^n \sum_{t_1, \ldots, t_r = 0}^{m-1} G_{n}^{(r)} \left( x + \frac{\sum_{i=1}^{n} t_i}{m} \right),\tag{2.16}
\]

where \( (e^t/(e^t+1))^r e^{xt} = \sum_{n=0}^{\infty} G_{n}^{(r)}(x) t^n/n! \).
(2) If one sets $w = 1$ and $r, n \in \mathbb{N}$, then one obtains Carlitz’s multiplication formula for the multiple generalized Genocchi polynomials $G_n^{(r)}(x; a_1, \ldots, a_r)$ as follows:

$$G_n^{(r)}(mx; a_1, \ldots, a_r) = m^n \sum_{t_1, \ldots, t_r=0}^{n-1} G_n^{(r)} \left( x + \sum_{i=1}^{n} a_i \frac{t_i}{m}; a_1, \ldots, a_r \right),$$

(2.17)

where $(2t^r / (\prod_{j=1}^{r} (e^{at_j} + 1)))e^{xt} = \sum_{n=0}^{\infty} G_n^{(r)}(mx; a_1, \ldots, a_r)(t^n/n!)$.

Finally, we discuss the recurrence formula for the multiple generalized $w$-Genocchi polynomials as follows. Let $r \in \mathbb{N}$ and $a_1, \ldots, a_r$ be strictly positive real numbers. For any $k = 1, \ldots, r$, we can directly derive the following equation:

$$\sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \binom{n}{j} G_{j,w}^{(k)}(x | a_1, \ldots, a_k) G_{n-j,w}^{(r-k)}(a_{k+1}, \ldots, a_r) \right) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left( \sum_{m+i=n,m,i \geq 0} G_{m,w}^{(k)}(x | a_1, \ldots, a_k) \frac{t^m}{m!} G_{i,w}^{(r-k)}(a_{k+1}, \ldots, a_r) \frac{t^i}{i!} \right)$$

$$= \left( \sum_{n=0}^{\infty} G_{n,w}^{(k)}(x | a_1, \ldots, a_k) \frac{t^n}{n!} \right) \left( \sum_{i=0}^{\infty} G_{i,w}^{(r-k)}(a_{k+1}, \ldots, a_r) \frac{t^i}{i!} \right)$$

$$= \left( \prod_{j=1}^{k} \frac{(2t)^j}{(e^{at_j} + 1)} \right) \left( \prod_{j=k+1}^{r} \frac{(2t)^j}{(e^{at_j} + 1)} \right) = \prod_{j=1}^{r} \frac{(2t)^j}{(e^{at_j} + 1)}$$

$$= \sum_{n=0}^{\infty} G_n^{(r)}(mx | a_1, \ldots, a_r) \frac{t^n}{n!}.$$

By comparing the coefficients on both sides in (2.18), we obtain the recurrence formula for the multiple generalized $w$-Genocchi polynomials.

**Theorem 2.6 (recurrence formula).** Let $r \in \mathbb{N}$ and $a_1, \ldots, a_r$ be strictly positive real numbers. For any $k = 1, \ldots, r$, one has

$$G_n^{(r)}(mx | a_1, \ldots, a_r) = \sum_{j=0}^{n} \binom{n}{j} G_{j,w}^{(k)}(x | a_1, \ldots, a_k) G_{n-j,w}^{(r-k)}(a_{k+1}, \ldots, a_r).$$

(2.19)

**Acknowledgment**

This paper was supported by Konkuk University in 2011.
References

Submit your manuscripts at http://www.hindawi.com