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Research Article

Specific Differential Equations for Generating Pulse Sequences

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This study presents nonlinear differential equations capable to generate continuous functions similar to pulse sequences. First are studied some basic properties of second-order differential equations with time-dependent coefficients generating bounded oscillating functions F similar to test-functions (the function F and its derivative F' being equal to zero at the same time moments). The necessary intercorrelations between the phase of generated oscillations and the time-dependent coefficients is presented, being shown also that the external command function should be set to a constant value at these time moments so as to determine the amplitude and the sign of generated oscillations. Then some possibilities of using previous differential equation for generating positive-definite functions with null values for the function and its derivative at the same time moments and with constant slope for its amplitude are presented, being shown that the corresponding external command function should present also alternating components. Finally all previous results are used for determining a set of second-order differential equations with time dependent coefficients and a set of external command and corrective functions for generating a pulse sequence useful for modelling time series.

1. Introduction

As was presented in [1], practical test-functions are important in signal analysis due to the fact that such a function f and a finite number of its derivatives $f', f'', \ldots$ are equal to zero at the beginning and the end of a certain working interval and sampling procedures can be performed in a robust manner. In the same paper were studied invariance properties of differential equations able to generate such practical test-functions on a limited time interval. However, aspects presented in that initial study should be extended to functions defined on extended time intervals, so as to investigate possibilities of generating sequences of such practical test functions. Stress should be laid upon sequence of sharp mathematical functions similar to pulses, while in the limit case pulse sequences can be considered as time series.

2. Differential Equations for Generating Bounded Oscillating Functions

For a limited time interval it was shown [2] that a second-order system working at the limit of stability and an integrator can generate a function represented by

$$z = \int_0^{2\pi T} A(1 - \cos(\omega t))dt \tag{2.1}$$

for an external constant input $u = A$ (the integration time interval being represented by a period of the oscillating second-order system, while $T = 1/\omega$) in a robust manner. At the end of the integration, the value $z(2\pi T)$ can be sampled in a robust manner, while both $z(t)$ and $z'(t)$ are equal to zero. Moreover, filtering properties of such a structure are similar to those of a structure consisting of an asymptotically stable second-order system and an integrator.

Let us check the possibility of obtaining a more general positive-definite oscillating function as a result of a differential equation. We consider the function $F(t)$ written as

$$F(t) = \left[1 - \cos f(t)\right] \tag{2.2}$$

(its average value is equal to unity on an unlimited time interval). This implies

$$\begin{gathered} F'(t) = f'(t)\sin f(t), \\ F''(t) = f''(t)\sin f(t) + \left(f'(t)\right)^2 \cos f(t). \end{gathered} \tag{2.3}$$

We are looking for a second-order differential equation under the form

$$F''(t) = a(t)F'(t) + b(t)(1 - F(t)) \tag{2.4}$$

(similar to linear differential equations with time-invariant coefficients generating damped oscillations, it is considered that $F'$and F'' are equal to zero when $F(t)$ equals its average value, this means when $F(t) = 1$).

By substituting in the previous equation $F(t)$, respectively, $F'(t)$, $F''(t)$ with $1-\cos f(t)$, respectively, $f'(t)\sin f(t)$, $f''(t)\sin f(t) + (f'(t))^2 \cos f(t)$ it results

$$f''\sin f + f'^2 \cos f = af'\sin f + b\cos f. \tag{2.5}$$

For a robust solution we should equate functions multiplying $\sin f$, respectively $\cos f$ on both sides. It results

$$b = f'^2, \qquad a = \frac{f''}{f'}. \tag{2.6}$$

By considering $f' = \pm b^{1/2}$ it results

$$f'' = \pm \frac{b'}{2b^{1/2}}, \qquad a = \frac{f''}{f'} = \frac{b'}{2b}. \tag{2.7}$$

Thus the differential equation generating the output $F = 1 - \cos f(t)$ (and its derivative $F' = f'(t) \sin f(t)$) can be written as

$$F'' = \frac{b'}{2b} F' + b(1 - F) \tag{2.8}$$

or

$$F'' = \frac{f'}{f''} F' + f'^2 (1 - F), \tag{2.9}$$

where b equals f'^2. Since $b = f'^2$ is a positive-definite function, at time moments when both $F(t)$ and $F'(t)$ vanishes (implying $\cos f(t) = 1$, $\sin f(t) = 0$) and the previous equation becomes

$$F'' = f'^2, \tag{2.10}$$

so for $|f'(t)| \approx 0$ at this time moments the second derivative $F'' \approx 0$ and thus function F can be approximated as zero on a large time interval.

Next step consists in studying the possibilities of generating a function $F(t)$ represented by

$$F = G\big(1 - \cos p(t)\big). \tag{2.11}$$

It results that

$$\begin{gathered} F = G - G \cos p, \\ F' = G' - G' \cos p + G p' \sin p, \\ F'' = G'' - G'' \cos p + G' p' \sin p + G p'' \sin p + G' p' \sin p + G p'^2 \cos p. \end{gathered} \tag{2.12}$$

We consider a slightly changed differential equation—the term $(1 - F)$ in previous differential equations being substituted by $(G - F)$, where G is an external command function (it can be noticed that the amplitude of the external command G should be proportional to the amplitude of function F). Thus, the differential equation becomes

$$F'' = \frac{b'}{2b} F' + b(G - F). \tag{2.13}$$

Substituting the mathematical expressions of $F(t)$ and $F'(t)$, $F''(t)$ previously obtained in this modified differential equation, it results

$$G'' + \left(Gp'^2 - G''\right)\cos p + (2G'p' + Gp'')\sin p = \left(\frac{b'G'}{2b}\right) + \left(bG - \frac{b'G'}{2b}\right)\cos p + \left(\frac{b'Gp'}{2b}\right)\sin p. \tag{2.14}$$

By equating the terms which do not depend on $\sin p$, $\cos p$ and the functions multiplying $\cos p, \sin p$ on both sides of this equation, it results

(i)

$$G'' = \frac{b'G'}{2b}, \tag{2.15}$$

(ii)

$$Gp'^2 - G'' = bG - \frac{b'G'}{2b}, \tag{2.16}$$

which implies

$$Gp'^2 - G'' = bG - G'' \tag{2.17}$$

(by substituting $b'G'/2b$ from previous equation), and

$$Gp'^2 = bG, \tag{2.18}$$

then

$$p'^2 = b, \quad p' = \pm b^{1/2} \tag{2.19}$$

(similar to the analysis of a possible function $F = 1 - \cos f(t)$ presented at the beginning of this paragraph), and

(iii)

$$2G'p' + Gp'' = \frac{b'Gp'}{2b}. \tag{2.20}$$

Since $p'^2 = b$, it results $b' = 2p'p''$, and by substituting b' with this mathematical expression in last equation it results

$$2G'p' + Gp'' = p''G \tag{2.21}$$

(the right-hand side of the equation was reduced by dividing both the numerator and the denominator with $2p'^2$) and

$$2G'p' = 0, \quad G'p' = 0 \tag{2.22}$$

with possible solutions: $G' = 0$ (G is a constant function) and/or $p' = 0$ (p is a constant function). However, we should notice that p cannot be a constant function, while function $b = p'^2$ cannot be equal to zero (this would correspond to a differential equation which does not depend on the external command G).

Substituting $b = p'^2$ in equation $G'' = b'G'/2b$ previously obtained (see (i)) it results

$$G'' = \frac{p''G'}{p'} \tag{2.23}$$

(the right-hand side of the equation was reduced by dividing both the numerator and the denominator with $2p'^2$) and

$$G''p' - p''G' = 0 \tag{2.24}$$

which can be written also as

$$(G'p')' - 2G'p'' = 0. \tag{2.25}$$

Substituting $G'p' = 0$ (according to the previous conclusion; see (iii)) it results

$$2G'p'' = 0, \quad G'p'' = 0 \tag{2.26}$$

with possible solutions $G' = 0$ and/or $p'' = 0$.

By studying possible solutions for the equations presented at (i), (ii) and (iii), it results that ($b = p'^2$), and ($G' = 0$ and/or $p' = 0$) and ($G' = 0$ and/or $p'' = 0$). This implies that $b = p'^2$ (as in the previous case), and $G' = 0$, so $G(t)$ should be a constant function.

A more general second-order differential equation generating a bounded positive-definite oscillating function $F(t) = G(t)(1 - \cos p(t))$ is represented by

$$F'' = \frac{b'}{2b}F' + b(H - F) \tag{2.27}$$

(it can be noticed that the external command denoted by $H(t)$ could be different from the function $G(t)$ corresponding to the amplitude of F). The analysis is similar to the one performed in the previous case (when the external command and the amplitude of the generated function F were considered to be equal). In the same manner, F' and F'' are computed and substituted in the second-order differential equation. By equating the functions multiplying $\cos p$, $\sin p$ and on both sides of this mathematical expression, it results the same equations as in the previous case (see (ii), (iii)), with the same consequences: $b = p'^2$

(by equating functions multiplying $\cos p$), and $G' = 0$ (by equating functions multiplying $\sin p$).

By equating the terms which do not depend on $\sin p, \cos p$ on both sides of this mathematical expression (for a robust solution) it results

$$G'' = \frac{b'G'}{2b} + b(H - G) \tag{2.28}$$

(which differs to equation obtained in the previous case; see (i)).

Since $G' = 0$, it results $G'' = 0$. Substituting $G' = 0$, $G'' = 0$ in previous equation it results

$$b(H - G) = 0, \tag{2.29}$$

which implies

$$H - G = 0, \quad H = G \tag{2.30}$$

(it was shown that b cannot be equal to zero). So for a robust solution it is necessary for the external command H to be equal to the amplitude $G(t)$ of $F(t)$, both being constant functions ($H = G$, so $H' = G' = 0$).

It is also possible to equate just the functions multiplying $\cos p$ on both sides of the mathematical expression (this implies $b = p'^2$ and, consequently, $b' = 2p'p''$, as has been shown). Performing a reduction of similar terms on both sides of equation, it results

$$G'' + (2G'p' + Gp'')\sin p = \frac{b'G'}{2b} + b(H - G) + \left(\frac{b'Gp'}{2b}\right)\sin p. \tag{2.31}$$

Substituting $b = p'^2$, $b' = 2p'p''$ it results

$$G'' + (2G'p' + Gp'')\sin p = \left(\frac{p''G'}{p'}\right) + p'^2(H - G) + (p''G)\sin p. \tag{2.32}$$

The first fraction on right side has been obtained by dividing both numerator and denominator by $2p'$ (a nonzero expression, since $p'^2 \neq 0$, $p' \neq 0$ as was shown) and the function multiplying $\sin p$ on right side has been obtained by dividing both numerator and denominator by $2p'^2$ (a nonzero value, since $p'^2 \neq 0$ as was shown). It results

$$G'' - \left(\frac{p''G'}{p'}\right) + p'^2(G - H) + 2G'p'\sin p = 0, \tag{2.33}$$

so the function H (the external command) can be written as

$$H = \frac{G'' - p''G'/p' + 2G'p'\sin p}{p'^2} + G. \tag{2.34}$$

This result can be used for determining the external command function $H(t)$ if the function $G(t)$ (the amplitude of the function F to be generated) is known. However, it can be noticed that H contains an oscillating part $(2G'/p')\sin p$, and so the differential equation is far of being robust.

It can be noticed that the function $F = G(t)(1 - \cos p(t))$ has an important property: the function $F(t)$ and its derivative $F'(t) = G' - G'\cos p(t) + Gp'\sin p(t)$ are equal to zero at the same time moments t_k, when $\cos p(t_k) = 1$ and $\sin p(t_k) = 0$ (supposing $G(t)$ to be a constant function, as has been shown). Inspecting the second-order differential equation $F'' = (b'/2b)F' + b(H - F)$ (where $b = p'^2$ and H is a constant function—as has been shown) it results that at this time moments the differential equation can be written as

$$F''(t_k) = \left(\frac{b'}{2b}\right)F'(t_k) + b(H(t_k) - F(t_k)) = bH(t_k). \tag{2.35}$$

So at this time moments (when the state variables F and F' of the second-order differential equations are equal to zero) the second derivative is determined by the external command H. This suggests that at this time moments a new function:

$$F_k(t) = G_k\left(1 - \cos p_k(t)\right) \tag{2.36}$$

can be generated by a differential equation:

$$F_k''(t) = \left(\frac{b_k'(t)}{2b_k(t)}\right)F_k'(t) + b_k(t)(H_k - F_k(t)), \tag{2.37}$$

where the time moment t_k has become the origin of time for the new working interval, with the external command function represented by the constant function H_k. The amplitude and the sign of the external command H at the time moments when F and F' are equal to zero determine the amplitude and sign of the function F on next working time interval (until both F, F' become zero again). The phase $p(t)$ of the function F and the function $b(t)$ which determines the differential equations on each working interval are intercorrelated by $b = p'^2$, as has been shown.

It can be noticed that the previous differential equations can be easily implemented (using either analogue circuits if a high working frequency is necessary or digital circuits if more accurate results are required). In case of analogue circuits, mathematical operations as multiplying and/or dividing functions are available due to high performances of operational amplifiers. The errors between the required function and the function generated by analogue circuits cannot be avoided; however, the cumulative errors can be easily set to zero at the end of each working interval t_k, by simply setting to zero the functions F, F' at these time moments (when they should present a null value, according to theory). By performing this operation before applying a new external command function H_k, a robust implementation of the mathematical model can be acquired.

The same aspect is valid also if digital circuits are used. A higher accuracy can be obtained, but numerical errors cannot be avoided and thus numerical values of F and F' should be set to zero at the end of each working interval (as in case of analogue circuits).

3. Functions Similar to Pulse Sequences Generated by Second-Order Differential Equations

The mathematical expression for the second-order equation presented in previous paragraph should be used for generating functions similar to pulse sequences or time series. For this purpose, at first step it will be presented a differential equation able to generate a function with a constant slope for its amplitude, this means a function which can be written as

$$F = G(t)\left(1 - \cos p(t)\right) = t\left(1 - \cos p(t)\right) \tag{3.1}$$

(the slope of the amplitude $G(t)$ has been considered to be equal to unity). It results

$$\begin{gathered} F' = 1 - \cos p + tp' \sin p, \\ F'' = 2p' \sin p + tp'' \sin p + tp'^2 \cos p. \end{gathered} \tag{3.2}$$

By substituting the previous expression for F'' into the left side of the differential equation $F'' = aF' + b(H - F)$ (where $b = p'^2$, $a = b'/2b = p''/p'$ as was shown), by substituting the previous expressions for F and F'' on the right side of the differential equation and by considering that the external command function H can be represented by a sum:

$$H = h_1(t) + h_2(t) \sin p(t) + h_3 \cos p(t), \tag{3.3}$$

it results

$$\left(2p' + tp''\right) \sin p + tp'^2 \cos p = (a + bh_1 - bt) + \left(atp' + bh_2\right) \sin p + (bh_3 + bt - a) \cos p. \tag{3.4}$$

By equating the terms which do not depend on $\sin p$, $\cos p$ and the functions multiplying $\cos p, \sin p$ on both sides of this equation, it results

(i)

$$a + bh_1 - bt = 0, \tag{3.5}$$

(ii)

$$atp' + bh_2 = 2p' + tp'', \tag{3.6}$$

(iii)

$$bh_3 + bt - a = tp'^2. \tag{3.7}$$

Since $b = p'^2$, $a = p''/p'$ (as was shown) from previous three equations it results that

$$h_1 = \frac{bt - a}{b} = t - \frac{p''}{p'^3},$$
$$h_2 = \frac{2p' + tp'' - atp'}{b} = \frac{2p''' + tp''' - p'''t}{p'^2} = \frac{2}{p'}, \tag{3.8}$$
$$\frac{tp'^2 - bt + a}{b} = -t + \frac{tp'^2 + p''/p'}{p'^2} = -t + t + \frac{p''}{p'^3} = \frac{p''}{p'^3}.$$

So the external command function $H(t)$ necessary for generating a function $F = t(1-\cos p(t))$ by a differential second-order equation $F'' = (p''/p')F' \quad + (p')^2(H - F)$ is represented by

$$H = h_1(t) + h_2(t)\sin p(t) + h_3 \cos p(t) = t - \left(\frac{p''}{p'^3}\right)(1 - \cos p) + \left(\frac{2}{p'}\right)\sin p. \tag{3.9}$$

However, pulse sequences are best represented by functions as

$$F_n(t) = A\left(\sin p(t)\right)^n = A\sin^n p, \tag{3.10}$$

which is similar to a constant time series for $n \to \infty$ (the functions differs to zero just when $\sin p = 1$).

For $A = 1$, it results

$$F_n = \sin^n p\ ,$$
$$F_n' = p' n \sin^{n-1} p \cos p, \tag{3.11}$$
$$F_n'' = -np'^2 \sin^n p + n(n-1)p'^2 \sin^{n-2} p\cos^2 p + np'' \sin^{n-1} p \cos p.$$

By substituting $\cos^2 p = 1 - \sin^2 p$ in the second term of right-hand side, it results

$$F_n'' = -n^2 p'^2 \sin^n p + n(n-1)p'^2 \sin^{n-2} p + np'' \sin^{n-1} p \cos p. \tag{3.12}$$

The first term on right-hand side can be written as

$$-n^2 p'^2 \sin^n p = -b_n F_n \tag{3.13}$$

(where $b_n = n^2 p'^2$).

The second term on right-hand side can be written as

$$n(n-1)p'^2 \sin^{n-2} p = n^2 p'^2 \sin^{n-2} p - np'^2 \sin^{n-2} p = b_n H_{n-2} - b_n h_{n-1}, \tag{3.14}$$

where $b_n = n^2 p'^2$, $H_n = \sin^{n-2} p$, $h_{n-1} = (\sin^{n-1} p)/n$.

The third term on right side can be written under the form

$$np'' \sin^{n-1} p \cos p = a_n F'_n, \tag{3.15}$$

where $a_n = p''/p'$ and $F'_n = p' n \sin^{n-1} p \cos p$ as was shown.

So the second derivative F''_n can be written under the form

$$F''_n = a_n F'_n + b_n (H_{n-2} - h_{n-1} - F_n). \tag{3.16}$$

It can be noticed that the function F_n can be generated by a second-order differential equation similar to the one presented in previous paragraph. The time-dependent coefficient a_n (multiplying F'_n) equals the time-dependent coefficient a of the differential equation presented in the previous paragraph for any n; the time-dependent coefficient b_n (multiplying the difference between the external command $H_{n-2} - h_{n-1}$ and the function F_n to be generated) is represented by time-dependent coefficient $b = p'^2$ of the differential equation presented in the previous paragraph divided by n^2.

Two important aspects should be emphasized.

(i) The first term of the external command function $H_{n-2} - h_{n-1}$ is represented by $H_{n-2} = F_{n-2}$ (since $F_{n-2} = \sin^{n-2} p = H_{n-2}$).

(ii) The second term of the external command function $H_{n-2} - h_{n-1}$ is represented by a corrective function h_{n-1} which can be written as: $h_{n-1} = F_{n-1}/n$; so, for $n \to \infty$ the function $h_{n-1} \to 0$ and thus the external command function can be approximated by F_{n-2}.

This suggests that a set of differential equations

$$F''_{k+2} = a F'_{k+2} + b_{k+2} \left(F_k - \frac{F_{k+1}}{k+2} - F_k \right), \quad k = 2, 3, \ldots \tag{3.17}$$

can be used for generating the complete set of functions $F_n = \sin^n p$ starting from F_2, F_3. First F_2 and F_3 are used for generating F_4, then F_3 and F_4 are used for generating F_5 and so on.

Such a set of differential equations can be also implemented using analogue or digital circuits for multiplying and/or dividing functions. The errors between the required set of functions and the functions generated by analogue or digital circuits become larger as n increases, because the number of functions which should be previously generated also increases (it is necessary to generate functions corresponding to $n - 1$, $n - 2$, $n - 3$ before generating a function corresponding to a certain n and the errors are cumulative). For this reason, the operation of setting to zero all generated functions should be done more often and the working interval should be shortened (it can include just a few alternating components of generated signal).

4. Conclusions

This study has presented nonlinear differential equations capable to generate continuous mathematical functions similar to pulse sequences. First were studied some basic properties

of second-order differential equations $F'' = aF' + b(H - F)$ with time-dependent coefficients $a(t), b(t)$ generating bounded oscillating functions $F = G(1-\cos p(t))$ similar to test-functions (the function F and its derivative F' being equal to zero at the same time moments) and similar to wavelets also [3]. The necessary intercorrelations between the phase of generated oscillations and the time-dependent coefficients have been obtained as $b = p'^2$, $a = p''/p'$, being shown also that the external command function should be set to a constant value at these time moments so as to determine the amplitude and sign of generated oscillations. Then some possibilities of using previous differential equation for generating positive-definite functions with null values for the function and its derivative at the same time moments and with constant slope for its amplitude are presented (similar to solitary waves, see also [4]), being shown that the corresponding external command function should present also alternating components as $h_2 \sin p(t), h_3 \cos p(t)$. Finally all previous results are used for determining a set of second-order differential equations with time dependent coefficients and a set of external command and corrective functions for generating a pulse sequence useful for modelling time series (similar to time series application to surface analysis [5] or to fracture phenomena [6]) or for computing autocorrelation function [7] with extension to network traffic [8]). The study is also suitable for extending wavelets dynamical aspects (see [9]) to correlation-based computational methods (see [10]).

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